MATHEMATICAL COMMUNICATIONS
Math. Commun. **20**(2015), 193–200

Some families of identities for the integer partition function

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Received January 26, 2015; accepted September 8, 2015

Abstract. We give a series of recursive identities for the number of partitions with exactly k parts and with constraints on both the minimal difference among the parts and the minimal part. Using these results, we demonstrate that the number of partitions of n is equal to the number of partitions of $2n + d\binom{n}{2}$ of length n, with d-distant parts. We also provide a direct proof for this identity. This work is a result of our aim to find a bijective proof for Rogers-Ramanujan identities.

AMS subject classifications: 05A17, 11P84

Key words: partition identity, partition function, Euler function, pentagonal numbers, Rogers-Ramanujan identities

1. Introduction

The sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ in weakly decreasing order is called a *partition*. The numbers λ_i are *parts* of λ. The number of parts is the *length* of λ, denoted by $l(\lambda)$, and the sum of parts $|\lambda|$ is the *weight* of λ. Having $|\lambda| = n$, it is said that λ is a partition of n, denoted $\lambda \vdash n$.

We let $\mathcal{P}$ denote the set of all partitions and $\mathcal{P}_n$ the set of partitions of n. Furthermore, we let $p(n)$ denote the number of elements in $\mathcal{P}_n$. The number of partitions of n with k parts is denoted by $p_k(n)$, while $p_{\leq k}(n)$ is the number of partitions of n with at most k parts, $p_{\leq k}(n) = \sum_{i=0}^{k} p_i(n)$. The number of partitions of n having a minimal part $\geq r$ is denoted by $p(n; r)$. Furthermore, partitions with distant parts are of our interest. Let $p^{(d)}(n)$ be the number of partitions of n with the property that the difference between any two parts is at least d. The number of partitions with 2-distant parts, that represent the left-hand side of the first Rogers-Ramanujan identity coincides with $p^{(2)}(n)$. One may say that such partitions have super-distant parts.

Now we recall *Euler's identity* for the ordinary partition function $P(x)$,

$$\psi(x)P(x) = 1, \tag{1}$$

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where

$$\psi(x) = \prod_{k\geq 1}(1-x^k)$$

is the *Euler function* [3]. More precisely,

$$(1-x-x^2+x^5+x^7-x^{12}-\cdots)(1+p(1)x+p(2)x^2+p(3)x^3+\cdots)=1.$$

By equating the coefficients of x^n, $n>0$ we obtain the recurrence relation

$$p(n)=p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+\cdots, \qquad (2)$$

where the general term involves the j-th pentagonal number

$$\frac{j(3j-1)}{2}$$

and its reflection (obtained by $j \leftarrow -j$)

$$\frac{j(3j+1)}{2}.$$

There is a basic recurrence relation that provides computing the number of partitions with exactly k parts,

$$p_k(n)=p_{k-1}(n-1)+p_k(n-k). \qquad (3)$$

This recurrence relation relates two alternatives for the minimal part of a partition (to be 1 or >1).

In this paper, we extend these ideas, searching for similar recurrences for other types of partitions. In particular, we are interested in partitions of length k having 1-distant parts, 2-distant parts, etc. The underlying motivation for this work is to find a new bijective proof of the Rogers-Ramanujan identities. Namely, recurrences for both the l.h.s. and the r.h.s. of these identities may possibly give an insight into the matching of related partitions. Recall that the first bijective proof is done by Garsia and Milne, while a more recent one is provided by Pak and Boulet [2].

It is worth mentioning that an efficient way of computing the number of partitions of n with at most k parts is provided by partial fraction decomposition of the generating function. This idea dates back to Cayley, it is developed by Munagi [4] and formulae for $1\leq k\leq 70$ are recently derived by Sills and Zeilberger [5].

2. Partitions of length k with 1-distant parts

Proposition 1. *The number of partitions $\lambda \vdash n$ of length k with 1-distant parts and the minimal part at least 2 is equal to the alternating sum of the number of partitions μ_i of $n-i$ with* 1*-distant parts and such that $l(\mu_i)=k-i$, $0\leq i\leq k$,*

$$p_k^{(1)}(n;2)=\sum_{i=0}^{k}(-1)^i p_{k-i}^{(1)}(n-i). \qquad (4)$$

Proof. The number of partitions of n of length k and 1-distant parts and with the minimal part at least 2 equals the difference

$$p_k^{(1)}(n;2) = p_k^{(1)}(n) - p_{k-1}^{(1)}(n-1;2). \tag{5}$$

Namely, if one adds number 1 as a part to the partition of n–1 of length k–1 with 1-distant parts and the minimal part at least 2, the resulting partition will be a partition of n with k parts and with 1-distant property having the minimal part equal to 1. Now, the above statement follows immediately. Further, relation

$$p_k^{(1)}(n;2) = p_k^{(1)}(n) - p_{k-1}^{(1)}(n-1) + p_{k-2}^{(1)}(n-2;2)$$

holds true, which implies (4). □

There is an analogue of relation (3) for the partitions with 1-distant parts and it is given by the following Proposition 2.

Proposition 2. *The number of partitions $\lambda \vdash n$ of length k with 1-distant parts is equal to the sum of the numbers of partitions $\mu \vdash n-1$ of length k–1 with 1-distant parts having the minimal part equal to or greater than 2 and the number of partitions $\nu \vdash n-k$ of length k with 1-distant parts*

$$p_k^{(1)}(n) = p_{k-1}^{(1)}(n-1;2) + p_k^{(1)}(n-k). \tag{6}$$

Proof. Let us separate the partitions of n with k 1-distant parts into two sets, one having 1 as a part and another one with parts equal to or greater than 2. The first set can be built from partitions of length k–1 of n–1 with 1-distant parts greater than or equal to 2, by adding 1 as a part. The second set is obtained from the set of all partitions of $n-k$ of length k with 1-distant parts by increasing every part by 1. Clearly, these correspondences are invertible and this completes the proof. □

Using Propositions 1 and 2 we obtain the recurrence relation for the numbers $p_k^{(1)}(n)$. Later, we shall provide a shorter proof.

Theorem 1. *The number of partitions $\lambda \vdash n$ of length k with 1-distant parts is equal to the sum of numbers of partitions $\mu_i \vdash n-ik$, $i = 1, 2, \ldots$ of length k–1 and 1-distant parts*

$$p_k^{(1)}(n) = \sum_{i\geq 1} p_{k-1}^{(1)}(n-ki). \tag{7}$$

Proof. By adding 1 to each part ν_i, $1 \leq i \leq k$, of a partition $\nu \vdash n$, we get a partition $\nu' \vdash n+k$ with the minimal part ≥ 2. Thus

$$p_k^{(1)}(n) = p_k^{(1)}(n+k;2) \tag{8}$$

holds true. According to Proposition 2, we have

$$\begin{aligned} p_k^{(1)}(n) &= p_{k-1}^{(1)}(n-1;2) + p_k^{(1)}(n-k) \\ &= p_{k-1}^{(1)}(n-1;2) + p_{k-1}^{(1)}(n-k-1;2) + p_k^{(1)}(n-2k) \\ &= p_{k-1}^{(1)}(n-1;2) + p_{k-1}^{(1)}(n-k-1;2) + \cdots + p_{k-1}^{(1)}(n-qk-1;2), \end{aligned}$$

where $q = \lfloor (n-1)/k \rfloor$. When applying (8) to the above terms, the statement of the theorem follows immediately. □

In particular, when $k = 4$, Theorem 1 gives these relations:

$$\begin{aligned} p_4^{(1)}(n) &= \sum_{i\geq 1} p_3^{(1)}(n-4i) \\ &= \sum_{i,j\geq 1} p_2^{(1)}(n-3i-4j) \\ &= \sum_{i,j,l\geq 1} p_1^{(1)}(n-2i-3j-4l). \end{aligned}$$

In order to additionally illustrate these results, we calculate the number of partitions of 22 with four 1-distant parts. According to Theorem 1, it follows that

$$p_4^{(1)}(22) = p_3^{(1)}(18) + p_3^{(1)}(14) + p_3^{(1)}(10) + p_3^{(1)}(6) + p_3^{(1)}(2) = 34.$$

In the second case, $p_4^{(1)}(22)$ can be represented as the sum of the numbers of partitions of length 2:

$$\begin{aligned} p_4^{(1)}(22) = {} & p_2^{(1)}(15) + p_2^{(1)}(12) + p_2^{(1)}(9) + p_2^{(1)}(6) + p_2^{(1)}(3) \\ & + p_2^{(1)}(11) + p_2^{(1)}(8) + p_2^{(1)}(5) + p_2^{(1)}(2) \\ & + p_2^{(1)}(7) + p_2^{(1)}(4) + p_2^{(1)}(1) + p_2^{(1)}(3) \\ = {} & 34. \end{aligned}$$

In the third case, we end up with 39 terms, five of them being zero.

3. Partitions of the Rogers-Ramanujan type of fixed length

The previous facts for partitions with 1-distant parts can be extended to partitions with any difference d among the parts.

Adding d to every part of a partition of n with k parts, we obtain a partition of $n + dk$ of length k with the saved difference among the parts. The additional characteristic of the resulting partition is that the minimal part becomes at least $d+1$. Since the inverse operation gives the original partition, Proposition 3 follows.

Proposition 3. *The number of partitions $\lambda \vdash n$ of length k with d-distant parts is equal to the number of partitions $\mu \vdash n + dk$ of length k with d-distant parts and the smallest part $\geq (d+1)$,*

$$p_k^{(d)}(n) = p_k^{(d)}(n + dk; d+1). \tag{9}$$

Proposition 4. *The number of partitions $\lambda \vdash n$ of length k with d-distant parts is the sum of the number of partitions $\mu \vdash n-1$ with d-distant parts that are at least $d+1$ and the number of partitions $\nu \vdash n-k$ of length k with d-distant parts*

$$p_k^{(d)}(n) = p_{k-1}^{(d)}(n-1; d+1) + p_k^{(d)}(n-k). \tag{10}$$

Proposition 4 can be proved by similar reasoning to the proof of Proposition 2. As a generalization of Theorem 1, we have the following

Theorem 2. *The number of partitions $\lambda \vdash n$ of length k with d-distant parts is equal to the sum of the numbers of partitions $\mu_i \vdash n - ik + d - 1$, $i = d, d+1, \dots$ of length k-1 and d-distant parts*

$$p_k^{(d)}(n) = \sum_{i \geq d} p_{k-1}^{(d)}(n - ki + d - 1). \tag{11}$$

Proof. Using Proposition 4 iteratively, we have

$$p_k^{(d)}(n) = p_{k-1}^{(d)}(n-1; d+1) + p_{k-1}^{(d)}(n-k-1; d+1) + \cdots + p_{k-1}^{(d)}(n-qk-1; d+1),$$

where $q = \lfloor (n-1)/k \rfloor$. According to Proposition 3 we have to decrease the argument in every term by $d(k-1)$ in order to get the partitions with no constraints on the minimal part. This gives the following identity

$$p_k^{(d)}(n) = p_{k-1}^{(d)}(n-dk+d-1) + p_{k-1}^{(d)}(n-(d+1)k+d-1) + \cdots + p_{k-1}^{(d)}(n-(d+q)k+d-1).$$

and Theorem 2 follows. □

As an illustration of Theorem 2, there are 12 partitions of 18 with three 2-distant parts

$$\begin{aligned} p_3^{(2)}(18) &= p_2^{(2)}(13) + p_2^{(2)}(10) + p_2^{(2)}(7) + p_2^{(2)}(4) + p_2^{(2)}(1) \\ &= 5 + 4 + 2 + 1 = 12. \end{aligned}$$

Remark 1. *Theorem 2 provides a direct computation of the number of partitions representing the l.h.s. of the first Rogers-Ramanujan identity, $p^{(2)}(n) = \sum_{k \geq 1} p_k^{(2)}(n)$. Having in mind that there is a one-to-one correspondence between these partitions and partitions whose Young diagram begins with a square [1], there is one more approach to compute $p^{(2)}(n)$. This bijection provides that one can count partitions $\lambda \vdash n - i^2$ of length at most i to get the value of $p^{(2)}(n)$,*

$$p^{(2)}(n) = \sum_{i \geq 1} p_{\leq i}(n - i^2).$$

A similar remark holds for the partitions representing the l.h.s. of the second Rogers-Ramanujan identity. More precisely, it can be shown that

$$p^{(2)}(n; 2) = \sum_{i \geq 1} p_{\leq i}(n - i(i+1)).$$

Note that in our first example the recurrence for $p_4^{(1)}(22)$ begins with the number of partitions of 18 and continues with the difference 4. However, in our second example the first difference is 5, $(= 18 - 13)$ while the other differences are equal to 3. So, for the partitions with 1-distant parts, the first difference is always equal to any other difference, for every k, assuming we deal with the recurrence relation for the number of partitions with k parts expanded into terms of the number of partitions with k–1 parts. On the other hand, for the partitions with 2-distant parts when $k = 3$, the first difference is 5 and it increases by 2 as k increases by 1.

Clearly, Theorem 2 shows that the difference among arguments on the right-hand side of the equality is always k, while the difference δ between the argument on the left-hand side and the first argument on the right-hand side depends on both k and d. The difference δ increases by d when k increases by 1. Table 1 presents these first differences for the partitions with 0, 1, 2, and 3-distant parts, denoted by $\delta_0, \delta_1, \delta_2$, and δ_3, respectively.

k	δ_0	δ_1	δ_2	δ_3
3	1	3	5	7
4	1	4	7	10
5	1	5	9	13

Table 1.

4. Families of identities for $p(n)$

An immediate consequence of the basic recurrence relation (3) is a nice identity between $p(n)$ and the number of partitions of $2n$ with n parts. Moreover, there is an equality between $p(n)$ and the number of partitions of $mn + n$ with mn parts,

$$p(n) = p_{mn}(mn + n), \;\; m \geq 1. \tag{12}$$

A natural question is whether there is a similar identity between $p(n)$ and the number of partitions of a certain natural number with 1-distant parts, and in general with d-distant parts. Here we prove that identity

$$p(n) = p_n^{(1)}(2n + \binom{n}{2}) \tag{13}$$

holds true. This result is generalized in Theorem 3. Following the manner of our previous proofs, it is firstly shown algebraically. Than we provide a direct bijection that proves this identity. Note that, similarly, Theorem 2 can also be proved combinatorially. In what follows, we use the fact that there is an explicit formula for the number of partitions with two d-distant parts,

$$p_2^{(d)}(n) = \left\lfloor \frac{n-d}{2} \right\rfloor, n \geq d. \tag{14}$$

Namely, from partitions $\lambda \vdash n$ of length 2, $n = \lambda_1 + \lambda_2$, with d-distant parts the smallest part λ_2 gets values from 1 to $p_2^{(d)}(n)$ because

$$n = \lambda_1 + \lambda_2 \geq 2\lambda_2 + d$$

implies

$$\lambda_2 \leq \left\lfloor \frac{n-d}{2} \right\rfloor.$$

This means that relation (14) holds true.

By iterative usage of Theorem 2, it follows that $p_k^{(d)}(n)$ is equal to the sum

$$p_k^{(d)}(n) = \sum_{i_1,i_2,\dots,i_{k-2}\geq d} p_2^{(d)}(n - 3i_1 - \cdots - ki_{k-2} + (k-2)(d-1)). \tag{15}$$

By shifting all indexes i_j by 1, $p_k^{(d)}(n)$ is also equal to

$$\sum_{i_1,i_2,\dots,i_{k-2}\geq d-1} p_2^{(d)}(n - 3i_1 - \cdots - ki_{k-2} + (k-2)(d-1) - \frac{k(k+1)}{2} + 3). \tag{16}$$

According to (14), a particular term in sum (16) is

$$\left\lfloor \frac{n - 3i_1 - \cdots - ki_{k-2} + (k-2)(d-1) - \frac{k(k+1)}{2} + 3 - d}{2} \right\rfloor,$$

which reduces to

$$\left\lfloor \frac{n - 3i_1 - \cdots - ki_{k-2} + kd - 3d - k + 5 - \frac{k(k+1)}{2}}{2} \right\rfloor. \tag{17}$$

On the other hand, the value $p_k^{(d-1)}(n)$ is equal to the sum

$$\sum_{i_1,i_2,\dots,i_{k-2}\geq d-1} p_2^{(d-1)}(n - 3i_1 - 4i_2 - \cdots - ki_{k-2} + (k-2)(d-2)) \tag{18}$$

and a particular term in this sum is

$$\left\lfloor \frac{n - 3i_1 - \cdots - ki_{k-2} + kd - 3d - 2k + 5}{2} \right\rfloor. \tag{19}$$

Note that the difference between numerators in (19) and (17) is $\binom{k}{2}$,

$$kd - 3d - k + 5 - \frac{k(k+1)}{2} + \frac{k(k-1)}{2} = kd - 3d - 2k + 5.$$

This means that by adding $\binom{k}{2}$ to every term in sum (16) we establish the equality between this sum and sum (18). Consequently,

$$p_k^{(d-1)}(n) = p_k^{(d)}(n + \binom{k}{2}) \tag{20}$$

holds true. Applying this equality to relation (12), for $m = 1$ we get

$$\begin{aligned} p(n) = p_n(2n) &= p_n^{(1)}(2n + \binom{n}{2}) \\ &= p_n^{(2)}(2n + 2\binom{n}{2}) \\ &\ \vdots \\ &= p_n^{(d)}(2n + d\binom{n}{2}) \end{aligned}$$

which completes the following statement.

Theorem 3. *The number of partitions $\lambda \vdash n$ is equal to the number of partitions $\mu \vdash 2n + d\binom{n}{2}$ of length n with d-distant parts*

$$p(n) = p_n^{(d)}(2n + d\binom{n}{2}). \tag{21}$$

Proof. We present a bijection between partitions of n and partitions of $2n + \binom{n}{2}d$ of length n. Let $n = \lambda_1 + \cdots + \lambda_n$, $\lambda_n \geq 0$ be a partition of n. Clearly, the equality

$$2n = (\lambda_1 + 1) + \cdots + (\lambda_n + 1)$$

holds true. Adding $(n-1)d, \ldots, 2d, d, 0$ to the parts of λ, respectively, makes parts d distant. The fact that the resulting partition has weight $2n + \binom{n}{2}d$ and n parts completes the proof.

□

Theorem 3 can be easily extended so that

$$p(n) = p_{mn}^{(d)}(mn + n + d\binom{n}{2}), \quad m \geq 1. \tag{22}$$

In other words, for every $n, d \in \mathbb{N}_0$ there are infinitely many sets of partitions whose length is a multiple of n and whose parts are d-distant, that are equinumerous to $\mathcal{P}_n$.

Acknowledgement

Work on this paper started during the first author's stay at the Isaac Newton Institute for Mathematical Science in Cambridge. He thanks the Institute's personnel for their hospitality.

The authors thank the anonymous referee for valuable suggestions that improved the final version of the paper.

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