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Some facts about zero divisors of triangular infinite matrices

Roksana Słowik*

Institute of Mathematics, Silesian University of Technology, Kaszubska 23, 44-100 Gliwice, Poland

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Abstract. We are interested in the elements that are zero divisors in $\mathcal{T}_\infty(F)$ - the ring of $\mathbb{N} \times \mathbb{N}$ upper triangular matrices over a field F. It is known that a matrix from $\mathcal{T}_\infty(F)$ is a left zero divisor if and only if it contains at least one zero on the main diagonal. The problem when an infinite triangular matrix is a right zero divisor stays unsolved. In the paper, we give some sufficient conditions for a matrix from $\mathcal{T}_\infty(F)$ to be a right zero divisor. We also present some properties of infinite matrices that help us investigate the problem.

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1. Introduction

If $\mathcal{A}$ is an algebra, then $a \in \mathcal{A}$ is called a left or right zero divisor if there exist $x \in \mathcal{A}$ such that these two matrices satisfy

$$ax = 0$$

or

$$xa = 0,$$

respectively.

It can be easily noticed that if a is an $n \times n$ matrix over a field, then a is a zero divisor if and only if it is singular. In particular, a triangular $n \times n$ matrix is a left or right zero divisor if and only if at least one of its coefficients from the main diagonal is equal to 0. However, if one considers an $\mathbb{N} \times \mathbb{N}$ triangular matrix, the situation is different. This problem was studied by Suškevič [1]. He proved that every upper triangular matrix containing at least one zero on the main diagonal is a left zero divisor. The case when a matrix is a right zero divisor turned out to be far more complicated. It was proved that if a matrix has only a finite (but nonzero) number of zero coefficients on the main diagonal, then this matrix is a right zero divisor. Moreover, some examples of triangular matrices were given having an infinite number of nonzero coefficients on the main diagonal that are not right zero divisors.

In this paper, we wish to improve the results of Suškevič. We denote by $\mathcal{T}_\infty(F)$ the algebra of upper triangular $\mathbb{N} \times \mathbb{N}$ matrices. First, we will prove that every

*Corresponding author. *Email address:* `roksana.slowik@gmail.com` (R. Słowik)

infinite triangular matrix a is a conjugate of some sort of a sum of two matrices a_1, a_2, where a_1 has only zeros on the main diagonal and a_2 has no zeros on the main diagonal. Then we will show that such sum is a zero divisor if and only if a_1 is either a zero divisor or a zero matrix. Thus we will focus on matrices whose first diagonals contain only zeros. In particular, we will prove the following theorem.

Theorem 1. *Let F be a field and let $a \in \mathcal{T}_\infty(F)$. Suppose k is the smallest number such that the k-th diagonal contains some nonzero coefficients.*

1. *If $k \geq 1$ and the k-th diagonal of a does not contain any zero coefficient, then a is not a right zero divisor.*
2. *If the k-th diagonal of a contains a finite but nonzero number of zero coefficients, then a is a right zero divisor.*

2. Results

First, we introduce some notation. The symbol e_n stands for the $n \times n$ identity matrix and e_∞ for the $\mathbb{N} \times \mathbb{N}$ identity matrix. If we deal with an identity matrix whose dimension does not have a meaning or whose dimension may be finite as well as infinite, we will simply write e.

By a^T we will denote the transpose of a.

If a is a matrix and b is an invertible matrix, then by a^b we understand the conjugate $b^{-1}ab$.

By $\mathcal{N}T^k_\infty(F)$ we will understand the subring of all matrices x satisfying condition

$$\forall\, i \in \mathbb{N} \quad \forall\, 0 \leq j \leq k \qquad x_{i,i+j} = 0. \tag{1}$$

Instead of $\mathcal{N}T^0_\infty(F)$, we will write $\mathcal{N}T_\infty(F)$.

We start with an easy lemma.

Lemma 1. *Let F be any field. If $a \in \mathcal{T}_\infty(F)$ is a right zero divisor, then there exists $n \in \mathbb{N}$ such that $a_{nn} = 0$.*

Proof. Suppose the opposite – that for all $n \in \mathbb{N}$ we have $a_{nn} \neq 0$. Assume that $xa = 0$. We get now

$$0 = (xa)_{ii} = x_{ii}a_{ii}.$$

Since $a_{ii} \neq 0$, we get $x_{ii} = 0$ for all $i \in \mathbb{N}$.

Suppose now that for some $k \geq 0$ we have proved that the main, first, …, the k-th diagonal of x is equal to 0. Consider the $k+1$-st diagonal. From $(xa)_{i,i+k+1} = 0$ we obtain

$$\begin{aligned} 0 &= \sum_{j=0}^{k+1} x_{i,i+j}a_{i+j,i+k+1} \\ &= \sum_{j=0}^{k} x_{i,i+j}a_{i+j,i+k+1} + x_{i,i+k+1}a_{i+k+1,i+k+1} \\ &= x_{i,i+k+1}a_{i+k+1,i+k+1}, \end{aligned}$$

so since $x_{i,i+k+1}a_{i+k+1,i+k+1} \neq 0$, we have $x_{i,i+k+1} = 0$. Thus, all the diagonals of x are zero and $x = 0$. This contradicts the fact that a is a zero divisor. $\square$

Now we will concentrate on similarity of matrices. We wrote in the introduction that we would prove that every triangular matrix is similar to some sort of sum of two matrices. First, we need to clarify what kind of sum we mean.

Definition 1. *Suppose that a is an infinite matrix. Let M be a nonempty subset of $\mathbb{N}$ and let $\{I_m\}_{m\in M}$ be a family of nonempty disjoint subsets of $\mathbb{N}$ such that $\cup_{m\in M} I_m = \mathbb{N}$. If a satisfies condition*

$$\forall\, m, m' \in M \quad m \neq m',\ i \in I_m,\ j \in I_{m'} \quad \Rightarrow \quad a_{ij} = 0,$$

then we say that a is a generalized direct sum of infinite matrices a_m ($m \in M$) defined as follows

$$(a_m)_{ij} = \begin{cases} a_{ij} & \text{if } i, j \in I_m \\ 0 & \text{otherwise,} \end{cases}$$

and we write $a = \oplus_{m\in M}(a_m)$.

This definition is based on the definition of a generalized direct sum of finite matrices given in [2].

Now we will focus on matrices that are similar to generalized direct sums. First, we will prove a lemma.

Lemma 2. *Suppose that F is a field and a is triangular, finite or infinite, matrix of the form*

$$\left(\begin{array}{c|c} a_1 & b \\ \hline 0 & a_2 \end{array}\right),$$

where either $(a_1)_{ii} = 0$ for all i and $(a_2)_{jj} \neq 0$ for all j, or $(a_1)_{ii} \neq 0$ for all i and $(a_2)_{jj} = 0$ for all j. Then there exists a triangular matrix t of the form

$$t = \left(\begin{array}{c|c} e & y \\ \hline 0 & e \end{array}\right)$$

such that

$$a^t = \left(\begin{array}{c|c} a_1 & 0 \\ \hline 0 & a_2 \end{array}\right).$$

Proof. The equation

$$\left(\begin{array}{c|c} a_1 & b \\ \hline 0 & a_2 \end{array}\right)\left(\begin{array}{c|c} e_{k_1} & y \\ \hline 0 & e_{k_2} \end{array}\right) = \left(\begin{array}{c|c} e_{k_1} & y \\ \hline 0 & e_{k_2} \end{array}\right)\left(\begin{array}{c|c} a_1 & 0 \\ \hline 0 & a_2 \end{array}\right)$$

is equivalent to

$$a_1 y - y a_2 = -b.$$

If $(a_1)_{ii} = 0$ and $(a_2)_{jj} \neq 0$, then we can find the rows of y in a recursive way, starting from the last one and going up. Note that even in the case when a_2 is infinite, the matrix y consists of a finite number of rows, so this procedure has sense.

Analogously, if $(a_1)_{ii} \neq 0$ and $(a_2)_{jj} = 0$, we can find the columns of y starting from the firstone and moving to the next in each step. Clearly, if a_2 is infinite, then y has an infinite number of columns. However, when evaluating each of them, we make use of only a finite number of the preceding columns. Therefore, we can find y from the claim. $\square$

Now, using the above lemma, we will prove our theorem about decomposition.

Theorem 2. *Let F be a field. For every $a \in \mathcal{T}_\infty(F)$ there exist disjoint sets of indices I_1, I_2, $I_1 \cup I_2 = \mathbb{N}$ and triangular matrices t, a_1, a_2, such that the following conditions are fulfilled*

$$a^t = (a_1)_{I_1} \oplus (a_2)_{I_2},$$

$$\text{for all } i \in I_1 \text{ we have} \quad (a_1)_{ii} = 0, \tag{2}$$

$$\text{for all } i \in I_2 \text{ we have} \quad (a_2)_{ii} \neq 0. \tag{3}$$

Proof. Let us write a in a block form, i.e.,

$$a = \left(\begin{array}{c|c|c|c} a_{(1)} & a_{(12)} & a_{(13)} & \cdots \\ \hline & a_{(2)} & a_{(23)} & \\ \hline & & a_{(3)} & \\ \hline & & & \ddots \end{array}\right).$$

Without loss of generality, we will assume that for all $n \in \mathbb{N}$ $(a_{(2n-1)})_{ii} = 0$ and $(a_{(2n)})_{jj} \neq 0$ for all i, j.

If a has only two blocks on the main block-diagonal, then by Lemma 2, the claim holds.

Now we will proceed inductively. Suppose that for some $n \geq 1$ for al $1 \leq i \leq n$ we have $a_i \in \mathcal{T}_{m_i}(F)$ and there exists $t^{(i)}$ of the form

$$t^{(i)} = \left(\begin{array}{c|c|c|c|c|c|c|c} e_{m_1} & 0 & 0 & 0 & \cdots & 0 & b^{(1,i)} & 0 \\ \hline & e_{m_2} & 0 & 0 & & 0 & 0 & 0 \\ \hline & & e_{m_3} & 0 & & 0 & b^{(3,i)} & 0 \\ \hline & & & e_{m_4} & & 0 & 0 & 0 \\ \hline & & & & \ddots & 0 & \vdots & \vdots \\ \hline & & & & & e_{m_{i-1}} & b^{(i-1,i)} & 0 \\ \hline & & & & & & e_{m_i} & 0 \\ \hline & & & & & & & e_\infty \end{array}\right) \quad \text{if } i \text{ is even}$$

and

$$t^{(i)} = \left(\begin{array}{c|c|c|c|c|c|c|c} e_{m_1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \hline & e_{m_2} & 0 & 0 & & 0 & c^{(2,i)} & 0 \\ \hline & & e_{m_3} & 0 & & 0 & 0 & 0 \\ \hline & & & e_{m_4} & & 0 & c^{(4,i)} & 0 \\ \hline & & & & \ddots & 0 & \vdots & \vdots \\ \hline & & & & & e_{m_{i-1}} & c^{(i-1,i)} & 0 \\ \hline & & & & & & e_{m_i} & 0 \\ \hline & & & & & & & e_\infty \end{array}\right) \quad \text{if } i \text{ is odd}$$

such that for any $1 \leq i \leq n$

$$a^{\prod_{j=1}^{i} t^{(j)}} = \left(\begin{array}{c|c|c|c|c|c|c}
a_{(1)} & 0 & a_i^{(1,3)} & 0 & \cdots & \triangle & * \\ \hline
 & a_{(2)} & 0 & a_i^{(2,4)} & & \square & * \\ \hline
 & & a_{(3)} & 0 & & \triangle & * \\ \hline
 & & & a_{(4)} & & \square & * \\ \hline
 & & & & \ddots & \vdots & * \\ \hline
 & & & & & a_{(i)} & * \\ \hline
 & & & & & & *
\end{array}\right),$$

where, depending on parity of the i, either $\triangle$ or $\square$ are zero matrices. Note that from Lemma 2 it follows that for $i = 2$ such $t^{(i)}$ exists. Now we will prove that it exists for $n+1$. We will assume that n is even. For the case when n is odd, the inductive step is analogous.

We know that for all matrices $a_{(j)}$, $1 \leq j \leq n$, $2 \nmid j$, there exist matrices $y_j^{(n+1)}$ such that

$$\left(\begin{array}{c|c} a_{(j)} & a^{(j,n+1)} \\ \hline 0 & a_{(n+1)} \end{array}\right)^{\left(\begin{array}{c|c} e_{m_j} & y_j^{(n+1)} \\ \hline 0 & e_{m_{n+1}} \end{array}\right)} = \left(\begin{array}{c|c} a_{(j)} & 0 \\ \hline 0 & a_{(n+1)} \end{array}\right).$$

We put then

$$t^{(n+1)} = \left(\begin{array}{c|c|c|c|c|c|c|c}
e_{m_1} & 0 & 0 & 0 & \cdots & 0 & y_1^{(n+1)} & 0 \\ \hline
 & e_{m_2} & 0 & 0 & & 0 & 0 & 0 \\ \hline
 & & e_{m_3} & 0 & & 0 & y_3^{(n+1)} & 0 \\ \hline
 & & & e_{m_4} & & 0 & 0 & 0 \\ \hline
 & & & & \ddots & 0 & \vdots & 0 \\ \hline
 & & & & & e_{m_n} & y_n^{(n+1)} & 0 \\ \hline
 & & & & & & e_{m_{n+1}} & 0 \\ \hline
 & & & & & & & e_\infty
\end{array}\right)$$

and have

$$a^{\prod_{j=1}^{n+1} t^{(j)}} = \left(\begin{array}{c|c|c|c|c|c|c}
a_{(1)} & 0 & a_i^{(1,3)} & 0 & \cdots & \triangle & * \\ \hline
 & a_{(2)} & 0 & a_i^{(2,4)} & & 0 & * \\ \hline
 & & a_{(3)} & 0 & & \triangle & * \\ \hline
 & & & a_{(4)} & & 0 & * \\ \hline
 & & & & \ddots & \vdots & * \\ \hline
 & & & & & a_{(n+1)} & * \\ \hline
 & & & & & & *
\end{array}\right).$$

Define now the sequence of matrices $t_n \in \mathcal{T}_\infty(F)$ as follows.

$$t_1 = t^{(1)}, \qquad t_{n+1} = t_n t^{(n+1)} \quad \text{for } n \geq 1$$

Note that from the form of matrices $t^{(i)}$ $(1 \leq i \leq n)$ it follows that

$$(t_{i+1})_{kl} = (t_i)_{kl} \qquad \text{for all } 1 \leq k, l \leq \sum_{j=1}^{i} m_j. \tag{4}$$

If a has a finite number, say k, of blocks on the main diagonal, it suffices to put $t = t_k$. Otherwise, we define t by the rule $t_{ij} = (t_{\max(i,j)})_{ij}$. As (4) is satisfied, this t is well defined and since we have $(a^t)_{ij} = (a^{t_{max(i,j)}})_{ij}$, it fulfills the desired condition. $\square$

Theorem 2 states that we can somehow decompose a triangular matrix. For the components of this 'decomposition', the following property holds.

Lemma 3. *Let F be a field. If $a \in \mathcal{T}_\infty(F)$ is of the form $a = (a_1)_{I_1} \oplus (a_2)_{I_2}$, $I_1 \neq 0$, where a_1 and a_2 satisfy* (2) *and* (3)*, respectively, and $I_1, I_2 \neq \emptyset$, then a is a zero divisor if and only if a_1 is either a zero divisor or a zero matrix.*

Proof. We will make a proof for the case of a right zero divisor. The case of a left zero divisor is exactly the same.

Let $a = (a_1)_{I_1} \oplus (a_2)_{I_2}$ and suppose that $(a_1)_{ii} = 0$ for all $i \in I_1$ and $(a_2)_{ii} \neq 0$ for all $i \in I_2$. Our a is a right zero divisor if and only if there exist $x \in \mathcal{T}_\infty(F)$ such that

$$(xa)_{nm} = \sum_{j=n}^{m} x_{nj} a_{jm} = 0 \qquad \text{for all } n \leq m. \tag{5}$$

Suppose that a is a zero divisor. Then, in particular, (5) is satisfied for the pairs n, m such that $n, m \in I_1$, so as have

$$0 = \sum_{j=n}^{m} x_{nj} a_{jm} = \sum_{\substack{n \,\leq\, j \,\leq\, m \\ j \,\in\, I_1}} x_{nj} \cdot ((a_1)_{I_1} \oplus (0)_{I_2})_{jm},$$

there exists a matrix x of the form $x = (x_1)_{I_1} \oplus (0)_{I_2}$ such that $xa = 0$. Hence, there exists x_1 such that $x_1 a_1 = 0$, so a_1 is a right zero divisor.

Analogously, if for some x_1 we have $x_1 a_1 = 0$, then we put $x = (x_1)_{I_1} \oplus (0)_{I_2}$ and we can easily check that (5) is fulfilled. $\square$

As we have proved Theorem 2 and Lemma 3, we will focus on matrices from $\mathcal{N}T_\infty(F)$.

Let us introduce the matrix defined as below.

$$s = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}.$$

One can see that if $a \in \mathcal{N}T_\infty(F)$, then

$$a = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & \cdots \\ 0 & 0 & a_{23} & a_{24} & \\ 0 & 0 & 0 & a_{34} & \\ \vdots & & & & \ddots \end{pmatrix} = \begin{pmatrix} a_{12} & a_{13} & a_{14} & \cdots \\ 0 & a_{23} & a_{24} & \\ 0 & 0 & a_{34} & \\ \vdots & & & \ddots \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}.$$

Analogously, if $a \in \mathcal{N}T_\infty^{k-1}(F)$ $(k \geq 1)$, then

$$a = \begin{pmatrix} a_{1,1+k} & a_{1,2+k} & a_{1,3+k} & \cdots & \\ 0 & a_{2,2+k} & a_{2,3+k} & & \\ 0 & 0 & a_{3,3+k} & & \\ \vdots & & & & \ddots \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}^k .$$

Thus, $xbs^k = 0$ if and only if $xb = 0$. Hence we have

Lemma 4. *Let F be a field and let $a \in \mathcal{N}T_\infty^{k-1}(F)$ for some $k \geq 1$. Then a is a right zero divisor if and only if $b \in \mathcal{T}_\infty(F)$ defined by the rule $a = bs^k$ is a right zero divisor.*

Now it easy to prove the theorem presented in the introduction.

Proof of Theorem 1: From Lemma 4 we know that a is a right zero divisor if and only if b defined by the rule $a = bs^{k+1}$ is. Hence, we can assume that $a \in \mathcal{T}_\infty(F) \setminus \mathcal{N}T_\infty(F)$. If $a_{ii} \neq 0$ for all $i \in \mathbb{N}$, then by Lemma 1 the matrix a is not a right zero divisor. Thus, the first claim is proved.

Suppose that $a_{ii} \neq 0$ for a finite but nonzero number of indices i. From Theorem 2 we know that for some $t \in \mathcal{T}_\infty(F)$ we have $a^t = (a_1)_{I_1} \oplus (a_2)_{I_2}$, where $a_1 \in \mathcal{N}T_k(F)$ for some $k \in \mathbb{N}$. This means that a^t satisfies the assumptions of Lemma 3. Hence, a^t is a right zero divisor and so is a. $\square$

From our considerations we get the following.

Corollary 1. *Suppose F is a field and either $a \in \mathcal{T}_\infty(F) \setminus \mathcal{N}T_\infty(F)$ or $a \in \mathcal{N}T_\infty^{k-1}(F) \setminus \mathcal{N}T_\infty^{k}(F)$ for some $k \in \mathbb{N}$. Assume that a contains only a finite number of nonzero coefficients on its first nonzero diagonal. If*

1. $0 < |\{i : a_{ii} = 0 \wedge a_{i+1,i+1} = 0 \wedge a_{i,i+1} = 0\}| < \infty$ *in the case when* $a \in \mathcal{T}_\infty(F) \setminus \mathcal{N}T_\infty(F)$,
2. $0 < |\{i : a_{i,i+k} = 0 \wedge a_{i+1,i+k+1} = 0 \wedge a_{i,i+k+1} = 0\}| < \infty$ *in the case when* $a \in \mathcal{N}T_\infty^{k-1}(F) \setminus \mathcal{N}T_\infty^{k}(F)$,

then a is a right zero divisor.

Proof. By Lemma 4, if $a \in \mathcal{N}T_\infty^{k-1}(F) \setminus \mathcal{N}T_\infty^{k}(F)$ we can consider b such that $a = bs^{k+1}$. We will assume then that $a \in \mathcal{T}_\infty(F) \setminus \mathcal{N}T_\infty(F)$. We know that there exist $t \in \mathcal{T}_\infty(F)$ such that $a^t = (a_1)_{I_1} \oplus (a_2)_{I_2}$ and a is a right zero divisor if and only if a_1 is. Hence, we will concentrate on a_1.

Since $a_1 \in \mathcal{N}T_\infty(F) \setminus \mathcal{N}T_\infty^{1}(F)$, from Theorem 1 it follows that it suffices to prove that a_1 has only a finite number of zeros on the first diagonal.

The coefficients $(a_1)_{i,i+1}$ can be divided into two ,sorts'.

- $(a_1)_{i,i+1} = (a^t)_{k,k+1}$ for some k such that $k, k+1 \in I_1$,
- $(a_1)_{i,i+1} = (a^t)_{kl}$ for some k, l such that $l > k+1$.

Since we have only afinite number of nonzero coefficients on the main diagonal, we have also a finite number of diagonal blocks (When we say diagonal blocks, we mean the blocks $a_{(i)}$ from the proof of Theorem 2.). Thus, we also have only a finite number of $a_{i,i+1}$ of the second 'sort'. Therefore, we can focus on the first one.

Note that for these entries we have

$$(a_1)_{i,i+1} = (a^t)_{k,k+1} = t_{kk}^{-1} a_{k,k+1} t_{k+1,k+1},$$

so $(a_1)_{i,i+1} = 0$ if and only if $a_{k,k+1} = 0$. Hence, if a has a finite but nonzero number of coefficients such that $a_{kk} = a_{k+1,k+1} = a_{k,k+1} = 0$, then a_1 also satisfies this condition. Then, by Theorem 1, the matrix a_1 is a right zero divisor, and by Lemma 3 so is a. □

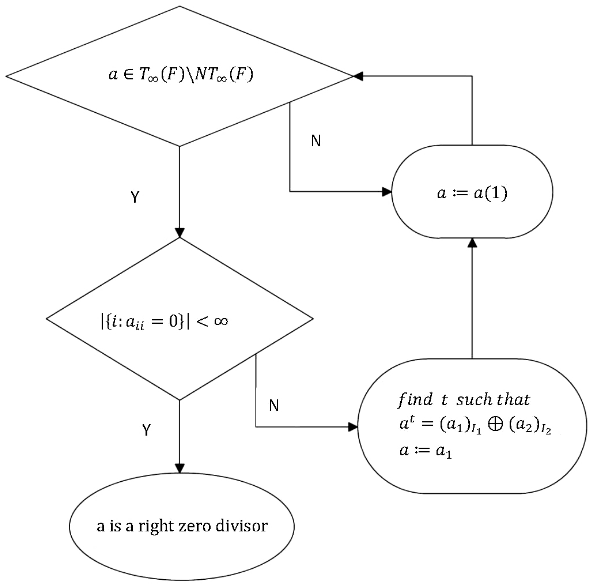

Figure 1: *Diagram of the proposed algorithm*

From what we have proved in this paper it follows that if we want to know whether an infinite triangular matrix is a right zero divisor, we can try to use the procedure in Figure 1. The matrix $a(1)$ is the matrix satsifying $a = a(1)s$.

Clearly, this algorithm does not have to finish.

Example 1. *Let a be defined as follows.*

$$a_{ij} = \begin{cases} 1 & \text{if } j = 2i - 1 \\ 0 & \text{otherwise.} \end{cases}$$

It can be observed that in each step we have $|\{i : a_{ii} = 0\}| = \infty$ and the procedure does not finish.

Let us notice that in this case it can be also observed that a is not a right zero divisor.

At the end of the paper, we would like to ask the following question.

Problem: Is $a \in \mathcal{T}_\infty(F)$ a right zero divisor if and only if the proposed above algorithm finishes for this a?

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