MATHEMATICAL COMMUNICATIONS 151
Math. Commun. **20**(2015), 151–160

Bridging Bernstein and Lagrange polynomials*

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Received February 17, 2015; accepted June 19, 2015

Abstract. Linear combinations of iterates of Bernstein polynomials exponentially converging to the Lagrange interpolating polynomial are given. The results are applied in CAGD to get an exponentially fast weighted progressive iterative approximation technique to fit data with finer and finer precision.

AMS subject classifications: 41A25, 41A36

Key words: Bernstein polynomials, Lagrange interpolation, Hotelling's method, Bézier curves, weighted progressive iterative approximation

1. Introduction

Many papers have been devoted to the study of linear combinations of Bernstein polynomials algebraically converging to the Lagrange interpolating polynomial (see, e.g. [1, 3, 7, 8, 9, 10, 12]). Such result found application among others in CAGD to get the so-called weighted progressive iterative approximation technique (WPIA in short) to fit data with finer and finer precision at the algebraic rate (see, e.g. [5, 6]).

The aim of the present paper is to introduce and study a linear combination of iterates of Bernstein polynomials exponentially converging to the Lagrange interpolating polynomial (Section 2). Approximation error estimates are given in Theorems 1 and 2. Theorem 3 establishes the optimal convergence result in some sense.

The results are applied in CAGD to get a WPIA technique exponentially converging (Section 3). The key idea is to iteratively change the control points of Bézier curves to generate sequences of Bézier curves exponentially converging to the Lagrange interpolating curve at the original control points. Convergence results optimal in some sense are shown in Theorem 4.

The proofs are based on the iterative approximation of the inverse of the collocation matrix of Bernstein polynomials and on their eigenstructure (Section 5). Numerical experiments are also shown, confirming our theoretical analysis.

*Dedicated to the memory of Prof. D. D. Stancu.

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2. From Bernstein to Lagrange polynomials

For any $f \in C^0([0,1])$ we denote by $B_n(f)$ the classical Bernstein polynomial of degree $\leq n$ defined by

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f(t_k), \quad x \in [0,1], \tag{1}$$

with $t_k = k/n$ and

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}. \tag{2}$$

Setting $\overline{f} = \left[f(0), f\left(\frac{1}{n}\right), \ldots, f(1)\right]^T$ and $\overline{b}_n(x) = [p_{n,0}(x), p_{n,1}(x), \ldots, p_{n,n}(x)]$, we can rewrite (1) in matrix representation by

$$B_n(f;x) = \overline{b}_n(x)\overline{f}.$$

The Lagrange polynomial interpolating f at t_k, $k = 0, \ldots, n$, is defined by

$$L_n(f;x) = \sum_{k=0}^{n} \ell_{n,k}(x) f(t_k), \tag{3}$$

with

$$\ell_{n,k}(t_i) = \begin{cases} 1, & i = k \\ 0, & i \neq k, \ 0 \leq i, k \leq n. \end{cases}$$

Setting

$$\overline{\ell}_n(x) = [\ell_{n,0}(x), \ell_{n,1}(x), \ldots, \ell_{n,n}(x)],$$

we can rewrite (3) by

$$L_n(f;x) = \overline{\ell}_n(x)\overline{f}.$$

An essential role in our further considerations is played by the following collocation matrix K

$$K := \begin{pmatrix} p_{n,0}(t_0) & p_{n,1}(t_0) & \ldots & p_{n,n}(t_0) \\ p_{n,0}(t_1) & p_{n,1}(t_1) & \ldots & p_{n,n}(t_1) \\ \ldots & \ldots & \ddots & \ldots \\ p_{n,0}(t_n) & p_{n,1}(t_n) & \ldots & p_{n,n}(t_n) \end{pmatrix}.$$

This matrix has been studied in many different fields of classical approximation theory, numerical analysis, Computer Aided Geometric Design, etc.. It is well-known (see, e.g. [2]) that if $\{\lambda_k^{(n)}\}_{k=0}^n$ denote the eigenvalues of K ordered decreasingly, this results in

$$\lambda_k^{(n)} = \frac{n!}{(n-k)!n^k}, \quad k = 0, 1, \ldots, n.$$

Therefore, $1 = \lambda_0^{(n)} = \lambda_1^{(n)} > \lambda_2^{(n)} > \cdots > \lambda_n^{(n)} > 0$. Hence, (see, e.g. [12])

$$\|I - K\| := \rho(I - K) = 1 - \lambda_n^{(n)} < 1, \tag{4}$$

with $\rho(U)$ being the spectral radius of the matrix U.

From the uniqueness of the Lagrange interpolating polynomial it results

$$L_n\left(f; t_k\right) = f\left(t_k\right) = \overline{b}_n\left(t_k\right)\overline{f}^I = K\overline{f}^I, \quad k = 0, \ldots, n, \tag{5}$$

with $\overline{f}^I = \left(f_0^I, f_1^I, \ldots, f_n^I\right)^T$ being the vector solution of system (5).

For $m \in \mathbb{N}$, we denote by $U^m = U \cdot U \cdots U$ m times the m-th iterate of the matrix U and we recall $B_n^{m+1}(f; x) = \overline{b}_n(x)K^m\overline{f}$, (see [9]). It is well-known that (see, e.g. [7])

$$\lim_m B_n^m(f; x) = f(0) + x(f(1) - f(0)).$$

Now we construct a linear combination of iterates of Bernstein polynomials exponentially converging to the Lagrange interpolating polynomial. Indeed, consider the polynomial operator $B_{n,m}$ defined by

$$\begin{aligned} B_{n,1}(f; x) &= B_n(f; x) = \overline{b}_n(x)\overline{f}, \\ B_{n,m+1}(f; x) &= \overline{b}_n(x)A_m\overline{f}, \ m \geq 1, \\ A_m &= A_{m-1}(2I - KA_{m-1}), \\ A_0 &= wI, \end{aligned} \tag{6}$$

with I the identity matrix and $0 < w$ a fixed parameter that can take any possible value as long as it can guarantee the convergence of the above polynomials (see later).

Since the matrix K is centrosymmetric (we say that the matrix D is centrosymmetric iff $d_{i,j} = d_{n-i,n-j}, i, j = 0, \ldots, n$), the computational cost for the computation of A_m at each iteration is $O(n^2/2)$. Moreover, $B_{n,m+1}(f)$ can be efficiently computed by the well-known De Casteljau algorithm.

From (6) we immediately deduce $B_{n,m}(f)$ is a polynomial of degree $\leq n$, generally not positive. It is easy to check that

$$B_{n,m+1} = I - \left(I - wB_n\right)^{2^m}, \quad m \geq 1. \tag{7}$$

If $w = 1$ in (7), the sequence $\{B_{n,m}\}_m$ is extracted from the sequence of a linear combination of iterates of Bernstein polynomials studied in [7, 10].

In particular,

$$\begin{aligned} B_{n,2}(f; x) &= \overline{b}_n(x)A_0(2I - KA_0)\overline{f} \\ &= \overline{b}_n(x)w(2I - wK)\overline{f} \\ &= 2wB_n(f; x) - w^2B_n^2(f; x), \end{aligned} \tag{8}$$

and

$$\begin{aligned}
B_{n,3}(f;x) &= \overline{b}_n(x)A_1(2I-KA_1)\overline{f} \\
&= \overline{b}_n(x)\left(2wI-w^2K\right)\left[2I-K\left(2wI-w^2K\right)\right]\overline{f} \\
&= \overline{b}_n(x)\left(2wI-w^2K\right)\left(2I-2wK+w^2K^2\right)\overline{f} \\
&= \overline{b}_n(x)\left(4wI-4w^2K+2w^3K^2-2w^2K+2w^3K^2-w^4K^3\overline{f}\right) \\
&= \overline{b}_n(x)\left(4wI-6w^2K+4w^3K^2-w^4K^3\right)\overline{f} \\
&= w\left[4B_n(f;x)-6wB_n^2(f;x)+4w^2B_n^3(f;x)-w^3B_n^4(f;x)\right].
\end{aligned}$$

Note that if

$$w \le 2 \min_{0\le k\le n} \frac{|f(t_k)|}{|B_n(f;t_k)|}, \tag{9}$$

then from (8) $B_{n,2}$ is a positive operator. In particular, if f is convex, then it is well-known that $f(x) \ge B_n(f;x), \forall x \in [0,1]$; hence, from (9) $B_{n,2}$ is a positive operator, $\forall\, 0 < w \le 2$.

In what follows C will denote a positive constant which may assume different values even in the same formula. Moreover, $\|f\|$ denotes the usual supremum norm of $f \in C([0,1])$ on $[0,1]$.

The following Theorem 1 gives a motivation for the construction of $B_{n,m}$ for fixed n and increasing m, i.e., it shows that $B_{n,m}(f)$ approximates f at the knots k/n, $k = 1,\ldots,n-1$, better than original $B_n(f)$. In other words, the loss of positivity is compensated by an exponential convergence rate at the knots. This result is important to study the WPIA property (see Section 3).

Indeed, if $\lambda_n^{(n)}$ denotes the smallest eigenvalue of K, then

Theorem 1. *Let $\rho(I-wK) < 1$. Then for any $f \in C([0,1])$ and for any fixed $m > 0$*

$$|f(t_i) - B_{n,m+1}(f;t_i)| \le \left(1 - w\lambda_n^{(n)}\right)^{2^m}, \quad i = 0,\ldots,n. \tag{10}$$

In particular, if $0 < w < 2$, then (10) holds true. Moreover, if $w = 1$, then for any $f \in C^{2^{m+1}}([0,1])$, $2^{m+1} < n$,

$$\|f - B_{n,m}(f)\| \le C \frac{||f|| + ||f^{(2^{m+1})}||}{n^{2^m}}. \tag{11}$$

Remark 1. *Estimate (11) can be easily derived from a result in [7]. If $w = 1$, more general estimates in terms of the $\phi-$modulus of Ditzian-Totik can be found in [4].*

Moreover, putting $B_{n,\infty} := \lim_{m\to\infty} B_{n,m}$, then

Theorem 2. *Let $\rho(I-wK) < 1$. If $f \in C([0,1])$, then for every $x \in [0,1]$ and n*

$$B_{n,\infty}(f;x) = L_n(f;x) = \overline{b}_n(x)\overline{f}^I = \overline{b}_n(x)K^{-1}\overline{f}. \tag{12}$$

In particular, if $0 < w < 2$, then (12) holds true for every $x \in [0,1]$ and n.

Remark 2. *Theorem 2 says that by the sequence (6) we can reach the Lagrange interpolating polynomial (3) without solving the linear system (5). In other words, the sequence $\{B_{n,m}\}$ continuously links B_n operator to L_n operator.*

Moreover, we give an estimate of the approximation error of $B_{n,\infty}$ by $B_{n,m}$.

Theorem 3. *Let $\rho(I - wK) < 1$. For any $f \in C([0,1])$*

$$\|B_{n,\infty}(f) - B_{n,m+1}(f)\| \le \|f\| \left(1 - w\lambda_n^{(n)}\right)^{2^m} \left(\lambda_n^{(n)}\right)^{-1}. \tag{13}$$

The fastest rate is attained when

$$w = \frac{2}{1 + \lambda_n^{(n)}}; \tag{14}$$

therefore,

$$\|B_{n,\infty}(f) - B_{n,m+1}(f)\| \le \|f\| \left(\frac{1 - \lambda_n^{(n)}}{1 + \lambda_n^{(n)}}\right)^{2^m} \left(\lambda_n^{(n)}\right)^{-1}.$$

Remark 3. *From Theorems 2–3 it follows that sequence $\{B_{n,m}\}_m$ exponentially converges to the Lagrange interpolating polynomial.*

Note that if $n \to \infty$, then w in (14) tends to 1.

Following [11] we can analogously construct sequences of Bernstein polynomials converging to Lagrange interpolation at the rate 3^m, 4^m or 7^m, but the computational cost is higher, so we omit the details.

3. Weighted progressive iterative approximation

The above results find application in CAGD to construct sequences of Bézier curves based on $B_{n,m}$ operator exponentially converging to the Lagrange interpolating curve.

Let us see in detail the WPIA process. Here, differently from [5, 6], the index m starts from 1 in analogy to the nonparametric case. Given the control polygon $P = [P_0, P_1, \ldots, P_n]^T$, $P_i \in \mathbb{R}^d$, $i = 0, \ldots, n$, $d \ge 2$, and the basis functions $p_{n,i}(x)$, $i = 0, \ldots, n$, defined by (2), we can generate the initial Bézier curve

$$\gamma_w^1(t) = \sum_{i=0}^{n} p_{n,i}(t) P_i^0 := B_n[P,t], \quad t \in [0,1], \tag{15}$$

with $P_i^0 = P_i$, $i = 0, \ldots, n$. Then we calculate the successive Bézier curves of the sequence $\gamma_w^{m+1}(t)$, for $m \ge 1$, as follows

$$\gamma_w^{m+1}(t) = \sum_{i=0}^{n} p_{n,i}(t) P_i^m, \tag{16}$$

with

$$P_i^m = P_i^{m-1} + \Delta_i^{m-1}$$

and Δ_i^{m-1} the adjusting vector given by

$$\Delta_i^{m-1} = A_{m-1}(I - KA_0)^{2^{m-1}} P_i^0, \quad i = 0, \ldots, n.$$

Remark 4. *In the nonparametric case, curve γ_w^m corresponds to the polynomial defined by (6).*

We say that curve γ_w^1 satisfies the WPIA property iff $\lim_m \gamma_w^m(t_i) = P_i$, $i = 0, \ldots, n$.

We have

Theorem 4. *If $\rho(I - wB) < 1$, curve γ_w^1 satisfies the WPIA property. Moreover, the WPIA process has the fastest convergence rate when*

$$w = \frac{2}{1 + \lambda_n^{(n)}}, \tag{17}$$

and in such case

$$\rho(I - wK) = \frac{1 - \lambda_n^{(n)}}{1 + \lambda_n^{(n)}}.$$

Remark 5. *The WPIA property makes it possible to construct a sequence of control polygons converging to the control polygon of an interpolating curve of Bézier type. Moreover, the parameter k can be used as a shape parameter in order to model different shapes, obtaining the Bézier curve and the interpolating Lagrange polynomial curve as an extreme case. By choosing the optimal value of the weight w, Theorem 4 shows that the WPIA shares the progressive iterative approximation property and has the fastest convergence rate. As remarked before, here the rate is exponential, while the known WPIA process has an algebraic rate (see [6]).*

If $w = 1$, we can consider the corresponding curves $\gamma_1^m(t)$. We say that γ_1^1 satisfies the progressive iterative approximation (PIA in short) property iff $\lim_m \gamma_1^m(t_i) = P_i$, $i = 0, \ldots, n$ (cfr. [5]). We have

Corollary 1. *Curve γ_1^1 satisfies the PIA property.*

Remark 6. *Based on PIA format, we can design an adaptive fitting method to fit data points, by adjusting the control points corresponding to these data points, if fitting precision is above a predefined threshold.*

The extension of the WPIA process to the tensor product surfaces case is immediate (see, e.g. [6]) and we omit the details.

The WPIA technique can be used to write the degree-elevation algorithm for $B_{n,m}$ curves. Indeed, from (6) and (15) – (16), if $Q = [Q_0, Q_1, \ldots, Q_{n+1}]^T$, $Q_i \in R^d, d \geq 2, i = 0, 1, \ldots, n+1$,

$$B_{n+1,m+1}[Q, t] - B_{n,m+1}[P, t] = \overline{b}_{n+1}(t) A_m Q - \overline{b}_n(t) A_m P,$$

and by the well-known degree-elevation technique for Bézier curves

$$B_{n+1,m+1}[Q,t] = B_{n,m+1}[P,t], \quad \forall t \in [0,1],$$

if

$$Q_i^m = \frac{i}{n+1} P_{i-1}^m + \left(1 - \frac{i}{n+1}\right) P_i^m, \quad i = 0,1,2,\dots,n+1. \tag{18}$$

Hence from (18), if Q_i^m is collinear with P_{i-1}^m and P_i^m, $1 \le i \le n+1$, then we can replace the Bézier curve $B_{n,m}[P]$ of degree n by the Bézier curve $B_{n+1,m}[Q]$ of degree $n+1$. Such algorithm, as well-known, allows us to increase the flexibility of the curve $B_{n,m}$. We remark that because of the nonsingularity of A_m, from (18) there exists one and only one vector Q solution of such problem.

Analogously, the subdivision algorithm for $B_{n,m}$ curves can be deduced from the well-known subdivision algorithm for Bézier curves.

4. Example

Let us consider the lemniscate of Gerono given by [6]

$$(x(t), y(t)) = (\cos t, \sin t \cos t), \quad t \in [0, 2\pi].$$

A sequence of 11 points $\{\mathbf{P}_i\}_{i=0}^{10}$ is sampled from the parameter curve as

$$\mathbf{P}_i = (x(s_i), y(s_i)), \; s_i = -\frac{\pi}{2} + i\frac{2\pi}{10}, \quad i = 0,1,\dots,10. \tag{19}$$

Starting with these control points we fit the lemniscate by two sequences of curves generated by the WPIA process defined by (15) – (16) and (17), with $w \simeq 1.99927$, and by the PIA process defined by (15) – (16) with $w = 1$.

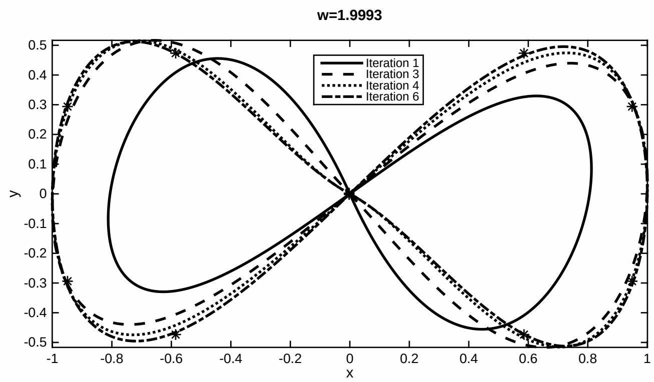

Figure 1: *Approximations of the lemniscate generated by the WPIA process (15) – (16) for the optimal* $w \simeq 1.99927$ *at the first, the third, the fourth and the sixth iterations; a star symbol denotes the control points given in (19).*

Figures 1 and 2 show the first, the third, the fourth and the sixth curves of such sequences for WPIA and PIA, respectively, and a star symbol denotes the control points given in (19). The fitting errors of the above WPIA and PIA processes (the maximum Euclidean norm of the corresponding adjusting vectors of such curves) are shown in Fig. 3 up to the machine precision. Fig. 3 shows that the WPIA process reaches a faster convergence than PIA, as expected from our theoretical results.

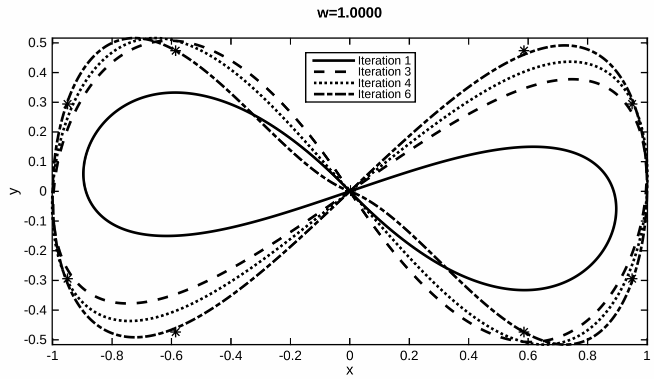

Figure 2: *Approximations of the lemniscate generated by the WPIA process (15) – (16) for* $w = 1$ *at the first, the third, the fourth and the sixth iterations; a star symbol denotes the control points given in (19).*

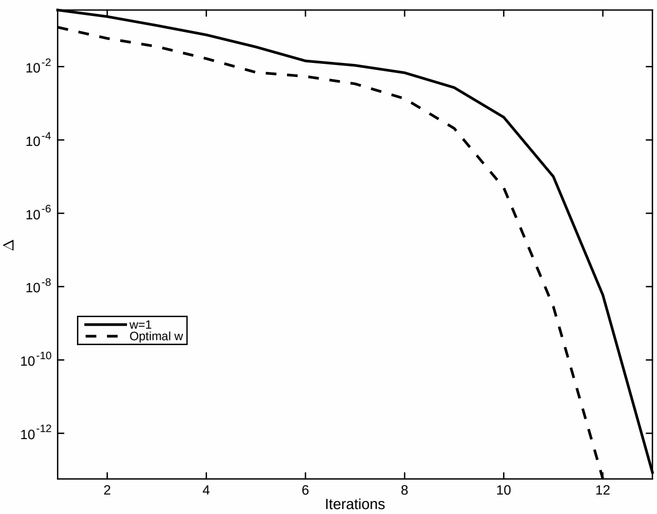

Figure 3: *Fitting error of the WPIA and PIA processes for the lemniscate example*

5. Proofs of the main results

We recall the Schultz-Hotelling's method to approximate the inverse of a matrix by an iterative process and some results on the eigenstructure of Bernstein polynomials.

Lemma 1 (See e.g. [11])**.** *Let $\rho(I - UT_0) < 1$, with U and T_0 $(n+1) \times (n+1)$ matrices and U invertible. Then the sequence*

$$T_{n+1} = T_n(2I - UT_n)$$

converges to U^{-1} and

$$I - UT_{n+1} = (I - UT_0)^{2^{n+1}}.$$

Lemma 2 (See e.g. [12])**.** *The matrix K is invertible and all rows of K and K^{-1} sum to 1.*

Lemma 3. *Let $\rho(I - wK) < 1$. Then the sequence of polynomials $\{B_{n,m}\}_{m=0}^{\infty}$ uniformly tends to its limiting operator $B_{n,\infty}$ on $[0,1]$ as $m \to \infty$ and*

$$B_{n,\infty}(f;x) = \overline{b}_n(x)K^{-1}\overline{f}. \tag{20}$$

Proof. From the assumption $\rho(I - wK) < 1$, by Lemmas 1 and 2 and (6) we know that

$$\lim_{m} A_m = K^{-1}.$$

(20) immediately follows by representation (6). □

Proof of Theorem 1. From (6), Lemmas 1 and 3 for $i = 0, \dots, n$,

$$\begin{aligned} |f(t_i) - B_{n,m+1}(f,t_i)| &= |B_{n,\infty}(f;t_i) - B_{n,m+1}(f,t_i)| \\ &\leq \|I - KA_m\|\|f\| = \|f\| \left\|(I - KA_0)^{2^m}\right\| \\ &= \|f\| \left(1 - w\lambda_n^{(n)}\right)^{2^m}. \end{aligned}$$

If $0 < w < 2$, then $\rho(I - KA_0) = \left(1 - w\lambda_n^{(n)}\right) < 1$ and we can follow the first part of the proof. Working as in [7, Theorem 2], (11) follows. □

Proof of Theorem 2. From Lemma 3

$$\overline{b}_n(t_i)K^{-1}\overline{f} = \overline{\ell}_n(t_i)\overline{f} = f(t_i),$$

that is, $B_{n,\infty} = L_n$. Finally, if $0 < w < 2$, then $\rho(I - KA_0) = \left(1 - w\lambda_n^{(n)}\right) < 1$ and we can follow the first part of the proof. □

Proof of Theorem 3. From (6) and Lemmas 1 and 3

$$\begin{aligned}\|B_{n,\infty}(f)-B_{n,m+1}(f)\| &\le \|b_n\| \left\|K^{-1}(I-KA_m)\right\| \|f\| \\ &\le \left(\lambda_n^{(n)}\right)^{-1} \left\|(I-KA_0)^{2^m}\right\| \|f\| \\ &\le \|f\| \left(1-\lambda_n^{(n)}w\right)^{2^m} \left(\lambda_n^{(n)}\right)^{-1},\end{aligned}$$

that is, (13).

Working as in [6], the assertion follows. □

Proof of Theorem 4. The WPIA property follows from Remark 4 and Theorem 1. From Theorem 3 we deduce the assertion. □

Proof of Corollary 1. From Theorem 4 and (4) we immediately obtain the PIA property for γ_1^0 curves. □

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