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UNIT-SPHERE PRESERVING MAPPINGS

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Abstract. We prove that if a one-to-one mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ $(n \geq 2)$ preserves the unit $n-1$ spheres (S^{n-1}), then f is a linear isometry up to translation.

1. Introduction

Let X and Y be normed spaces. A mapping $f : X \to Y$ is called an isometry if f satisfies the equality

$$\|f(x) - f(y)\| = \|x - y\|$$

for all $x, y \in X$. A distance $r > 0$ is said to be preserved (conservative) by a mapping $f : X \to Y$ if

$$\|f(x) - f(y)\| = r \quad \text{for all} \quad x, y \in X \quad \text{with} \quad \|x - y\| = r.$$

If f is an isometry, then every distance $r > 0$ is conservative by f, and conversely. We can now raise a question whether each mapping that preserves certain distances is an isometry. Indeed, A. D. Aleksandrov [1] had raised a question whether a mapping $f : X \to X$ preserving a distance $r > 0$ is an isometry, which is now known to us as the Aleksandrov problem. Without loss of generality, we may assume $r = 1$ when X is a normed space (see [15]).

F. S. Beckman and D. A. Quarles [2] solved the Aleksandrov problem for finite-dimensional real Euclidean spaces $X = \mathbb{R}^n$ (see also [3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 16, 17, 18, 19]):

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THEOREM 1.1 (Theorem of Beckman and Quarles). *If a mapping* $f : \mathbb{R}^n \to \mathbb{R}^n$ $(2 \leq n < \infty)$ *preserves a distance* $r > 0$*, then* f *is a linear isometry up to translation.*

It seems to be interesting to investigate whether the 'distance $r > 0$' in the above theorem can be replaced by some properties characterized by 'geometrical figures' without loss of its validity.

In [9], the first author proved that if a one-to-one mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ $(n \geq 2)$ maps the periphery of every regular triangle (quadrilateral or hexagon) of side length $a > 0$ onto the periphery of a figure of same type with side length $b > 0$, then there exists a linear isometry $I : \mathbb{R}^n \to \mathbb{R}^n$ up to translation such that

$$f(x) = (b/a)I(x).$$

In this note, we show further that if a one-to-one mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ $(n \geq 2)$ maps every unit $n-1$ sphere (S^{n-1}) onto a unit $n-1$ sphere (S^{n-1}), then f is a linear isometry up to translation.

2. Main theorem

Now, let us prove our main theorem.

THEOREM 2.1. *If a one-to-one mapping* $f : \mathbb{R}^n \to \mathbb{R}^n$ $(n \geq 2)$ *maps every unit* $n-1$ *sphere onto a unit* $n-1$ *sphere, then* f *is a linear isometry up to translation.*

PROOF. Assume $n \geq 3$ first. We show f preserves the distance 2. Assume $T_1, T_2 \in \mathbb{R}^n$ and $d(T_1, T_2) = 1$. Without loss of generality assume that $T_1 = ((1/\sqrt{2}), (1/\sqrt{2}), 0, \ldots, 0)$ and $T_2 = (0, (1/\sqrt{2}), (1/\sqrt{2}), 0, \ldots, 0)$. Define $S_1, \ldots, S_n, S_{n+1}$ to be the unit $n-1$ spheres (S^{n-1}) centered at $A_1 = (\sqrt{2}, 0, \ldots, 0)$, $A_2 = (0, \sqrt{2}, 0, \ldots, 0), \ldots, A_n = (0, \ldots, 0, \sqrt{2})$, and $A_{n+1} = (x, x, \ldots, x)$ respectively, where x is the unique negative real number satisfying $d(A_i, A_{n+1}) = 2$, $i = 1, \ldots, n$. The S_i's are all unit $n-1$ spheres such that any pair of these spheres meet each other at exactly one point. Then the same must be true for their image spheres $D_1, \ldots, D_n, D_{n+1}$. Denote the centers of these image spheres by $B_1, \ldots, B_n, B_{n+1}$. Because any pair of these spheres intersect each other at exactly one point, we have $d(B_i, B_j) = 2$ whenever $i \neq j$.

Now if we are given two sets in $\mathbb{R}^n$, each of which contain $n+1$ points whose mutual distances are all equal to 2, then there is an isometry $\phi : \mathbb{R}^n \to \mathbb{R}^n$ with $\phi(B_i) = A_i$, and consequently $(\phi \circ f)(S_i) = S_i$, $i = 1, \ldots, n+1$. Since $S_1 \cap S_2 = \{T_1\}$ and $S_2 \cap S_3 = \{T_2\}$, we have necessarily $(\phi \circ f)(T_1) = T_1$ and $(\phi \circ f)(T_2) = T_2$. Thus $d(f(T_1), f(T_2)) = 1$, as desired.

For $n = 2$, consider two points A and B in $\mathbb{R}^2$ which are separated from each other by the unit distance. Then we can draw three unit circles $\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3$ such that any two of them touch each other at one point as in Figure 1. If we call $c_i = f(C_i)$ $(i = 1, 2, 3)$, then we get the three contact points a, b, c which form the three vertices of a regular triangle with unit distance. Now since $f(A) = a$ and $f(B) = b$, the proof is complete (see Figure 1).

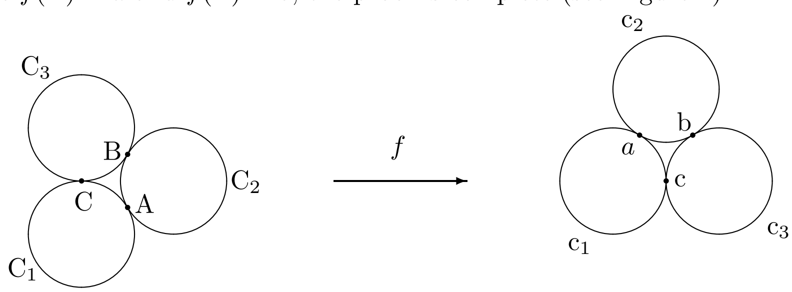

Figure 1

□

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