ISSN 1846-6168
UDK 534.131:511.34

ANALYSIS OF LONGITUDINAL OSCILLATIONS FOR SYSTEMS WITH CONTINUOUS VARIABLE PARAMETERS USING FORCE INTEGRATION METHOD

ANALYSIS OF LONGITUDINAL OSCILLATIONS FOR SYSTEMS WITH CONTINUOUS VARIABLE PARAMETERS USING FORCE INTEGRATION METHOD

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Pregledni rad

Abstract: *Force integration method for systems with continuous variable parameters analysis is described. The method is based on analytical solutions of according differential equations. Method substance is shown on the problem of free longitudinal vibrations of straight rod with random continuous variable of longitudinal stiffness and mass per unit length taking into account resistance. Equations for rod condition dynamic parameters and the efficient way of its numerical realization are shown. Free lateral vibrations of cantilever wedge are researched.*

Keywords: *analytical solutions, vibrations of cantilever wedge, force integration method, rod longitudinal vibrations, oscillation frequencies, vibrations factors.*

Review article

Sažetak: *U članku je opisana metoda integracija sile za analizu sustava s kontinuirano varijabilnim parametrima. Metoda se temelji na analitičkim rješenjima odgovarajućih diferencijalnih jednadžbi. Suština metode prikazana je na problemu slobodnih uzdužne vibracija ravnog štapa sa slučajnim kontinuiranim varijablama uzdužne krutosti i jedinične linijske mase, uzimajući u obzir čvrstoću. Prikazane su jednadžbe za dinamičke parametre stanja štapa i učinkovit način njihovog rješavanja. Istražene su slobodne uzdužne vibracije konzolnog klina.*

Ključne riječi: *analitičko rješenje, vibracije konzolnog klina, metoda integracije sile, uzdužne vibracije štapova, frekvencije titranja, faktori vibracije.*

1. INTRODUCTION

Systems with variable parameters occur in different branches of industry and agriculture. It is primarily rods with variable cross-section, which are applied as structural components or spatial patterns. Multipurpose steel poles, which are widely applied in municipal engineering, power engineering (power transmission lines supports, windmill towers, wind-powered generators supports), television, radio and astronautics (antennas of different structures), etc., are examples of such structures. Beam type structure elements should be particularly mentioned, whose stiffness, mass per unit length, parameters of elastic foundation (at it existence), loads or other parameters vary on length by some rule.

Systems with variable parameters occur at plate bending with thickness, which vary in the direction of one or both coordinates, or plates with variable stiffness laying on elastic foundation that can be described by Winkler model or by two elastic parameters. In construction, there are many objects with variable stiffness, such as chimneys, stacks, water towers, cooling towers etc.

From the mathematic point of view, problem of strength, rigidity, vibrations analysis of mentioned above systems lead to differential equations with variables factors. As a rule, if there is a general integral of equation found, initial problem has its solution too. In such cases, it is often said that the problem is solved by force integration method. This article is devoted problem of free vibrations of straight rod with random continuous variable lateral stiffness and random continuous variable muss per unit length taking into account resistance.

2. MAIN SECTION

2.1. Analysis of recent researches and publications

There are many papers devoted to the problem of rod vibrations. It means that it is an actual and applied problem. Among recent publications works [1-13] should be noted. Most attention is paid to vibrations of rods with continuous variable parameters [6, 8, 9, 12, 13]. Despite the fact that the analytical solutions have undoubtedly the advantage, they occur only for a few kinds of rods. In fact, it is about cases where the factors of the corresponding differential equations change by binomial rules [2, 6, 12]. In that cases solution is expressed through Bessel functions. For the general case it is stated, that analytical solutions are unknown [12] or noted that they cannot be found [9]. That is why researches are mainly based on approximate methods.

In papers [14-16] for the first time analytical solutions for general differential equation of structural mechanic with random variable parameters are formed. The essence of applied integration method is described detailed in [16]. In the author's view, the existence of such method opens up new perspectives how to solve a variety of the above-mentioned technical problems and, in the first place, the problems of calculating systems with continuous variable parameters.

2.2. Free longitudinal vibrations of rod

The general scheme of vibrations is shown in figure 1.

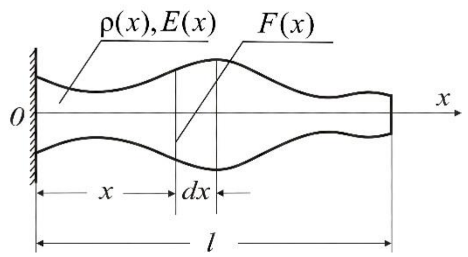

Figure 1. Rod design model on longitudinal vibrations

It is known [8, 12, 17], that differential equation of free longitudinal vibrations of rod, taking into account the resistance forces, generally looks as follows:

$$m(x)\frac{\partial^2 u}{\partial t^2}+p(x,t)=\frac{\partial}{\partial x}\left(E(x)F(x)\frac{\partial u}{\partial x}\right)+r(x,t) \tag{1}$$

where *E(x)F(x)* - longitudinal stiffness at the point *x*; *E(x)* – rod material's modulus of elaticity; *m(x)=p(x)F(x)* – mass per unit length intensity at the point *x*; *p(x)* – rod material's density; *p(x,t)* – external resistance forces intensity; *r(x,t)* – inner resistance forces intensity; *u(x,t)* – unknown function – longitudinal displacement of rod cross-section with coordinate *x* at the time of *t*.

This equation is right for the model, which is not taking into account inertial forces, which appears because of transverse strains.

There are many hypotheses to take into account resistance forces. Let the external resistance forces are proportional to the rod mass and velocity [17]. Inner friction will be taken into account by Kelvin-Voigt hypothesis [12, 17], corresponding to which the inner resistance force is proportional to the first velocity of deformation. In this case, equation (1) takes the form:

$$\begin{aligned}&m(x)\frac{\partial^2 u}{\partial t^2}+\alpha m(x)\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(E(x)F(x)\frac{\partial u}{\partial x}\right)+\\&+\beta\frac{\partial^2}{\partial t\partial x}\left(E(x)F(x)\frac{\partial u}{\partial x}\right)\end{aligned} \tag{2}$$

where *α, β* – factors of external and inner friction properly.

Applying the Fourier method, solution of equation (2) will be in form:

$$u(x,t)=v(x)T(t) \tag{3}$$

where *v(x)* – peak value of longitudinal displacement, *T(t)* – unknown function of time. Expression for longitudinal force we will get by known equation [12].

$$\begin{aligned}&N(x,t)=E(x)F(x)\left(\frac{\partial u}{\partial x}+\beta\frac{\partial^2 u}{\partial t\partial x}\right)=\\&=N(x)\left(T(t)+\beta\dot{T}(t)\right)\end{aligned} \tag{4}$$

where *N(x) = E(x)I(x)v'(x)* – peak value of longitudinal force. Dividing variables in equations (2), we will get two independent equations:

$$\ddot{T}(t)+2h\dot{T}(t)+\omega^2T(t)=0; \tag{5}$$

$$(E(x)F(x)v'(x))'+\omega^2m(x)v(x)=0 \tag{6}$$

where $2h = \alpha + \beta\omega^2$; ω^2 – Fourier method constant.

It is not difficult to write down general solution of equation (5). Expressed through parameters of initial conditions of movement $T(0), \dot{T}(0)$, it has form $T(t)=Ae^{-ht}\sin(\tilde{\omega}t+\delta)$, where

$$\tilde{\omega}=\sqrt{\omega^2-h^2},\ A=\sqrt{T^2(0)+\left(\frac{\dot{T}(0)+hT(0)}{\tilde{\omega}}\right)^2},$$

$$\delta=\mathrm{arctg}\left(\frac{T(0)\tilde{\omega}}{\dot{T}(0)+hT(0)}\right) \tag{7}$$

This solutions indicates that free lateral vibrations of rod with taking into account resistance forces are taken place in time by damping harmonic rule with frequency $\tilde{\omega}$ and that introduced constant value ω represents frequency of free vibrations without taking into account resistances (case $\alpha=\beta=0$).

It is important to express peak functions $v(x)$, $N(x)$ by dimensionless fundamental solutions. Because of this let us take for initial variable parameters expressions $F(x)=F_0\phi(x)$, $E(x)=E_0\psi_1(x)$, $\rho(x)=\rho_0\psi_2(x)$, where F_0, E_0, ρ_0 - constants, properly area of cross-section, modulus of elasticity and density of material in some point of rod; $\phi(x), \psi_1(x), \psi_2(x)$ - dimensionless functions. Then $E(x)F(x)=E_0F_0A(x)$, $m(x)=m_0B(x)$, where $m_0=\rho_0F_0$; $A(x)=\phi(x)\psi_1(x),\ B(x)=\phi(x)\psi_2(x)$ - dimensionless functions, which properly define rules of variation of stiffness and mass per unit length along the rod. This approach allows us to get analytical forms for vibrations frequency and operate only with dimensionless values at calculations.

Fundamental form of vibrations is defined as a solution of equation (6). General integral of this equation and peak function for longitudinal force, expressed through initial parameters, are formulated in [18]. Here we show them as:

$$v(x)=v(0)X_1(x)+N(0)\frac{l}{E_0F_0}X_2(x) \tag{8}$$

$$N(x)=v(0)\frac{E_0F_0}{l}\tilde{X}_1(x)+N(0)\tilde{X}_2(x) \tag{9}$$

where $X_n(x)(n=1,2)$ - fundamental functions of equation (6), which are defined by absolutely and uniform convergent serieses:

$$X_n(x)=\alpha_{n,0}(x)-K^2\alpha_{n,1}(x)+K^4\alpha_{n,2}(x)-\\-K^6\alpha_{n,3}(x)+... \tag{10}$$

$$K=\omega l\sqrt{\frac{\rho_0}{E_0}} \tag{11}$$

Let call functions $\alpha_{n,0}(x)\,(n=1,2)$ as *initial* and functions $\alpha_{n,k}(x)\,(n=1,2)\,(k=1,2,3,...)$ as *generating*. They define in following way:

$$\alpha_{1,0}(x)=1,\ \alpha_{2,0}(x)=\frac{1}{l}\int_0^x\frac{1}{A(x)}dx \tag{12}$$

$$\alpha_{n,k}(x)=\frac{1}{l^2}\int_0^x\frac{1}{A(x)}\int_0^x B(x)\alpha_{n,k-1}(x)dxdx\\(k=1,2,3,...) \tag{13}$$

$$\alpha_{n,k}(x)=\frac{1}{l^{2k}}\int_0^x\frac{1}{A(x)}\int_0^x B(x)...\\...\int_0^x\frac{1}{A(x)}\int_0^x B(x)\alpha_{n,0}(x)dxdx...dxdx \tag{14}$$

Here are two formulas for generating functions for clarity – recurrent and detailed. Number of integrals in recent equation without integrals, which could have initial function, is *2k*.

For fundamental functions in equations (9) it is valid equation:

$$\tilde{X}_n(x)=\tilde{\alpha}_{n,0}(x)-K^2\tilde{\alpha}_{n,1}(x)+K^4\tilde{\alpha}_{n,2}(x)-\\-K^6\tilde{\alpha}_{n,3}(x)+... \tag{15}$$

where

$$\tilde{\alpha}_{n,k}(x)=lA(x)\alpha'_{n,k}(x)\ (n=1,2)\ (k=0,1,2,...) \tag{16}$$

It is important that unknown parameter K and functions (12), (14), (16) are dimensionless [18]. Because of this fundamental functions (10), (15) are dimensionless too. As a result, dimensions of constant factors at dimensionless functions in right parts of equations (8), (9) are the same with dimensions of corresponding left parts.

Directly from equation (11) we get analytical form of free vibrations frequency without taking into account resistances:

$$\omega=\frac{K}{l}\sqrt{\frac{E_0}{\rho_0}} \tag{17}$$

where k – unknown dimensionless *vibration factor*. Due to this formula, the problem of determining of frequency is going to determining the vibration factor. Equations for peak vibrations of rod $v(x),\,N(x)$ exactly depend on vibration factor. That is why frequency equations will be used for its determining. These equations will be found after realization of specified initial boundary conditions.

There are cases when initial and generating functions are calculated in explicit form and, as a result, fundamental solutions transforms into primitive or special functions. However, such cases are rare. Therefore, in terms of generality of results it is important to indicate an effective way of numerical implementation of the above formulas for initial and generating functions, which would be suitable for any continuous variable stiffness and random continuous variable mass per unit length.

It is known that one of the methods of numerical integration is based on replacing the integrand by its approximating polynomial. This idea is useful in our case, because it avoids multiple numerical integration, which is prescribed by formulas (12), (13), (14). This is especially important for ease of software implementation of the method.

$$\frac{1}{A(x)}=A_0+A_1\left(\frac{x}{l}\right)+A_2\left(\frac{x}{l}\right)^2+...+A_s\left(\frac{x}{l}\right)^s \tag{18}$$

$$B(x)=B_0+B_1\left(\frac{x}{l}\right)+B_2\left(\frac{x}{l}\right)^2+...+B_p\left(\frac{x}{l}\right)^p \tag{19}$$

Then we easily get from equations (12), (16) forms of initial functions by polynomials:

$$\alpha_{1,0}(x)=1;\ \alpha_{2,0}(x)=\frac{1}{l}\int_0^x\frac{1}{A(x)}dx=\frac{x}{l}\sum_{j=0}^{s}\frac{A_j}{j+1}\left(\frac{x}{l}\right)^j;\\\tilde{\alpha}_{1,0}(x)=0;\ \tilde{\alpha}_{2,0}(x)=1$$

It is more difficult to get generating functions. It is clear from equation (14) that at conditions (18), (18) generating functions could be expressed as polynomials. However, calculating of factors of these polynomials is complicated problem. Required formulas were get in work [18] and have the form:

$$\alpha_{n,k}(x)=\left(\frac{x}{l}\right)^{2k+n-1}\sum_{j=0}^{kp+(k+n-1)s}c_{k,j}^{(n)}\left(\frac{x}{l}\right)^j \tag{20}$$

$$\tilde{\alpha}_{n,k}(x)=\left(\frac{x}{l}\right)^{2k+n-2}\sum_{j=0}^{kp+(k+n-2)s}\frac{d_{k-1,j}^{(n)}}{2k+j+n-2}\left(\frac{x}{l}\right)^j \tag{21}$$

Coefficients of these polynomials are calculated by formulas:

$$c_{k,j}^{(n)}=\frac{e_{k-1,j}^{(n)}}{2k+j+n-1}\ (k=1,2,3,...)\\(j=0,1,2,...,kp+(k-n+1)s) \tag{22}$$

$$e_{k-1,j}^{(n)}=\sum_{h=0}^{j}\frac{A_{j-h}\,d_{k-1,h}^{(n)}}{2k+h+n-2};\quad d_{k-1,j}^{(n)}=\sum_{r=0}^{j}B_{j-r}c_{k-1,r}^{(n)}$$

at that $A_{j-h}=0$, if $j-h>s$ and $d_{k-1,h}^{(n)}=0$, if $h>kp+(k+n-2)s$; $B_{j-r}=0$, if $j-r>p$ and $c_{k-1,r}^{(n)}=0$, if $r>(k-1)p+(k+n-2)s$. Here $e_{k-1,j}^{(n)}$ is calculated by values of $d_{k-1,h}^{(n)}$, which are calculated in its turn by values of $c_{k-1,r}^{(n)}$. It means that formula (22) is recurrent ratio on which every next value of $c_{k,j}^{(n)}$ is calculated by previous values. Initial parameters are given by formulas:

$$c_{0,0}^{(1)}=1;\quad c_{0,j}^{(2)}=\frac{A_j}{j+1}\ (j=0,1,...,s) \qquad (23)$$

Formula (22) can be written in detailed form:

$$c_{k,j}^{(n)}=\frac{1}{2k+j+n-1}\sum_{h=0}^{j}\left(\frac{A_{j-h}}{2k+h+n-2}\sum_{r=0}^{h}B_{h-r}\,c_{k-1,r}^{(n)}\right)$$
$$(k=1,2,3,...)\ (j=0,1,2,...,kp+(k-n+1)s). \qquad (24)$$

Expressed in analytical form formulas for dynamics parameters with indication of its numerical way of realization allows to define free dynamic vibrations *y(x,t)* and dynamic inner forces *N(x,t)* for the rod with random continuous variable stiffness and random continuous variable mass per unit length at any possible boundary conditions.

2.2. Example

Let consider uniform rod with a wedge form with constant thickness *d* and heights of bases $a,b\ (0\le a\le b)$. Let the minor base of wedge is in point *x=0* (free end), and major – in point *x = l* (fixed end) (Figure 2).

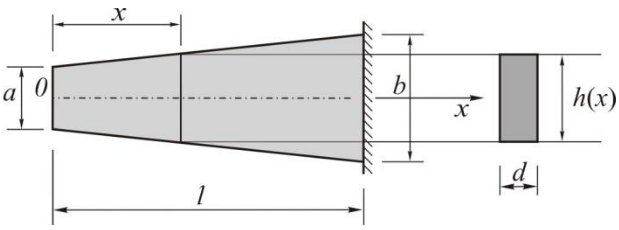

Figure 2. Cantilever wedge

Since the cross-section height in point *x* will be $h(x)=a+\frac{b-a}{l}x$, for the area of this cross-section we can write down $F(x)=db\left(\gamma+(1-\gamma)\frac{x}{l}\right)$, where $\gamma=\frac{a}{b}\ (0\le\gamma\le1)$. Than, with a glance to accepted above designations, let $F_0=bd;\ \phi(x)=\gamma+(1-\gamma)\frac{x}{l}$. Consequently $A(x)=B(x)=\gamma+(1-\gamma)\frac{x}{l}$.

It is obvious that at $\gamma=0$ wedge will be gabled, at $0<\gamma<1$ it will be truncated wedge and at $\gamma=1$ wedge degenerates into a rod with a constant cross-section.

This case corresponds to dynamic boundary conditions: $N(0,t)=0;\ u(l,t)=0$. Taking into account that they have to be performed for any time and using equations (3), (4), we get the equivalent boundary conditions in peak form: $N(0)=0;\ v(l)=0$.

Realizing boundary conditions by formula (7), we get the frequency equation $X_1(l)=0$, or

$$\alpha_{1,0}(l)-\alpha_{1,1}(l)K^2+\alpha_{1,2}(l)K^4-\alpha_{1,3}(l)K^6+\cdots=0 \qquad (25)$$

Left part of this equation is convergent series that is guaranteed by uniform convergence of series that defines fundamental functions. Its solutions can be found with any prescribed accuracy by the method of root comparison, which corresponds to different number of held series members.

For roots $K_j\ (j=1,2,3,...)$ of equation (25) according to formula (16) frequencies of free vibrations without taking into account resistances will correspond:

$$\omega_j=\frac{K_j}{l}\sqrt{\frac{E}{\rho}}\ (j=1,2,3,...) \qquad (26)$$

For this frequencies will correspond frequencies with resistances taken into account: $\tilde{\omega}_j=\sqrt{\omega_j^2-h_j^2},\ h_j=\frac{\alpha+\beta\omega_j^2}{2}\ (j=1,2,3,...)$.

Free vibrations forms, corresponding to frequencies (26), on the base of (8) let formulate as $v_j(x)=v_j(0)V_j\left(\frac{x}{l}\right)\ (j=1,2,3,...)$, where $V_j\left(\frac{x}{l}\right)=X_1(x,K_j)=\sum_{k=0}^{\infty}(-1)^kK_j^{2k}\alpha_{1,k}(x)$ - dimensionless function that defines the rule of general vibration form.

As can be seen a key role is played here by generating functions $\alpha_{1,k}(x)\ (k=1,2,3,...)$. In particular, when the wedge is gabled, this functions can be calculated in explicit form:

$$\alpha_{1,0}(x)=1,\ \alpha_{1,k}(x)=\frac{1}{l^2}\int_0^x\frac{1}{x}\int_0^x x\alpha_{k-1}(x)\,dxdx=$$
$$=\frac{1}{2^24^2...(2k)^2}\left(\frac{x}{l}\right)^{2k}\ (k=1,2,3,...) \qquad (27)$$

Then

$$X_1(x) = 1 - \frac{K^2}{2^2}\left(\frac{x}{l}\right)^2 + \frac{K^4}{2^2 4^2}\left(\frac{x}{l}\right)^4 - \\ - \frac{K^6}{2^2 4^2 6^2}\left(\frac{x}{l}\right)^6 + \cdots = J_0\left(K\frac{x}{l}\right) \quad (28)$$

where J_0 is Bessel's function with zero index [19]. Frequency equation and the rule of general vibration form of rod will be expressed by Bessel's function:

$$J_0(K) = 0; \ V_j\left(\frac{x}{l}\right) = J_0\left(K_j\frac{x}{l}\right)(j = 1,2,3...) \quad (28)$$

Here we come to the well-known conlusion [12] that vibration factors of gabled cantilever wedge are zeros of Bessel's functions that are well-known. First three zeros of this function are shown in table 1.

When the rod has a form of truncated wedge, generating functions $\alpha_{1,k}(x)$ $(k = 1,2,3,...)$ cannot be calculated in explicit form. For its software calculation quadrature formula (19) will be applied. It should only be noted that in this example function $B(x)$ is initially expressed by polinominal (19), where $p = 1,\ B_0 = \gamma,\ B_1 = 1 - \gamma$. Therefore it is not required to build for it approximation.

There are results of calculation of first three vibration factors of truncated cantilever wedge for every value of parameter γ with increment of 0.1.

Table 1. Vibration factors of truncated cantilever wedge

	Parameter γ				
	0	0.1	0.2	0.3	0.4
K_1	2.4048	2.2085	2.0594	1.9500	1.8640
K_2	5.5201	5.3009	4.9873	4.8959	4.8399
K_3	8.6537	8.1179	8.0307	7.9719	7.9334

	Parameter γ					
	0.5	0.6	0.7	0.8	0.9	1
K_1	2.4048	2.2085	2.0594	1.9500	1.8640	$\frac{\pi}{2}$
K_2	5.5201	5.3009	4.9873	4.8959	4.8399	$\frac{3\pi}{2}$
K_3	8.6537	8.1179	8.0307	7.9719	7.9334	$\frac{5\pi}{2}$

Let stop at case $\gamma = 0,5$. First three frequencies of free longitudinal vibrations of cantilever wedge without taking into account resistance corresponds to this case:

$$\omega_1 = \frac{1,7940}{l}\sqrt{\frac{E}{\rho}};\ \omega_2 = \frac{4,8021}{l}\sqrt{\frac{E}{\rho}}; \\ \omega_3 = \frac{7,9089}{l}\sqrt{\frac{E}{\rho}} \quad (29)$$

Research of longitudinal vibrations of wedge occurs in [17]. There are calculated by B.G. Galerkin's method first two frequencies of free vibrations. In the terms of taken here designations, calculated there vibrations are:

$$\omega_1 = \frac{1,794}{l}\sqrt{\frac{E}{\rho}};\ \omega_2 = \frac{5,033}{l}\sqrt{\frac{E}{\rho}} \quad (30)$$

As can be seen, with the full coincidence of the first frequency, the relative error between the second is about 4.6%.

Graphs of rules of free vibrations forms, which corresponds to frequencies (24) are shown in figure 3.

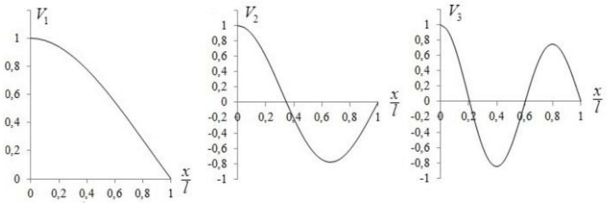

Figure 3. Graphs of rules of free vibrations form

Finally, the simplest situation arises when $\gamma = 1$. Then:

$$\alpha_{1,k}(x) = \frac{1}{(2k)!}\left(\frac{x}{l}\right)^{2k}, \quad X_1(x) = \cos K\frac{x}{l} \quad (31)$$

Hence we have frequency equation $\cos K = 0$ and the rule for general forms of free vibrations of rod $V_j\left(\frac{x}{l}\right) = \cos K_j \frac{x}{l}$ $(j = 1,2,3...)$.

3. CONCLUSION

The method of force integration, which can be used for calculation of systems with continuous variable parameters, was suggested. There are adduced all required formulas in analytical form for research of free lateral vibrations of rod with random continuous variable longitudinal stiffness and random continuous variable mass per unit length at any possible boundary conditions. Frequencies are calculated and formulas for main vibration forms of uniform cantilever wedge are found.

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