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EXTENSION DIMENSION OF INVERSE LIMITS. CORRECTION OF A PROOF

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Abstract. The erroneous proof of a lemma in a previous paper of the author on extension dimension of inverse limits is replaced by a correct one.

Recently I. Ivanšić and L. Rubin discovered an error in the proof of Lemma 4 of the author's paper [2]. In that proof, for a simplicial complex K, its geometric realization $|K|$ (endowed with the weak topology), a mapping $\phi\colon V \to I = [0,1]$ of a space V and two contiguous mappings $g, h\colon V \to |K|$, the author considered the function $k\colon V \to |K|$, defined by putting $k(x) = \phi(x)g(x) + (1 - \phi(x))h(x)$, for $x \in V$. Then he erroneously assumed that k is continuous, which is not always the case (see [1]). The purpose of this note is to give a correct proof of Lemma 4.

Lemma 4. *Let X be a normal space and K a simplicial complex. Let $A \subseteq X$ be a closed set and let $V, U \subseteq X$ be open sets such that $A \subseteq V \subseteq \overline{V} \subseteq U$. If $h\colon U \to |K|$ and $g\colon V \to |K|$ are mappings such that $h|V$ and g are contiguous mappings, then there exists a mapping $k\colon U \to |K|$, which is contiguous to h and is such that*

$$k|A = g|A, \tag{1}$$

$$k|U \backslash V = h|U \backslash V. \tag{2}$$

In the proof we will use the following Lemma.

Lemma 4'. *Let V be a topological space, K a simplicial complex and let $h, g\colon V \to |K|$ be contiguous mappings. Then there exists a homotopy*

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$\phi\colon V\times I\to|K|$ such that $\phi(x,0)=h(x)$ and $\phi(x,1)=g(x)$, for $x\in V$. Moreover, if for an $x\in V$, $g(x)$ and $h(x)$ belong to a simplex $\sigma\in K$, then $\phi(x\times I)\subseteq\sigma$.

Proof of Lemma 4. By normality of X, there exist an open set H, $A\subseteq H\subseteq\overline{H}\subseteq V$ and a mapping $\alpha\colon X\to I$ such that $\alpha|A=1$ and $\alpha|(X\backslash H)=0$. By Lemma 4′, there is a homotopy $\phi\colon V\times I\to|K|$ such that $\phi(x,0)=h(x)$ and $\phi(x,1)=g(x)$, for $x\in V$. Moreover, if for an $x\in V$, $g(x)$ and $h(x)$ belong to a simplex $\sigma\in K$, then $\phi(x\times I)\subseteq\sigma$. We define a mapping $k\colon U\to|K|$ by putting

$$k(x)=\begin{cases}\phi(x,\alpha(x)) & x\in V,\\ h(x), & x\in U\backslash\overline{H}.\end{cases}\tag{3}$$

Note that V and $U\backslash\overline{H}$ are open subsets of U, which cover U. Moreover, since $U\backslash\overline{H}\subseteq X\backslash\overline{H}$, we see that, for $x\in V\cap(U\backslash\overline{H})$, $\alpha(x)=0$, and thus, the first line of (3) yields the value $k(x)=\phi(x,0)=h(x)$. Therefore, k is indeed a well-defined mapping $k\colon U\to|K|$. If $x\in A$, then $\alpha(x)=1$. Since $x\in V$, we conclude that $k(x)=\phi(x,1)=g(x)$. If $x\in U\backslash V$, then $x\in U\backslash\overline{H}$ and thus, $k(x)=h(x)$. Finally, every $x\in V$ admits a simplex $\sigma\in K$ such that $h(x),g(x)\in\sigma$. Let us show that also $k(x)\in\sigma$. Indeed, by Lemma 4′, $\phi(x,t)\subseteq\sigma$, for every $t\in I$. In particular, $k(x)=\phi(x,\alpha(x))\in\sigma$. If $x\in U\backslash V$, then by definition (3), $k(x)=h(x)$. All this proves that h and k are contiguous mappings. □

Proof of Lemma 4′. Let $|K|_m$ denote the geometric realization of the complex K, endowed with the metric topology (see [3], Appendix 1.3). It is well known that the identity function $i\colon|K|\to|K|_m$ is continuous (see [3], Appendix 1.3, Corollary 5). Therefore, the mappings $h,g\colon V\to|K|$ can also be viewed as mappings $h,g\colon V\to|K|_m$. Since the mappings h and g are contiguous, the following formula defines a function $\psi\colon V\times I\to|K|_m$.

$$\psi(x,t)=(1-t)h(x)+tg(x),\ (x,t)\in V\times I.\tag{4}$$

Moreover, if for an $x\in V$, both points $h(x)$ and $g(x)$ belong to a simplex $\sigma\in K$, then also $\psi(x\times I)\subseteq\sigma$. By Theorem 8 of Appendix 1.3 of [3], $\psi\colon V\times I\to|K|_m$ is continuous and thus, it is a homotopy which connects h to g.

There exists a mapping $j\colon|K|_m\to|K|$ and a homotopy $J\colon|K|\times I\to|K|$, which connects the identity $1_{|K|}$ to ji. Moreover, for each simplex $\sigma\in K$, $J(\sigma\times I)\subseteq\sigma$ (see [3], Appendix 1.3, the proof of Theorem 10 and Remark 1 or Lemma 2.3 of [4]). We now define $\phi\colon V\times I\to|K|$ as the juxtaposition of three homotopies Jh, $j\psi$ and the reverse of Jg, i.e., for $(x,t)\in V\times I$, we put

$$\phi(x,t) = \begin{cases} J(h(x),3t) & t \in [0,1/3], \\ j\psi(x,3t-1), & t \in [1/3,2/3], \\ J(g(x),-3t+3), & t \in [2/3,1]. \end{cases} \tag{5}$$

The mapping ϕ is well defined, because for $t = 1/3$, the first and the second row in (5) yield the same value $\phi(x,1/3) = jh(x)$ and for $t = 2/3$, the second and the third row in (5) yield the same value $\phi(x,2/3) = jh(x)$. Furthermore, $\phi(x,0) = J(h(x),0) = h(x)$ and $\phi(x,1) = J(g(x),0) = g(x)$. Finally, let us show that whenever $g(x)$ and $h(x)$ belong to a simplex $\sigma \in K$, then $\phi(x \times I) \subseteq \sigma$. Indeed, $J(\sigma \times I) \subseteq \sigma$ and thus, the first and third row of (5) imply that $\phi(x,t) \in \sigma$, for $t \in [0,1/3] \cup [2/3,1]$. Moreover, by (4), $\psi(x \times I) \subseteq \sigma$. Since $j(\sigma) = J(\sigma \times 1) \subseteq J(\sigma \times I) \subseteq \sigma$, we conclude that also $j\psi(x \times I) \subseteq \sigma$. Consequently, by the second row in (5), $\phi(x,t) \in \sigma$, for $t \in [1/3,2/3]$. ∎

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