

## On a maximal subgroup of the Thompson simple group\*

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**Abstract.** The present paper deals with a maximal subgroup of the Thompson group, namely the group  $2_+^{1+8} \cdot A_9 := \overline{G}$ . We compute its conjugacy classes using the coset analysis method, its inertia factor groups and Fischer matrices, which are required for the computations of the character table of  $\overline{G}$  by means of Clifford-Fischer Theory.

**AMS subject classifications:** 20C15, 20C40

**Key words:** group extensions, character table, projective character, inertia groups, Fischer matrices

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### 1. Introduction

Let  $\overline{G}$  be the normalizer of the unique class of involutions  $2A$  of the sporadic Thompson group Th. Our group  $\overline{G}$  is the third largest maximal subgroup of Th (see [7]) and has the form  $\overline{G} = 2_+^{1+8} \cdot A_9$ , the non-split extension of the extraspecial 2-group of order 512 with an outer automorphism group isomorphic to  $O_8^+(2)$  by the Alternating group  $A_9$ . The group  $\overline{G}$  has order 92 897 280 and index 976 841 775 in Th. This is a very good example for the applications of Clifford-Fischer Theory since the group is a non-split extension with an extra-special 2-group as its kernel. Not many examples of this type have been studied via Clifford-Fischer Theory. In this paper, our main aims are to fully study this group, determine its inertia factor groups and compute all Fischer matrices. It will turn out that the character table of  $\overline{G}$  is a  $52 \times 52$  matrix. If one is only interested in the calculation of the character table, then it could be computed by using GAP or Magma and the generators  $x$  and  $y$  (see below) of  $\overline{G}$ . But Clifford-Fischer Theory provides much more interesting information on the group and on the character table; in particular, the character table produced by Clifford-Fischer Theory is in a special format that could not be achieved by direct computations using GAP or Magma. Also, providing examples of applications of Clifford-Fischer Theory to both split and non-split extensions is a sensible choice since each group requires an individual approach. The readers (particularly young researchers) will highly benefit from the theoretical background required for these computations. GAP and Magma are computational tools and would not replace good powerful and theoretical arguments.

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Let  $x$  and  $y$  be in the 248-dimensional matrix group over  $\mathbb{F}_2$ , that are generators of Th given by the electronic ATLAS of Wilson [18]. Using [6] and the program supplied by Wilson, it is possible to construct  $\overline{G}$  inside Th. We were also able to locate a normal subgroup  $N \triangleleft \overline{G}$  isomorphic to  $2_+^{1+8}$  by checking the normal subgroups of  $\overline{G}$  and the conditions for extraspecial  $p$ -groups.

**Remark 1.** Due to Havas, Soicher and Wilson [10], a presentation for  $\overline{G} = 2_+^{1+8} \cdot A_9$  has been restated in Lemmas 12.1.2 and 12.1.3 of [12].

For the notation used in this paper and the description of the Clifford-Fischer theory technique, we follow [3] and [4].

## 2. Conjugacy classes of group extensions and of $\overline{G} = 2_+^{1+8} \cdot A_9$

In this section, we calculate the conjugacy classes of  $\overline{G}$  using the coset analysis technique (for more details see [13] or [14]) as we are interested to organize the classes of  $\overline{G}$  corresponding to the classes of  $A_9$ . Note that in [12], G. Michler determined the conjugacy classes of  $\overline{G} = 2_+^{1+8} \cdot A_9$  using the Algorithm of Kratzer (see p. 294 in [12]). One can also use MAGMA or GAP, with the presentation of  $\overline{G}$  given by the ATLAS of Wilson, to compute its conjugacy classes.

**Example 1.** Consider the identity coset  $N1_{A_9} = N = 2_+^{1+8}$  as this coset gives much information about the structure of the character table of  $\overline{G}$ . If  $g = 1_{A_9}$ , then the action of  $N$  on  $N\overline{1_{A_9}} = N1_{A_9} = N$  produces the conjugacy classes of  $N$ , where we know that  $N$  has

- singleton conjugacy class consisting of  $1_N$ ,
- singleton conjugacy class consisting of the central involution  $\sigma$  of  $N$ ,
- 135 conjugacy classes, each of which consists of two non-central involutions,
- 120 conjugacy classes, each of which consists only of two elements of order 4.

Now using MAGMA, the action of  $\overline{G}$  on the preceding orbits leaves invariant  $\{1_N\} := \Delta_{11}$  and  $\{\sigma\} := \Delta_{12}$ , while fuses the 135 orbits of non-central involutions into a single orbit  $\Delta_{13}$  and also fuses the 120 orbits of elements of order 4 altogether into a single orbit  $\Delta_{14}$ . Thus the identity coset  $N$  produces four conjugacy classes  $\Delta_{11}$ ,  $\Delta_{12}$ ,  $\Delta_{13}$  and  $\Delta_{14}$  in  $\overline{G}$ , where  $|\Delta_{11}| = |\Delta_{12}| = 1$ ,  $|\Delta_{13}| = 270$  and  $|\Delta_{14}| = 240$ . We let  $g_{11} = 1_{\overline{G}}$ ,  $g_{12} = \sigma$ ,  $g_{13} \in \Delta_{13}$  and  $g_{14} \in \Delta_{14}$  be representatives of the  $\overline{G}$ -conjugacy classes obtained from  $N$ .

In Table 1, we list the conjugacy classes of  $\overline{G}$  together with the fusion of its classes into the classes of Thompson group Th. To each conjugacy class of  $\overline{G}$ , we have attached some weights  $m_{ij}$ , which will be used later in computing the Fischer matrices of  $\overline{G}$ . These weights are computed through the formula

$$m_{ij} = [N_{\overline{G}}(N\overline{g}_i) : C_{\overline{G}}(g_{ij})] = |N| \frac{|C_G(g_i)|}{|C_{\overline{G}}(g_{ij})|}, \text{ where } G = \overline{G}/N \cong A_9. \quad (1)$$

$[g_i]_{A_9}$	$m_{ij}$	$[g_{ij}]_{2_+^{1+8} \cdot A_9}$	$o(g_{ij})$	$ [g_{ij}]_{2_+^{1+8} \cdot A_9} $	$ C_{2_+^{1+8} \cdot A_9}^{(g_{ij})} $	$\leftrightarrow$ Th
$g_1 = 1A$	$m_{11} = 1$	$g_{11}$	1	1	92897280	1A
	$m_{12} = 1$	$g_{12}$	2	1	92897280	2A
	$m_{13} = 270$	$g_{13}$	2	270	344064	2A
	$m_{14} = 240$	$g_{14}$	4	240	387072	4A
$g_2 = 2A$	$m_{21} = 32$	$g_{21}$	4	12096	7680	4B
	$m_{22} = 480$	$g_{22}$	4	181440	512	4B
$g_3 = 2B$	$m_{31} = 32$	$g_{31}$	2	30240	3072	2A
	$m_{32} = 32$	$g_{32}$	4	30240	3072	4A
	$m_{33} = 192$	$g_{33}$	4	181440	512	4B
	$m_{34} = 256$	$g_{34}$	8	241920	384	8A
$g_4 = 3A$	$m_{41} = 256$	$g_{41}$	3	43008	2160	3C
	$m_{42} = 256$	$g_{42}$	6	43008	2160	6A
$g_5 = 3B$	$m_{51} = 64$	$g_{51}$	3	143360	648	3B
	$m_{52} = 64$	$g_{52}$	6	143360	648	6C
	$m_{53} = 384$	$g_{53}$	12	860160	108	12C
$g_6 = 3C$	$m_{61} = 16$	$g_{61}$	3	53760	1728	3A
	$m_{62} = 16$	$g_{62}$	6	53760	1728	6B
	$m_{63} = 288$	$g_{63}$	6	967680	96	6B
	$m_{64} = 96$	$g_{64}$	12	322560	288	12A
	$m_{65} = 96$	$g_{65}$	12	322560	288	12B
$g_7 = 4A$	$m_{71} = 128$	$g_{71}$	8	967680	96	8B
	$m_{72} = 384$	$g_{72}$	8	2903040	32	8B
$g_8 = 4B$	$m_{81} = 128$	$g_{81}$	4	1451520	64	4B
	$m_{82} = 128$	$g_{82}$	8	1451520	64	8A
	$m_{83} = 256$	$g_{83}$	8	2903040	32	8B
$g_9 = 5A$	$m_{91} = 256$	$g_{91}$	5	774144	120	5A
	$m_{92} = 256$	$g_{92}$	10	774144	120	10A
$g_{10} = 6A$	$m_{10,1} = 512$	$g_{10,1}$	12	3870720	24	12D
$g_{11} = 6B$	$m_{11,1} = 128$	$g_{11,1}$	6	3870720	24	6B
	$m_{11,2} = 64$	$g_{11,2}$	12	1935360	48	12A
	$m_{11,3} = 64$	$g_{11,3}$	12	1935360	48	12B
	$m_{11,4} = 128$	$g_{11,4}$	24	3870720	24	24A
	$m_{11,5} = 128$	$g_{11,5}$	24	3870720	24	24B
$g_{12} = 7A$	$m_{12,1} = 64$	$g_{12,1}$	7	1658880	56	7A
	$m_{12,2} = 64$	$g_{12,2}$	14	1658880	56	14A
	$m_{12,3} = 128$	$g_{12,3}$	14	3317760	28	14A
	$m_{12,4} = 128$	$g_{12,4}$	14	3317760	28	14A
	$m_{12,5} = 128$	$g_{12,5}$	28	3317760	28	28A
$g_{13} = 9A$	$m_{13,1} = 256$	$g_{13,1}$	9	5160960	18	9C
	$m_{13,2} = 256$	$g_{13,2}$	18	5160960	18	18B
$g_{14} = 9B$	$m_{14,1} = 64$	$g_{14,1}$	9	1290240	72	9A
	$m_{14,2} = 64$	$g_{14,2}$	18	1290240	72	18A
	$m_{14,3} = 128$	$g_{14,3}$	36	2580480	36	36B
	$m_{14,4} = 128$	$g_{14,4}$	36	2580480	36	36A
	$m_{14,5} = 128$	$g_{14,5}$	36	2580480	36	36C
$g_{15} = 10A$	$m_{15,1} = 512$	$g_{15,1}$	20	4644864	20	20A
$g_{16} = 12A$	$m_{16,1} = 256$	$g_{16,1}$	24	3870720	24	24C
	$m_{16,2} = 256$	$g_{16,2}$	24	3870720	24	24D
$g_{17} = 15A$	$m_{17,1} = 256$	$g_{17,1}$	15	3096576	30	15A
	$m_{17,2} = 256$	$g_{17,2}$	30	3096576	30	30A
$g_{18} = 15B$	$m_{18,1} = 256$	$g_{18,1}$	15	3096576	30	15B
	$m_{18,2} = 256$	$g_{18,2}$	30	3096576	30	30B

Table 1: The conjugacy classes of  $2_+^{1+8} \cdot A_9$

### 3. The theory of Clifford-Fischer matrices

We give a brief description on Clifford-Fischer theory for constructing the character table of a group extension  $\overline{G}$ .

Let  $\overline{H} \triangleleft \overline{G}$  and let  $\phi \in \text{Irr}(\overline{H})$ . For  $\overline{g} \in \overline{G}$ , define  $\phi^{\overline{g}}$  by  $\phi^{\overline{g}}(h) = \phi(\overline{g}h\overline{g}^{-1})$ ,  $\forall h \in \overline{H}$ . It follows that  $\overline{G}$  acts on  $\text{Irr}(\overline{H})$  by conjugation and we define the *inertia group* of  $\phi$  in  $\overline{G}$  by  $\overline{H}_\phi = \{\overline{g} \in \overline{G} \mid \phi^{\overline{g}} = \phi\}$ . Also, for a finite group  $K$ , we let  $\text{IrrProj}(K, \alpha^{-1})$  denote the set of irreducible projective characters of  $K$  with factor set  $\alpha^{-1}$ .

**Theorem 1** (Clifford Theorem). *Let  $\chi \in \text{Irr}(\overline{G})$  and let  $\theta_1, \theta_2, \dots, \theta_t$  be representatives of orbits of  $\overline{G}$  on  $\text{Irr}(N)$ . For  $k \in \{1, 2, \dots, t\}$ , let  $\theta_k^{\overline{G}} = \{\theta_k = \theta_{k1}, \theta_{k2}, \dots, \theta_{ks_k}\}$  and let  $\overline{H}_k$  be the inertia group in  $\overline{G}$  of  $\theta_k$ . Then*

$$\chi \downarrow_N^{\overline{G}} = \sum_{k=1}^t e_k \sum_{u=1}^{s_k} \theta_{ku}, \quad \text{where } e_k = \langle \chi \downarrow_N^{\overline{G}}, \theta_k \rangle.$$

Moreover, for fixed  $k$

$$\begin{aligned} \text{Irr}(\overline{H}_k, \theta_k) &:= \left\{ \psi_k \in \text{Irr}(\overline{H}_k) \mid \langle \psi_k \downarrow_N^{\overline{H}_k}, \theta_k \rangle \neq 0 \right\} \\ &\longleftrightarrow \left\{ \chi \in \text{Irr}(\overline{G}) \mid \langle \chi \downarrow_N^{\overline{G}}, \theta_k \rangle \neq 0 \right\} := \text{Irr}(\overline{G}, \theta_k) \end{aligned}$$

under the map  $\psi_k \mapsto \psi_k \uparrow_{\overline{H}_k}^{\overline{G}}$ .

**Proof.** See Theorems 4.1.5 and 4.1.7 of Ali [1] with the difference in notations.  $\square$

**Theorem 2.** *Further to the settings of Theorem 1, assume that for  $k \in \{1, 2, \dots, t\}$ , there exists  $\psi_k \in \text{Irr}(\overline{H}_k, \theta_k)$ . Then the irreducible characters of  $\overline{G}$  are given by*

$$\text{Irr}(\overline{G}) = \bigcup_{k=1}^t \left\{ (\psi_k \text{ inf}(\zeta)) \uparrow_{\overline{H}_k}^{\overline{G}} \mid \zeta \in \text{Irr}(\overline{H}_k/N) \right\}. \tag{2}$$

**Proof.** See Ali [1] or Whitley [17].  $\square$

**Remark 2.** *It is by no means necessarily the case that there exists an extension  $\psi_k$  of  $\theta_k$  to the inertia group (that is, the case  $\text{Irr}(\overline{H}_k, \theta_k) = \emptyset$ , the empty set, is feasible). However, there is always a projective extension  $\tilde{\psi}_k \in \text{IrrProj}(\overline{H}_k, \overline{\alpha}_k^{-1})$  for some factor set  $\overline{\alpha}_k$  of the Schur multiplier of  $\overline{H}_k$ . Thus a more appropriate formula for Equation (2) is (see Remark 4.2.7 of Ali [1])*

$$\text{Irr}(\overline{G}) = \bigcup_{k=1}^t \left\{ (\tilde{\psi}_k \text{ inf}(\zeta)) \uparrow_{\overline{H}_k}^{\overline{G}} \mid \tilde{\psi}_k \in \text{IrrProj}(\overline{H}_k, \overline{\alpha}_k^{-1}), \zeta \in \text{IrrProj}(\overline{H}_k/N, \alpha_k^{-1}) \right\}, \tag{3}$$

where the factor set  $\alpha_k$  is obtained from  $\overline{\alpha}_k$  as described in Corollary 7.3.3 of Whitley [17]. Hence the character table of  $\overline{G}$  is partitioned into  $t$  blocks  $\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_t$ , where each block  $\mathcal{K}_k$  of characters (ordinary or projective) is produced from the inertia group  $\overline{H}_k$ .

**Note 1.** Let  $[n]$  denote an equivalence class (containing  $n$ ) of the Schur multiplier of a finite group. It follows that if  $\alpha_k \sim [1]$  in Equation (3), then we get Equation (2). That is,  $\text{IrrProj}(\overline{H}_k, \overline{1}) = \text{Irr}(\overline{H}_k)$  and  $\text{IrrProj}(H_k, 1) = \text{Irr}(H_k)$ . By convention, we take  $\theta_1 = \mathbf{1}_N$ , the trivial character of  $N$ . Thus  $\overline{H}_{\theta_1} = \overline{H}_1 = \overline{G}$  and thus  $\overline{H}_1/N \cong G$ . Since  $\{\mathbf{1}_{\overline{G}}\} \subseteq \text{Irr}(\overline{G}, \mathbf{1}_N)$  and such that  $\mathbf{1}_{\overline{G}} \downarrow_N^{\overline{G}} = \mathbf{1}_N$ , the block  $\mathcal{K}_1$  will consist only of the ordinary irreducible characters of  $G$ .

We now fix some notations for the conjugacy classes.

- With  $\pi$  being the natural epimorphism from  $\overline{G}$  onto  $G$ , we use the notation  $U = \pi(\overline{U})$  for any subset  $\overline{U} \subseteq \overline{G}$ . Let us assume that  $\pi(g_{ij}) = g_i$  and by convention we may take  $g_{11} = \mathbf{1}_{\overline{G}}$ . Note that  $c(g_1)$  is the number of  $\overline{G}$ -conjugacy classes obtained from  $N$ .

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$$[g_{ij}]_{\overline{G}} \cap \overline{H}_k = \bigcup_{n=1}^{c(g_{ijk})} [g_{ijkn}]_{\overline{H}_k},$$

where  $g_{ijkn} \in \overline{H}_k$  and by  $c(g_{ijk})$  we mean the number of  $\overline{H}_k$ -conjugacy classes that form a partition for  $[g_{ij}]_{\overline{G}}$ . Since  $g_{11} = \mathbf{1}_{\overline{G}}$ , we have  $g_{11k1} = \mathbf{1}_{\overline{G}}$  and thus  $c(g_{11k1}) = 1$  for all  $1 \leq k \leq t$ .

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$$[g_i]_G \cap H_k = \bigcup_{m=1}^{c(g_{ik})} [g_{ikm}]_{H_k},$$

where  $g_{ikm} \in H_k$  and by  $c(g_{ik})$  we mean the number of  $H_k$ -conjugacy classes that form a partition for  $[g_i]_G$ . Since  $g_1 = \mathbf{1}_G$ , we have  $g_{1k1} = \mathbf{1}_G$  and thus  $c(g_{1k1}) = 1$  for all  $1 \leq k \leq t$ . Also,  $\pi(g_{ijkn}) = g_{ikm}$  for some  $m = f(j, n)$ .

**Proposition 1.** *With the notations of Theorem 2 and the above settings, we have*

$$(\tilde{\psi}_k \inf(\zeta)) \uparrow_{\overline{H}_k}^{\overline{G}}(g_{ij}) = \sum_{m=1}^{c(g_{ik})} \zeta(g_{ikm}) \sum_{n=1}^{c(g_{ijk})} \frac{|C_{\overline{G}}(g_{ij})|}{|C_{\overline{H}_k}(g_{ijkn})|} \tilde{\psi}_k(g_{ijkn}).$$

**Proof.** See Ali [1] or Barraclough [2]. □

We proceed to define the Fischer matrix  $\mathcal{F}_i$  corresponding to the conjugacy class  $[g_i]_G$ . We label the columns of  $\mathcal{F}_i$  by the representatives of  $[g_{ij}]_{\overline{G}}$ ,  $1 \leq j \leq c(g_i)$  obtained by the coset analysis, and below each  $g_{ij}$  we put  $|C_{\overline{G}}(g_{ij})|$ . Thus there are  $c(g_i)$  columns. To label the rows of  $\mathcal{F}_i$  we define the set  $\overline{J}_i$  to be (this is an equivalent to the notation  $R(g)$  used by Ali [1] (p. 49), where  $g$  is a representative of a conjugacy class of  $G$ )

$$\overline{J}_i = \{(k, g_{ikm}) \mid 1 \leq k \leq t, 1 \leq m \leq c(g_{ik}), g_{ikm} \text{ is an } \alpha_k^{-1}\text{-regular class}\},$$

or for more brevity we let

$$J_i = \{(k, m) \mid 1 \leq k \leq t, 1 \leq m \leq c(g_{ik}), g_{ikm} \text{ is an } \alpha_k^{-1}\text{-regular class}\}. \quad (4)$$

Then each row of  $\mathcal{F}_i$  is indexed by a pair  $(k, g_{ikm}) \in \bar{J}_i$  or  $(k, m) \in J_i$ . For fixed  $1 \leq k \leq t$ , we let  $\mathcal{F}_{ik}$  be a sub-matrix of  $\mathcal{F}_i$  with rows corresponding to the pairs  $(k, g_{ik1}), (k, g_{ik2}), \dots, (k, g_{ikr_{ik}})$  or for brevity  $(k, 1), (k, 2), \dots, (k, r_k)$ . Now let

$$a_{ij}^{(k,m)} := \sum_{n=1}^{c(g_{ijk})} \frac{|C_{\bar{G}}(g_{ij})|}{|C_{\bar{H}_k}(g_{ijkn})|} \tilde{\psi}_k(g_{ijkn}), \tag{5}$$

(for which  $\pi(g_{ijkn}) = g_{ikm}$ ). For each  $i$ , corresponding to the conjugacy class  $[g_i]_G$ , we define the Fischer matrix  $\mathcal{F}_i = \left( a_{ij}^{(k,m)} \right)$ , where  $1 \leq k \leq t$ ,  $1 \leq m \leq c(g_{ik})$ ,  $1 \leq j \leq c(g_i)$ . The Fischer matrix  $\mathcal{F}_i$ ,

$$\mathcal{F}_i = \left( a_{ij}^{(k,m)} \right) = \begin{pmatrix} \mathcal{F}_{i1} \\ \mathcal{F}_{i2} \\ \vdots \\ \mathcal{F}_{it} \end{pmatrix}$$

together with additional information required for their definition, is presented as follows:

		$\mathcal{F}_i$			
$g_i$		$g_{i1}$	$g_{i2}$	$\dots$	$g_{ic}(g_i)$
$ C_{\bar{G}}(g_{ij}) $		$ C_{\bar{G}}(g_{i1}) $	$ C_{\bar{G}}(g_{i2}) $	$\dots$	$ C_{\bar{G}}(g_{ic}(g_i)) $
$(k, m)$	$ C_{H_k}(g_{ikm}) $				
(1, 1)	$ C_G(g_i) $	$a_{i1}^{(1,1)}$	$a_{i2}^{(1,1)}$	$\dots$	$a_{ic}^{(1,1)}(g_i)$
(2, 1)	$ C_{H_2}(g_{i21}) $	$a_{i1}^{(2,1)}$	$a_{i2}^{(2,1)}$	$\dots$	$a_{ic}^{(2,1)}(g_i)$
(2, 2)	$ C_{H_2}(g_{i22}) $	$a_{i1}^{(2,2)}$	$a_{i2}^{(2,2)}$	$\dots$	$a_{ic}^{(2,2)}(g_i)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
(2, $r_2$ )	$ C_{H_2}(g_{i2r_{i2}}) $	$a_{i1}^{(2,r_2)}$	$a_{i2}^{(2,r_2)}$	$\dots$	$a_{ic}^{(2,r_2)}(g_i)$
( $u$ , 1)	$ C_{H_u}(g_{iu1}) $	$a_{i1}^{(u,1)}$	$a_{i2}^{(u,1)}$	$\dots$	$a_{ic}^{(u,1)}(g_i)$
( $u$ , 2)	$ C_{H_u}(g_{iu2}) $	$a_{i1}^{(u,2)}$	$a_{i2}^{(u,2)}$	$\dots$	$a_{ic}^{(u,2)}(g_i)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
( $u$ , $r_u$ )	$ C_{H_u}(g_{iur_{iu}}) $	$a_{i1}^{(u,r_u)}$	$a_{i2}^{(u,r_u)}$	$\dots$	$a_{ic}^{(u,r_u)}(g_i)$
( $t$ , 1)	$ C_{H_t}(g_{it1}) $	$a_{i1}^{(t,1)}$	$a_{i2}^{(t,1)}$	$\dots$	$a_{ic}^{(t,1)}(g_i)$
( $t$ , 2)	$ C_{H_t}(g_{it2}) $	$a_{i1}^{(t,2)}$	$a_{i2}^{(t,2)}$	$\dots$	$a_{ic}^{(t,2)}(g_i)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
( $t$ , $r_t$ )	$ C_{H_t}(g_{itr_{it}}) $	$a_{i1}^{(t,r_t)}$	$a_{i2}^{(t,r_t)}$	$\dots$	$a_{ic}^{(t,r_t)}(g_i)$
$m_{ij}$		$m_{i1}$	$m_{i2}$	$\dots$	$m_{ic}(g_i)$

Fischer matrices satisfy some interesting properties, which help in computations of their entries. We gather these properties in the following proposition.

**Proposition 2.**

- (i)  $\sum_{k=1}^t c(g_{ik}) = c(g_i)$ .
- (ii)  $\mathcal{F}_i$  is non-singular for each  $i$ .

- (iii)  $a_{ij}^{(1,1)} = 1, \forall 1 \leq j \leq c(g_i)$ .
- (iv) If  $N\bar{g}_i$  is a split coset, then  $a_{i1}^{(k,m)} = \frac{|C_G(g_i)|}{|C_{H_k}(g_{ikm})|}, \forall i \in \{1, 2, \dots, r\}$ . In particular, for the identity coset we have  $a_{i1}^{(k,m)} = [G : H_k]\theta_k(1_N), \forall (k, m) \in J_1$ .
- (v) If  $N\bar{g}_i$  is a split coset, then  $|a_{ij}^{(k,m)}| \leq |a_{i1}^{(k,m)}|$  for all  $1 \leq j \leq c(g_i)$ . Moreover, if  $|N| = p^\alpha$ , for some prime  $p$ , then  $a_{ij}^{(k,m)} \equiv a_{i1}^{(k,m)} \pmod{p}$ .
- (vi) For each  $1 \leq i \leq r$ , the weights  $m_{ij}$  satisfy the relation  $\sum_{j=1}^{c(g_i)} m_{ij} = |N|$ .
- (vii) Column orthogonality relation:

$$\sum_{(k,m) \in J_i} |C_{H_k}(g_{ikm})| a_{ij}^{(k,m)} \overline{a_{ij'}^{(k,m)}} = \delta_{jj'} |C_{\bar{G}}(g_{ij})|.$$

- (viii) Row orthogonality relation:

$$\sum_{j=1}^{c(g_i)} m_{ij} a_{ij}^{(k,m)} \overline{a_{ij}^{(k',m')}} = \delta_{(k,m)(k',m')} a_{i1}^{(k,m)} |N|.$$

**Proof.** See Basheer and Moori [4, 5]. □

#### 4. The inertia factors of $\bar{G} = 2_+^{1+8} \cdot A_9$

The action of  $\bar{G} = 2_+^{1+8} \cdot A_9$  on  $\text{Irr}(2_+^{1+8})$  produces four orbits of lengths 1, 120, 135 and 1 with representatives  $\theta_1, \theta_2, \theta_3$  and  $\theta_4$ , respectively, where  $\theta_1 = \mathbf{1}_N$  the trivial character of  $N$  and  $\theta_4$  is the unique faithful irreducible character of  $N$  of degree 16. Hence the inertia factor groups  $H_1, H_2, H_3$  and  $H_4$  of  $\theta_1, \theta_2, \theta_3$  and  $\theta_4$ , respectively, have indices 1, 120, 135 and 1, respectively, in  $G \cong A_9$ . Clearly,  $H_1 = H_4 = A_9$ . A subgroup of  $A_9$  of index 120 is a maximal subgroup isomorphic to  $PSL(2, 8):3$  (by considering the maximal subgroups of  $A_9$  given in the ATLAS). A subgroup of  $A_9$  of index 135 can only be contained in a maximal subgroup that is isomorphic to  $A_8$  and must have index 15 in  $A_8$ . Again, by considering the maximal subgroups of  $A_8$  given by the ATLAS, it turns out that this subgroup is isomorphic to  $2^3:GL(3, 2)$ . Hence we have determined all the inertia factor groups  $H_1, H_2, H_3$  and  $H_4$ . We list brief information on these inertia factors as follows:

$$\begin{array}{lll} H_1 = A_9, & [A_9 : H_1] = 1, & |\text{Irr}(H_1)| = 18, \\ H_2 = PSL(2, 8):3, & [A_9 : H_2] = 120, & |\text{Irr}(H_2)| = 11, \\ H_3 = 2^3:GL(3, 2), & [A_9 : H_3] = 135, & |\text{Irr}(H_3)| = 11, \\ H_4 = A_9, & [A_9 : H_4] = 1, & |\text{IrrProj}(H_4, 2)| = 12. \end{array}$$

Note that above we have listed the number of ordinary irreducible characters of  $H_1, H_2, H_3$  and the number of projective characters of  $H_4$  with the factor set

$\alpha^{-1}$ ,  $\alpha \sim [2]$  as we shall see later that this projective table is needed for the construction of the character table of  $\overline{G} = 2_+^{1+8}.A_9$ . Therefore, the character table of  $\overline{G}$  is composed of four blocks of characters which correspond to the ordinary character tables of  $A_9$ ,  $PSL(2, 8):3$ ,  $2^3:GL(3, 2)$  and the projective table of  $A_9$  with factor set  $\alpha^{-1}$ ,  $\alpha \sim [2]$ . Hence  $\overline{G}$  has altogether 52 irreducible characters, which equals the number of conjugacy classes we obtained in Section 2.

**Remark 3.** In [3], the character table of  $H_2 = PSL(2, 8):3$  has been constructed using Clifford-Fischer theory. The three Fischer matrices of  $H_2$  have also been listed in [3]. For the sake of convenience, in Table 2 we list the character table of  $H_2$ , in the format of Clifford-Fischer theory. For the character table of  $H_3$  we refer to Table 5.4 of [17].

	$1_{\mathbb{Z}_3}$					$\omega$			$\omega^{-1}$		
	$h_{11}$	$h_{12}$	$h_{13}$	$h_{14}$	$h_{15}$	$h_{21}$	$h_{22}$	$h_{23}$	$h_{31}$	$h_{32}$	$h_{33}$
$o(h_{ij})$	1	2	3	7	9	3	6	9	3	6	9
$ C_{H_2}(h_{ij}) $	1512	24	27	7	9	18	6	9	18	6	9
$ [h_{ij}]_{H_2} $	1	63	56	216	168	84	252	168	84	252	168
$\chi_1$	1	1	1	1	1	1	1	1	1	1	1
$\chi_2$	1	1	1	1	1	$A$	$A$	$A$	$\overline{A}$	$\overline{A}$	$\overline{A}$
$\chi_3$	1	1	1	1	1	$\overline{A}$	$\overline{A}$	$\overline{A}$	$A$	$A$	$A$
$\chi_4$	7	-1	-2	0	1	1	-1	1	1	-1	1
$\chi_5$	7	-1	-2	0	1	$A$	$-A$	$A$	$\overline{A}$	$-\overline{A}$	$\overline{A}$
$\chi_6$	7	-1	-2	0	1	$\overline{A}$	$-\overline{A}$	$\overline{A}$	$A$	$-A$	$A$
$\chi_7$	8	0	-1	1	-1	2	0	-1	2	0	-1
$\chi_8$	8	0	-1	1	-1	$2A$	0	$-A$	$2\overline{A}$	0	$-\overline{A}$
$\chi_9$	8	0	-1	1	-1	$2\overline{A}$	0	$-\overline{A}$	$2A$	0	$-A$
$\chi_{10}$	21	-3	3	0	0	0	0	0	0	0	0
$\chi_{11}$	27	3	0	-1	0	0	0	0	0	0	0

Table 2: The character table of  $H_2 = PSL(2, 8):3$

where in Table 2,  $A = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ .

### 5. Fusions of the inertia factors into $A_9$

In this section, we determine the fusions of the inertia factor groups  $H_2 = PSL(2, 8):3$  and  $H_3 = 2^3:GL(3, 2)$  into  $A_9$  using the permutation characters of  $A_9$  on these groups. From the ATLAS, the permutation character of  $A_9$  on  $H_2$  is of the form

$$\chi(A_9|H_2) = \underline{1a} + \underline{35b} + \underline{84a}, \tag{6}$$

where  $\underline{1a}$ ,  $\underline{35b}$  and  $\underline{84a}$  denote the characters  $\chi_1$ ,  $\chi_8$  and  $\chi_{12}$  of  $A_9$ , respectively (in the order listed in ATLAS). Also note that  $H_3$  is a maximal subgroup of  $A_8$  of index 15, which is in turn a maximal subgroup of  $A_9$  of index 9.

**Proposition 3.** Let  $K_1 \leq K_2 \leq K_3$  and let  $\psi$  be a class function on  $K_1$ . Then

$$(\psi \uparrow_{K_1}^{K_2}) \uparrow_{K_2}^{K_3} = \psi \uparrow_{K_1}^{K_3}.$$

More generally, if  $K_1 \leq K_2 \leq \dots \leq K_n$  is a nested sequence of subgroups of  $K_n$  and  $\psi$  is a class function on  $K_1$ , then

$$(\psi \uparrow_{K_1}^{K_2}) \uparrow_{K_2}^{K_3} \dots \uparrow_{K_{n-1}}^{K_n} = \psi \uparrow_{K_1}^{K_n}.$$



**Proof.** See Proposition 3.5.6 of [3]. □

Since we know the permutation characters of  $A_9$  on  $A_8$  and of  $A_8$  on  $H_3$ , together with the fusions of classes of  $H_3$  into  $A_8$  and of  $A_8$  into  $A_9$ , and by Proposition 3 and the ATLAS, we are able to calculate the permutation character  $\chi(A_9|H_3)$  as follows

$$\begin{aligned} \chi(A_9|H_3) &= \mathbf{1}_{H_3}^{A_9} = (\mathbf{1}_{H_3}^{A_8})\uparrow_{A_8}^{A_9} = (\underline{1a} + \underline{14a})\uparrow_{A_8}^{A_9} \\ &= \underline{1a}\uparrow_{A_8}^{A_9} + \underline{14a}\uparrow_{A_8}^{A_9} = \underline{1a} + \underline{8a} + \underline{14a}\uparrow_{A_8}^{A_9}. \end{aligned}$$

Now it is easy to evaluate the values of  $\underline{14a}\uparrow_{A_8}^{A_9}$  using the induction formula for characters. In fact, we have found that  $\underline{14a}\uparrow_{A_8}^{A_9} = \underline{42a} + \underline{84a}$  and thus

$$\chi(A_9|H_3) = \underline{1a} + \underline{8a} + \underline{42a} + \underline{84a}. \tag{7}$$

Using Equations (6) and (7), in Table 3 we list the values of  $\chi(A_9|H_2)$  and  $\chi(A_9|H_3)$  on those classes of  $A_9$ , where there are possible fusions from classes of  $H_2$  or  $H_3$ . Using the permutation characters of  $A_9$  on  $H_2$  and  $H_3$  together with the size of

$[g]_{A_9}$	1A	2A	2B	3A	3B	3C	4A	4B	5A	6A	6B	7A	9A	9B
$\chi(A_9 H_2)$	120	0	8	0	3	6	0	0	0	0	2	1	0	3
$\chi(A_9 H_3)$	135	15	7	0	0	9	3	3	0	0	1	2	0	0

Table 3: The values of the permutation characters of  $A_9$  on  $H_2$  and  $H_3$

centralizers, the fusions of  $H_2$  and  $H_3$  into  $A_9$  are completely determined. Let  $g_1, g_2, \dots, g_{18}$  be as in Table 1. To be consistent with the notations of [3], we rename the classes of  $H_2$  and  $H_3$  according to the fusions of these classes into classes of  $A_9$ . We list these fusions in Table 4.

Class of $H_2$	$\hookrightarrow$	Class of $G$	Class of $H_3$	$\hookrightarrow$	Class of $G$
$h_{11} = g_{121}$		1A	$1a = g_{131}$		1A
$h_{12} = g_{321}$		2B	$2a = g_{331}$		2B
$h_{13} = g_{521}$		3B	$2b = g_{231}$		2A
$h_{14} = g_{12,21}$		7A	$2c = g_{332}$		2B
$h_{15} = g_{14,22}$		9B	$3a = g_{631}$		3C
$h_{21} = g_{621}$		3C	$4a = g_{831}$		4B
$h_{22} = g_{11,21}$		6B	$4b = g_{731}$		4A
$h_{23} = g_{14,22}$		9B	$4c = g_{832}$		4B
$h_{31} = g_{622}$		3C	$6a = g_{11,31}$		6B
$h_{32} = g_{11,22}$		6B	$7a = g_{12,31}$		7A
$h_{33} = g_{14,23}$		9B	$7b = g_{12,32}$		7A

Table 4: The fusions of classes of  $H_2$  and  $H_3$  into classes of  $G$

### 6. Fischer matrices of $\overline{G} = 2_+^{1+8} \cdot A_9$

In this section, we calculate the Fischer matrices of  $\overline{G} = 2_+^{1+8} \cdot A_9$ . We have seen in Section 4 that the action of  $\overline{G} = 2_+^{1+8} \cdot A_9$  on  $\text{Irr}(2_+^{1+8})$  produces four orbits with representatives  $\theta_1, \theta_2, \theta_3$  and  $\theta_4$ , where  $|\theta_1^{\overline{G}}| = 1$ ,  $|\theta_2^{\overline{G}}| = 120$ ,  $|\theta_3^{\overline{G}}| = 135$  and  $|\theta_4^{\overline{G}}| = 1$ . By Proposition 2, it follows that  $c(g_1) = 4$ . We determine the identity Fischer matrix  $\mathcal{F}_1$  in detail as this matrix gives some information about the degrees of the irreducible characters of  $\overline{G}$ .

**Lemma 1.** *The identity Fischer matrix  $\mathcal{F}_1$  is an integral matrix.*

**Proof.** The result follows since the irreducible characters of  $N$  are integer-valued characters and the rows of  $\mathcal{F}_1$  are the orbit sums of characters of  $N$ .  $\square$

For  $\mathcal{F}_1$ , which is a  $4 \times 4$  matrix, let its columns correspond to  $g_{11} = 1_N$ ,  $g_{12} = \sigma$ ,  $g_{13}$  and  $g_{14}$  in this order, where  $\sigma$  is the central involution of  $N$  and  $g_{13}$  and  $g_{14}$  are elements of orders 2 and 4, respectively. Also, let its rows correspond to  $H_1 = A_9$ ,  $H_2 = PSL(2, 8):3$ ,  $H_3 = 2^3:GL(3, 2)$  and  $H_4 = A_9$  in this order (recall that  $\mathcal{F}_1$  corresponds to  $g_1 = 1_{A_9}$  and there is a fusion from each inertia factor and hence each of these inertia factors contributes with exactly one row to  $\mathcal{F}_1$ ). Next, we determine some of the entries of  $\mathcal{F}_1$ . The first row and column of  $\mathcal{F}_1$  are determined by properties of Fischer matrices given in Proposition 2. Recall that the first column is

$$[|\theta_1^{\overline{G}}| \deg(\theta_1) \quad |\theta_2^{\overline{G}}| \deg(\theta_2) \quad |\theta_3^{\overline{G}}| \deg(\theta_3) \quad |\theta_4^{\overline{G}}| \deg(\theta_4)]^T = [1 \quad 120 \quad 135 \quad 16]^T.$$

The last row of  $\mathcal{F}_1$  is

$$\left[ \sum_{\theta_i \in \theta_4^{\overline{G}}} \theta_i(g_{11}) \quad \sum_{\theta_i \in \theta_4^{\overline{G}}} \theta_i(g_{12}) \quad \sum_{\theta_i \in \theta_4^{\overline{G}}} \theta_i(g_{13}) \quad \sum_{\theta_i \in \theta_4^{\overline{G}}} \theta_i(g_{14}) \right] \\ = [\theta_4(g_{11}) \quad \theta_4(g_{12}) \quad \theta_4(g_{13}) \quad \theta_4(g_{14})] = [16 \quad -16 \quad 0 \quad 0].$$

Recall that the orbits  $\theta_1^{\overline{G}} = \{1_N\}$ ,  $\theta_2^{\overline{G}}$  and  $\theta_3^{\overline{G}}$  consist of irreducible characters of  $N$  that contain  $Z(N) = \{1_N, \sigma\}$  in their kernel, while  $\theta_4^{\overline{G}} = \{\theta_4\}$ , where  $\theta_4$  is the unique faithful irreducible character of  $N$  of degree 16, which does not contain  $Z(N)$  in its kernel. Since the second column of  $\mathcal{F}_1$  corresponds to  $\sigma$ , entries of this column will coincide with entries of the first column point-wise, except in the last row, where  $\theta_4(\sigma) = -16 \neq 16 = \theta_4(1_N)$ . Therefore, the second column of  $\mathcal{F}_1$  is  $[1 \quad 120 \quad 135 \quad -16]^T$ . Thus we have found the first two columns together with the first and last row of  $\mathcal{F}_1$ . So far, the identity Fischer matrix  $\mathcal{F}_1$  has the form

		$\mathcal{F}_1$			
		$g_{11}$	$g_{12}$	$g_{13}$	$g_{14}$
$o(g_{1j})$		1	2	2	4
$ C_{\overline{G}}(g_{1j}) $		92897280	92897280	344064	387072
$(k, m)$	$ C_{H_k}(g_{1km}) $				
(1, 1)	181440	1	1	1	1
(2, 1)	1512	120	120	$a_{13}^{(2,1)}$	$a_{14}^{(2,1)}$
(3, 1)	1344	135	135	$a_{13}^{(3,1)}$	$a_{14}^{(3,1)}$
(4, 1)	181440	16	-16	0	0
$m_{1j}$		1	1	270	240

For simplicity of notation, let  $s = a_{13}^{(2,1)}$ ,  $t = a_{14}^{(2,1)}$ ,  $u = a_{13}^{(3,1)}$  and  $v = a_{14}^{(3,1)}$ . Using the orthogonality relations given in Proposition 2, we get 10 equations in the unknowns  $s, t, u$  and  $v$ . In fact, the first four of the following five equations formed using the column orthogonality relations suffice to find the values of  $s, t, u$  and  $v$ .

$$\begin{aligned} s + u &= -1, & 9s^2 + 8u^2 &= 968, \\ t + v &= -1, & 9t^2 + 8v^2 &= 1224 \quad \text{and} \quad 9st + 8uv &= -1080. \end{aligned}$$

The unique integral solution of these simultaneous equations reveals

$$s (= a_{13}^{(2,1)}) = -8, \quad t (= a_{14}^{(2,1)}) = 8, \quad u (= a_{13}^{(3,1)}) = 7, \quad v (= a_{14}^{(3,1)}) = -9.$$

Hence the identity Fischer matrix  $\mathcal{F}_1$  will have the form:

		$\mathcal{F}_1$			
$g_1$		$g_{11}$	$g_{12}$	$g_{13}$	$g_{14}$
$o(g_{1j})$		1	2	2	4
$ C_{\overline{G}}(g_{1j}) $		92897280	92897280	344064	387072
$\hookrightarrow$ Th		1A	2A	2A	4A
$(k, m)$	$ C_{H_k}(g_{1km}) $				
(1, 1)	181440	1	1	1	1
(2, 1)	1512	120	120	-8	8
(3, 1)	1344	135	135	7	-9
(4, 1)	181440	16	-16	0	0
$m_{1j}$		1	1	270	240

We proceed to calculate all the Fischer matrices of  $\overline{G} = 2_+^{1+8} \cdot A_9$ . In what follows, by an  $\alpha^{-1}$ -regular Fischer matrix we mean a Fischer matrix that corresponds to an  $\alpha^{-1}$ -regular class of  $A_9$ . Also, a class of  $A_9$  is  $\alpha^{-1}$ -irregular if it is not an  $\alpha^{-1}$ -regular class. Since in our case  $\alpha \sim [2]$ , we only use the terms  $\alpha$ -regular class and  $\alpha$ -regular Fischer matrix.

**Lemma 2.** For every  $\alpha$ -regular Fischer matrix  $\mathcal{F}_i$  of size  $c(g_i)$ , the sum of the first  $c(g_i) - 1$  rows equals the (componentwise) square of the last row.

**Proof.** The proof is similar to the proof of Lemma 6 of [15]. □

**Lemma 3.** For every  $\alpha$ -regular Fischer matrix  $\mathcal{F}_i$ , we can order the  $g_{ij}$  for  $1 \leq j \leq c(g_i)$  so that the last row of  $\mathcal{F}_i$  is of the form  $[q_i \ -q_i \ 0 \ \cdots \ 0]$  with  $q_i$  a power of 2 and we may choose the  $g_{i2} = \sigma g_{i1}$ , where  $\sigma$  is the central involution in  $\overline{G}$ . Furthermore,

$$a_{i1}^{(k,m)} = a_{i2}^{(k,m)} = \frac{|C_{H_k}(g_{i11})|}{|C_{H_k}(g_{ikm})|} \quad \text{for } 1 \leq k \leq 3, 1 \leq m \leq c(g_{ik}). \quad (8)$$

**Proof.** The proof is similar to the proof of Lemma 7 of [15], except that the ordinary character  $\eta$  of degree  $2^{11}$  of  $2_+^{1+22} \cdot Co_2$  (in [15]) is replaced, in our Group  $\overline{G} = 2_+^{1+8} \cdot A_9$ , by a projective character  $\xi$ , with a factor set  $\alpha \sim [2]$  of degree 16. □

**Note 2.** The proof of Lemma 7 of [15] contained a very important piece of information in that the last row of every Fischer matrix of  $2_+^{1+22} \cdot Co_2$  is  $[\eta(g_{i1}) \ \eta(g_{i2}) \ \cdots \ \eta(g_{is_i})]$ , where  $s_i$  in the author's notation has the same meaning of  $c(g_i)$  in our notation. In our group  $\overline{G} = 2_+^{1+8} \cdot A_9$ , the last row of every  $\alpha$ -regular Fischer matrix  $\mathcal{F}_i$  is given by  $[\xi(g_{i1}) \ \xi(g_{i2}) \ \cdots \ \xi(g_{ic(g_i)})]$ . Also, the values of  $\xi$  on elements of  $[g_{ij}]_{\overline{G}}$ , where  $g_i$  is a representative of an  $\alpha$ -irregular conjugacy class, are zeros. That is,  $\xi(g_{ij}) = 0, \forall 1 \leq j \leq c(g_i), i \in \{2, 3, 7, 8, 10, 15\}$ .

**Note 3.** Observe that with Lemma 2, Equation (8) and Note 2 we know the first two columns and the last row of every  $\alpha$ -regular Fischer matrix  $\mathcal{F}_i$ .

As an example, we compute the Fischer matrix  $\mathcal{F}_6$ , where  $g_6 \in [3C]_{A_9}$ . From Table 1 we see that  $g_6$  produces 5 conjugacy classes in  $\overline{G}$ ; namely,  $g_{61}, g_{62}, g_{63}, g_{64}$  and  $g_{65}$  with respective orders 3, 6, 6, 12 and 12 and respective centralizer sizes 1728,

1728, 96, 288 and 288. Therefore,  $\mathcal{F}_6$  is a  $5 \times 5$  matrix. Note that  $|C_{A_9}(g_6)| = 54$ . Now from Table 4 we infer that there are two classes; namely,  $h_{21} = g_{621}$  and  $h_{31} = g_{622}$  of  $H_2 = PSL(2, 8):3$  that fuse to  $[g_6]_{A_9}$ . Note that  $|C_{H_2}(g_{621})| = |C_{H_2}(g_{622})| = 18$ . Also, from Table 4 we deduce that there is only one class; namely,  $3a = g_{631}$  of  $H_3 = 2^3:GL(3, 2)$  that fuse to  $[g_6]_{A_9}$ . Note that  $|C_{H_3}(g_{631})| = 6$ . Furthermore,  $g_6$  is an  $\alpha$ -regular class (see the ATLAS) and therefore there is a row of  $\mathcal{F}_6$  which corresponds to  $H_4 = A_9$ . Now, using Equation (1) we find that  $m_{61} = m_{62} = 16$ ,  $m_{63} = 288$  and  $m_{64} = m_{65} = 96$ . It follows from the above assertions that  $\mathcal{F}_6$  has the form

		$\mathcal{F}_6$				
$g_6$		$g_{61}$	$g_{62}$	$g_{63}$	$g_{64}$	$g_{65}$
$o(g_{6j})$		3	6	6	12	12
$ C_{\overline{G}}(g_{6j}) $		1728	1728	96	288	288
$(k, m)$	$ C_{H_k}(g_{6km}) $					
(1, 1)	54	$a_{61}^{(1,1)}$	$a_{62}^{(1,1)}$	$a_{63}^{(1,1)}$	$a_{64}^{(1,1)}$	$a_{65}^{(1,1)}$
(2, 1)	18	$a_{61}^{(2,1)}$	$a_{62}^{(2,1)}$	$a_{63}^{(2,1)}$	$a_{64}^{(2,1)}$	$a_{65}^{(2,1)}$
(2, 2)	18	$a_{61}^{(2,2)}$	$a_{62}^{(2,2)}$	$a_{63}^{(2,2)}$	$a_{64}^{(2,2)}$	$a_{65}^{(2,2)}$
(3, 1)	6	$a_{61}^{(3,1)}$	$a_{62}^{(3,1)}$	$a_{63}^{(3,1)}$	$a_{64}^{(3,1)}$	$a_{65}^{(3,1)}$
(4, 1)	54	$a_{61}^{(4,1)}$	$a_{62}^{(4,1)}$	$a_{63}^{(4,1)}$	$a_{64}^{(4,1)}$	$a_{65}^{(4,1)}$
$m_{6j}$		16	16	288	96	96

Now using Proposition 2(iii), we get that  $a_{61}^{(1,1)} = a_{62}^{(1,1)} = a_{63}^{(1,1)} = a_{64}^{(1,1)} = a_{65}^{(1,1)} = 1$ . Moreover, using Equation (8) we get

$$a_{61}^{(2,1)} = a_{62}^{(2,1)} = \frac{|C_{H_2}(g_6)|}{|C_{H_2}(g_{621})|} = \frac{54}{18} = 3, \quad a_{61}^{(2,2)} = a_{62}^{(2,2)} = \frac{|C_{H_2}(g_6)|}{|C_{H_2}(g_{622})|} = \frac{54}{18} = 3 \text{ and}$$

$$a_{61}^{(3,1)} = a_{62}^{(3,1)} = \frac{|C_{H_2}(g_6)|}{|C_{H_2}(g_{631})|} = \frac{54}{6} = 9.$$

Using Lemmas 2 and 3 we get

$$(a_{61}^{(4,1)})^2 = q_6^2 = \sum_{k=1}^3 \sum_{m=1}^{c(g_{6k})} a_{61}^{(k,m)} = 1 + 3 + 3 + 9 = 16.$$

Without loss of generality, we may assume that  $a_{61}^{(4,1)} = 4$  and therefore  $a_{62}^{(4,1)} = -q_6 = -4$ . Also, by Lemma 3 we have  $a_{63}^{(4,1)} = a_{64}^{(4,1)} = a_{65}^{(4,1)} = 0$ . So far, the Fischer matrix  $\mathcal{F}_6$  has the form

		$\mathcal{F}_6$				
$g_6$		$g_{61}$	$g_{62}$	$g_{63}$	$g_{64}$	$g_{65}$
$o(g_{6j})$		3	6	6	12	12
$ C_{\overline{G}}(g_{6j}) $		1728	1728	96	288	288
$(k, m)$	$ C_{H_k}(g_{6km}) $					
(1, 1)	54	1	1	1	1	1
(2, 1)	18	3	3	$a_{63}^{(2,1)}$	$a_{64}^{(2,1)}$	$a_{65}^{(2,1)}$
(2, 2)	18	3	3	$a_{63}^{(2,2)}$	$a_{64}^{(2,2)}$	$a_{65}^{(2,2)}$
(3, 1)	6	9	9	$a_{63}^{(3,1)}$	$a_{64}^{(3,1)}$	$a_{65}^{(3,1)}$
(4, 1)	54	4	-4	0	0	0
$m_{6j}$		16	16	288	96	96

From the columns and rows orthogonality relations (Proposition 2(vii) and (viii)) we obtain a set of 18 simultaneous equations. With consideration of Lemma 2, these equations give

$$1 + a_{63}^{(2,1)} + a_{63}^{(2,2)} + a_{63}^{(3,1)} = 0, \quad 1 + a_{64}^{(2,1)} + a_{64}^{(2,2)} + a_{64}^{(3,1)} = 0 \text{ and}$$

$$1 + a_{65}^{(2,1)} + a_{65}^{(2,2)} + a_{65}^{(3,1)} = 0.$$

We solve the equations using Maxima [11] and obtain

$$\begin{aligned} a_{63}^{(2,1)} &= a_{63}^{(2,2)} = a_{64}^{(2,2)} = a_{65}^{(2,1)} = -1, & a_{64}^{(2,1)} &= a_{65}^{(2,2)} = 3, \\ a_{64}^{(3,1)} &= a_{65}^{(3,1)} = -3 \text{ and } a_{63}^{(3,1)} &= 1. \end{aligned}$$

Thus the Fischer matrix  $\mathcal{F}_6$  has the form

		$\mathcal{F}_6$				
$g_6$		$g_{61}$	$g_{62}$	$g_{63}$	$g_{64}$	$g_{65}$
$o(g_{6j})$		3	6	6	12	12
$ C_{\overline{G}}(g_{6j}) $		1728	1728	96	288	288
$\hookrightarrow \text{Th}$		3A	6B	6B	12A	12B
$(k, m)$	$ C_{H_k}(g_{6km}) $					
(1, 1)	54	1	1	1	1	1
(2, 1)	18	3	3	-1	3	-1
(2, 2)	18	3	3	-1	-1	3
(3, 1)	6	9	9	1	-3	-3
(4, 1)	54	4	-4	0	0	0
$m_{6j}$		16	16	288	96	96

In Table 1, we supplied  $|C_{\overline{G}}(g_{ij})|$  and  $m_{ij}$ ,  $1 \leq i \leq 18$ ,  $1 \leq j \leq c(g_i)$ . Also, we have obtained the fusions of the inertia factors  $H_2$  and  $H_3$  into  $G \cong A_9$  (Table 4). Now, using the properties of the Fischer matrices given in Proposition 2, Lemmas 2 and 3 we plan to compute all Fischer matrices of  $\overline{G}$ .

Now for any  $\chi_n \in \text{Irr}(\text{Th})$ ,  $1 \leq n \leq 48$ , let  $\chi_n^{(k)}$  denote a character of  $\overline{H}_k$  that  $\chi_n^{(k)} \uparrow_{\overline{H}_k}^{\overline{G}}$  is a constituent or a zero of  $\chi_n \downarrow_{\overline{G}}^{\text{Th}}$ . Since we have the fusions of  $N = 2_+^{1+8}$  and  $Z(N) = \{1, \sigma\}$  into Th, we have

$$\begin{aligned} \deg(\chi_n^{(1)}) &= \left\langle \chi_n \downarrow_N^{\text{Th}}, \mathbf{1}_N \right\rangle_N, \\ \deg(\chi_n^{(4)}) &= \left\langle \chi_n \downarrow_{Z(N)}^{\text{Th}}, -\mathbf{1}_{Z(N)} \right\rangle_{Z(N)} = \frac{1}{2} (\deg(\chi_n) - \chi_n(\sigma)), \end{aligned}$$

where  $\mathbf{1}_N$  and  $-\mathbf{1}_{Z(N)}$  denote the trivial character of  $N$  and the non-trivial character of  $Z(N)$ , respectively. In Table 5, we compute  $\deg(\chi_n^{(1)})$  and  $\deg(\chi_n^{(4)})$  for  $2 \leq n \leq 48$ . We use this table to determine whether we are required to use projective characters of inertia factor groups. From Table 5 we can see that  $\deg(\chi_n^{(4)}) = k \times 16$ ,

$n$	$\deg(\chi_n)$	$\langle \chi_n \downarrow_{\overline{G}}^{\text{Th}}, \chi_n \downarrow_{\overline{G}}^{\text{Th}} \rangle$	$\deg(\chi_n^{(1)})$	$\deg(\chi_n^{(4)})$
2	248	2	0	$8 \times 16$
3	4123	7	35	$128 \times 16$
4	27000	16	120	$840 \times 16$
5	27000	16	120	$840 \times 16$
6	30628	23	28	$960 \times 16$
.	.	.	.	.
.	.	.	.	.
48	190373976	390135998	370008	$5949288 \times 16$

Table 5: Degrees of some particular characters of some inertia groups

for some  $k \in \mathbb{N}$ ,  $k \geq 2$ . That is,  $\deg(\chi_n^{(4)}) \neq 16$  for all  $n \in \{2, 3, \dots, 48\}$ . We deduce that  $\overline{H}_4 = 2_+^{1+8} \cdot A_9 = \overline{G}$  has no character of degree 16 that is extended from the faithful irreducible character  $\theta_4$  (of degree 16) of  $N$ . Thus we have to use a projective character table of  $H_4 = A_9$  to construct the character table of  $\overline{G}$ . Since the Schur multiplier of  $A_9$  is  $\mathbb{Z}_2$ , we have to consider the projective character table of  $A_9$ , which is available in the ATLAS.

**Corollary 1.** *The Schur multiplier of  $\overline{G}$  is non-trivial and has a factor set  $\alpha \sim [2]$ .*

Since we will use the  $\text{IrrProj}(A_9, 2^{-1}) = \text{IrrProj}(A_9, 2)$  to construct the character table of  $\overline{G}$ , there exists (see Remark 2 or Theorem 4.3.2 of [16]) a character  $\xi \in \text{IrrProj}(\overline{G}, 2)$  such that  $\xi \downarrow_N^{\overline{G}} = \theta_4$ . Furthermore,  $\xi^2 \downarrow_N^{\overline{G}} = \theta_4^2$  is the lift of the regular character of  $N/Z(N) \cong 2^8$  and hence the sum of the orbits of  $\theta_1, \theta_2$  and  $\theta_3$  under the action  $\overline{G}$ . We deduce that there are uniquely determined linear characters  $\psi_k \in \text{Irr}(\overline{H}_k), 1 \leq k \leq 3$ , such that

$$\xi^2 = \sum_{k=1}^3 \psi_k \uparrow_{\overline{H}_k}^{\overline{G}} \quad \text{and} \quad \psi_k \downarrow_N^{\overline{H}_k} = \theta_k \quad \text{for } k \in \{1, 2, 3\}. \tag{9}$$

Hence  $\theta_1, \theta_2$  and  $\theta_3$  are extendible to ordinary characters of their respective inertia groups. In summary, to construct the character table of  $\overline{G}$ , we need the ordinary character tables of  $H_1 = A_9, H_2 = PSL(2, 8):3, H_3 = 2^3:GL(3, 2)$  and the projective character table of  $H_4 = A_9$  with the factor set  $\alpha^{-1}, \alpha \sim [2]$ . The character tables of  $A_9$  and  $2 \cdot A_9$  are available in the ATLAS. The character table of  $H_3$  can be constructed easily with GAP, while the character table of  $H_2$  appears as Table 2 of this paper. Note that from Section 4 we get 52 irreducible characters of  $\overline{G}$ , which is exactly the number of conjugacy classes of  $\overline{G}$  given in Table 1.

**Note 4.** *Observe that the character table of  $2 \cdot A_9$  indicates that classes of  $A_9$  represented by  $g_2 \sim 2A, g_3 \sim 2B, g_7 \sim 4A, g_8 \sim 4B, g_{10} \sim 6A$  and  $g_{15} \sim 10A$  are  $\alpha^{-1}$ -irregular classes, where  $\alpha \sim [2]$ . This shows that the number of projective characters is exactly the number of  $\alpha^{-1}$ -regular classes of  $A_9$ .*

Now note that with the help of Note 3 the computations of Fischer matrices  $\mathcal{F}_i$ , correspond to  $g_i$ , for  $i \in \{1, 4, 5, 6, 9, 11, 12, 13, 14, 16, 17, 18\}$  are reduced to computations of  $1 \times 1, 2 \times 2$  and  $3 \times 3$  matrices. Also, note that all  $\alpha$ -irregular Fischer matrices are of small size (the size of every Fischer matrix (regular or irregular) is  $c(g_i)$  and we have  $c(g_i) \in \{1, 2, 3, 4\}, \forall i \in \{2, 3, 7, 8, 10, 15\}$ ). The columns and rows orthogonality relations are sufficient to compute both regular and irregular Fischer matrices of  $\overline{G}$ . Hence we have all the Fischer matrices of  $\overline{G}$ , which we list below.

$\mathcal{F}_1$					
$g_1$		$g_{11}$	$g_{12}$	$g_{13}$	$g_{14}$
$o(g_{1j})$		1	2	2	4
$ C_{\overline{G}}(g_{1j}) $		92897280	92897280	344064	387072
$\hookrightarrow$ Th		1A	2A	2A	4A
$(k, m)$	$ C_{H_k}(g_{1km}) $				
(1, 1)	181440	1	1	1	1
(2, 1)	1512	120	120	-8	8
(3, 1)	1344	135	135	7	-9
(4, 1)	181440	16	-16	0	0
$m_{1j}$		1	1	270	240

  

$\mathcal{F}_2$		
$g_2$		$g_{21}$ $g_{22}$
$o(g_{2j})$		4   4
$ C_{\overline{G}}(g_{2j}) $		7680   5120
$\hookrightarrow$ Th		4B   4B
$(k, m)$	$ C_{H_k}(g_{2km}) $	
(1, 1)	480	1   1
(2, 1)	32	15   -1
$m_{2j}$		32   480

$\mathcal{F}_3$					
$g_3$		$g_{31}$	$g_{32}$	$g_{33}$	$g_{34}$
$o(g_{3j})$		4	2	4	8
$ C_{\overline{G}}(g_{3j}) $		3072	3072	512	384
$\hookrightarrow$ Th		4A	2A	4B	8A
$(k, m)$	$ C_{H_k}(g_{3km}) $				
(1, 1)	192	1	1	1	1
(2, 1)	24	8	-8	0	0
(3, 1)	192	1	1	1	-1
(3, 2)	32	6	6	-2	0
$m_{3j}$		32	32	192	256

$\mathcal{F}_4$		
$g_4$	$g_{41}$	$g_{42}$
$o(g_{4j})$	3	6
$ C_{\overline{G}}(g_{4j}) $	2160	2160
$\hookrightarrow$ Th	$3C$	$6A$
$(k, m)$	$ CH_k(g_{4km}) $	
(1, 1)	1080	1
(4, 1)	1080	1
$m_{4j}$	256	256

$\mathcal{F}_5$			
$g_5$	$g_{51}$	$g_{52}$	$g_{53}$
$o(g_{5j})$	6	3	12
$ C_{\overline{G}}(g_{5j}) $	648	648	108
$\hookrightarrow$ Th	$6C$	$3B$	$12C$
$(k, m)$	$ CH_k(g_{5km}) $		
(1, 1)	81	1	1
(2, 1)	27	3	3
(4, 1)	81	2	-2
$m_{5j}$	64	64	384

$\mathcal{F}_6$					
$g_6$	$g_{61}$	$g_{62}$	$g_{63}$	$g_{64}$	$g_{65}$
$o(g_{6j})$	3	6	6	12	12
$ C_{\overline{G}}(g_{6j}) $	1728	1728	96	288	288
$\hookrightarrow$ Th	$3A$	$6B$	$6B$	$12A$	$12B$
$(k, m)$	$ CH_k(g_{6km}) $				
(1, 1)	54	1	1	1	1
(2, 1)	18	3	3	-1	3
(2, 2)	18	3	3	-1	3
(3, 1)	6	9	9	1	-3
(4, 1)	54	4	-4	0	0
$m_{6j}$		16	16	288	96

$\mathcal{F}_7$		
$g_{71}$	$g_{72}$	
$o(g_{7j})$	8	8
$ C_{\overline{G}}(g_{7j}) $	96	32
$\hookrightarrow$ Th	$8B$	$8B$
$(k, m)$	$ CH_k(g_{7km}) $	
(1, 1)	24	1
(3, 1)	8	3
$m_{7j}$	128	384

$\mathcal{F}_8$			
$g_8$	$g_{81}$	$g_{82}$	$g_{83}$
$o(g_{8j})$	8	4	8
$ C_{\overline{G}}(g_{8j}) $	64	64	32
$\hookrightarrow$ Th	$8A$	$4B$	$8B$
$(k, m)$	$ CH_k(g_{8km}) $		
(1, 1)	16	1	1
(3, 1)	16	1	-1
(3, 2)	8	2	-2
$m_{8j}$	128	128	256

$\mathcal{F}_9$		
$g_9$	$g_{91}$	$g_{92}$
$o(g_{9j})$	5	10
$ C_{\overline{G}}(g_{9j}) $	120	120
$\hookrightarrow$ Th	$5A$	$10A$
$(k, m)$	$ CH_k(g_{9km}) $	
(1, 1)	60	1
(4, 1)	60	1
$m_{9j}$	256	256

$\mathcal{F}_{10}$	
$g_{10}$	$g_{10,1}$
$o(g_{10j})$	12
$ C_{\overline{G}}(g_{10j}) $	24
$\hookrightarrow$ Th	$12D$
$(k, m)$	$ CH_k(g_{10km}) $
(1, 1)	24
$m_{10j}$	512

$\mathcal{F}_{11}$					
$g_{11}$	$g_{11,1}$	$g_{11,2}$	$g_{11,3}$	$g_{11,4}$	$g_{11,5}$
$o(g_{11j})$	12	12	6	24	24
$ C_{\overline{G}}(g_{11j}) $	48	48	24	24	24
$\hookrightarrow$ Th	$12B$	$12A$	$6B$	$24A$	$24B$
$(k, m)$	$ CH_k(g_{11km}) $				
(1, 1)	6	1	1	1	1
(2, 1)	6	1	1	-1	-1
(2, 2)	6	1	1	-1	-1
(3, 1)	6	1	1	-1	-1
(4, 1)	6	2	-2	0	0
$m_{11j}$	64	64	128	128	128

$\mathcal{F}_{12}$					
$g_{12}$	$g_{12,1}$	$g_{12,2}$	$g_{12,3}$	$g_{12,4}$	$g_{12,5}$
$o(g_{12j})$	7	14	14	14	28
$ C_{\overline{G}}(g_{12j}) $	56	56	28	28	28
$\hookrightarrow$ Th	$7A$	$14A$	$14A$	$14A$	$28A$
$(k, m)$	$ CH_k(g_{12km}) $				
(1, 1)	7	1	1	1	1
(2, 1)	7	1	1	-1	-1
(3, 1)	7	1	1	-1	-1
(3, 2)	7	1	1	1	-1
(4, 1)	7	2	-2	0	0
$m_{12j}$	64	64	128	128	128

$\mathcal{F}_{13}$		
$g_{13}$	$g_{13,1}$	$g_{13,2}$
$o(g_{13j})$	9	18
$ C_{\overline{G}}(g_{13j}) $	18	18
$\hookrightarrow$ Th	$9C$	$18B$
$(k, m)$	$ CH_k(g_{13km}) $	
(1, 1)	9	1
(4, 1)	9	1
$m_{13j}$	256	256

$\mathcal{F}_{14}$					
$g_{14}$	$g_{14,1}$	$g_{14,2}$	$g_{14,3}$	$g_{14,4}$	$g_{14,5}$
$o(g_{14j})$	18	9	36	36	36
$ C_{\overline{G}}(g_{14j}) $	72	72	36	36	36
$\hookrightarrow$ Th	18A	9A	36B	36A	36C
$(k, m)$	$ C_{H_k}(g_{14km}) $				
(1, 1)	9	1	1	1	1
(2, 1)	9	1	1	-1	-1
(2, 2)	9	1	1	1	-1
(2, 3)	9	1	1	-1	1
(4, 1)	9	2	-2	0	0
$m_{14j}$	64	64	128	128	128

  

$\mathcal{F}_{15}$		$\mathcal{F}_{16}$	
$g_{15}$	$g_{15,1}$	$g_{16}$	$g_{16,1}$ $g_{16,2}$
$o(g_{15j})$	20	$o(g_{16j})$	24 24
$ C_{\overline{G}}(g_{15j}) $	20	$ C_{\overline{G}}(g_{16j}) $	24 24
$\hookrightarrow$ Th	20A	$\hookrightarrow$ Th	24D 24C
$(k, m)$	$ C_{H_k}(g_{15km}) $	$(k, m)$	$ C_{H_k}(g_{16km}) $
(1, 1)	20	(1, 1)	12 1
		(4, 1)	12 1
$m_{15j}$	512	$m_{16j}$	256 256

  

$\mathcal{F}_{17}$			$\mathcal{F}_{18}$		
$g_{17}$	$g_{17,1}$	$g_{17,2}$	$g_{18}$	$g_{18,1}$	$g_{18,2}$
$o(g_{17j})$	15	30	$o(g_{18j})$	15	30
$ C_{\overline{G}}(g_{17j}) $	30	30	$ C_{\overline{G}}(g_{18j}) $	30	30
$\hookrightarrow$ Th	15A	30A	$\hookrightarrow$ Th	15B	30B
$(k, m)$	$ C_{H_k}(g_{17km}) $		$(k, m)$	$ C_{H_k}(g_{18km}) $	
(1, 1)	15	1	(1, 1)	15	1
(4, 1)	15	1	(4, 1)	15	-1
$m_{17j}$	256	256	$m_{18j}$	256	256

Using the Fischer matrices we can now compute the character table of  $\overline{G}$ . For example, the part  $\mathcal{K}_{62}\mathcal{F}_{62}$  of the character table of  $\overline{G}$  can be derived as follows: From Table 4, two conjugacy classes; namely,  $h_{21} = g_{621}$  and  $h_{31} = g_{622}$  of  $H_2 = PSL(2, 8):3$ , fuse to class  $[g_6]_{A_9} = 3C$ . Let  $\mathcal{K}_{62}$  be the fragment of the character table of  $H_2$  on these two classes, which can be extracted from Table 2. The Fischer matrix  $\mathcal{F}_6$  has two rows corresponding to the classes  $h_{21}$  and  $h_{31}$  of  $H_2$ . Let  $\mathcal{F}_{62}$  be the partial Fischer matrix of  $\mathcal{F}_6$  consisting of these two rows. Therefore the values of the irreducible characters of  $\overline{G}$  (corresponding to  $H_2$ ) on the classes  $g_{61}, g_{62}, g_{63}, g_{64}$  and  $g_{65}$  are given by

$$\mathcal{K}_{62}\mathcal{F}_{62} = \begin{pmatrix} 1 & 1 \\ A & \overline{A} \\ \overline{A} & A \\ 1 & 1 \\ A & \overline{A} \\ \overline{A} & A \\ 2 & 2 \\ 2A & 2\overline{A} \\ 2\overline{A} & 2A \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 3 & -1 & 3 & -1 \\ 3 & 3 & -1 & -1 & 3 \end{pmatrix} = \begin{pmatrix} g_{61} & g_{62} & g_{63} & g_{64} & g_{65} \\ 6 & 6 & -2 & 2 & 2 \\ -3 & -3 & 1 & B & \overline{B} \\ -3 & -3 & 1 & \overline{B} & B \\ 6 & 6 & -2 & 2 & 2 \\ -3 & -3 & 1 & B & \overline{B} \\ -3 & -3 & 1 & \overline{B} & B \\ 12 & 12 & -4 & 4 & 4 \\ -6 & -6 & 2 & 2B & 2\overline{B} \\ -6 & -6 & 2 & 2\overline{B} & 2B \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where  $A = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$  and  $B = -1 + 2i\sqrt{3}$ . Similarly one can obtain all the 72 blocks  $\mathcal{K}_{ik}\mathcal{F}_{ik}$ ,  $1 \leq i \leq 18$ ,  $1 \leq k \leq 4$  and hence the full character table of  $\overline{G}$  is computed, which is supplied in the format of Clifford-Fischer Theory as Table 11.10 of [3].

**Corollary 2.** *The group  $\overline{G}$  has 12 faithful irreducible characters.*



**Proof.** Note that  $a_{11}^{(k,m)} = a_{12}^{(k,m)}$  for  $1 \leq k \leq 3$ , while  $a_{11}^{(4,m)} = 16 \neq -16 = a_{12}^{(4,m)}$ . Thus  $\ker(\chi) \supset [g_{12}]_{\overline{G}}$ ,  $\forall \chi \in \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ , where  $\mathcal{K}_1, \mathcal{K}_2$  and  $\mathcal{K}_3$  are the blocks of irreducible characters of  $\overline{G}$  obtained through characters of the inertia factor groups  $H_1, H_2$  and  $H_3$ , respectively. Hence if  $\overline{G}$  has a faithful irreducible character  $\chi$ , then  $\chi \in \mathcal{K}_4 = \{\xi\beta_d \mid \beta_d \in \text{IrrProj}(A_9, 2)\}$ . Since the values of  $\xi$  are completely known (see Note 2) and since the character table of  $2 \cdot A_9$  is also known, we can see that for all  $d \in \{1, 2, \dots, 12\}$

$$\xi\beta_d(g_{ij}) \neq \deg(\xi\beta_d), \forall 1 \leq i \leq 18, 1 \leq j \leq c(g_i), (i, j) \neq (1, 1).$$

Thus  $\overline{G}$  has 12 faithful irreducible characters as claimed.  $\square$

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## References

- [1] F. ALI, *Fischer-Clifford theory for split and non-split group extensions*, PhD Thesis, University of Natal, Pietermaritzburg, 2001.
- [2] R. W. BARRACLOUGH, *Some calculations related to the monster group*, PhD Thesis, University of Birmingham, Birmingham, 2005.
- [3] A. B. M. BASHEER, *Clifford-Fischer theory applied to certain groups associated with symplectic, unitary and Thompson groups*, PhD Thesis, University of KwaZulu-Natal, Pietermaritzburg, 2012.
- [4] A. B. M. BASHEER, J. MOORI, *Fischer matrices of Dempwolff group  $2^5 \cdot GL(5, 2)$* , Int. J. Group Theory **1**(2012), 43–63.
- [5] A. B. M. BASHEER, J. MOORI, *On the non-split extension group  $2^6 \cdot Sp(6, 2)$* , Bull. Iran. Math. Soc. **39**(2013), 1189–1212.
- [6] W. BOSMA, J. J. CANNON, *Handbook of Magma functions*, Department of Mathematics, University of Sydney, Sydney, 1994.
- [7] J. H. CONWAY, R. T. CURTIS, S. P. NORTON, R. A. PARKER, R. A. WILSON, *Atlas of finite groups*, Clarendon Press, Oxford, 1985.
- [8] M. R. DARAFSHEH, A. IRANMANESH, *Computation of the character table of affine groups using Fischer matrices*, Cambridge University Press, Cambridge, 1995.
- [9] THE GAP GROUP, *GAP – Groups, Algorithms, and Programming, Version 4.4.10*, available at <http://www.gap-system.org>.
- [10] G. HAVAS, L. H. SOICHER, R. WILSON, *A presentation for the Thompson sporadic simple group*, in: *Groups and Computation III* (William M. Kantor and Ákos Seress, Eds.), Ohio State University Math. Res. Inst. Berlin, Walter de Gruyter, New York, 1999, 192–200.
- [11] MAXIMA, *A computer algebra system. Version 5.18.1*, available at <http://maxima.sourceforge.net>.
- [12] G. MICHLER, *Theory of finite simple groups*, New Mathematical Monographs: 8, Cambridge University Press, Cambridge, 2006.

- [13] J. MOORI, *On the Groups  $G^+$  and  $\bar{G}$  of the form  $2^{10}:M_{22}$  and  $2^{10}:\bar{M}_{22}$* , PhD Thesis, University of Birmingham, Birmingham, 1975.
- [14] J. MOORI, *On certain groups associated with the smallest Fischer group*, J. London Math. Soc. (2) **23**(1981), 61–67.
- [15] H. PAHLINGS, *The character table of  $2_+^{1+22}:Co_2$* , J. Algebra **315**(2007), 301–325.
- [16] T. T. SERETLO, *Fischer-Clifford matrices and character tables of certain groups associated with simple groups  $O_8^+(2)$ ,  $HS$  and  $Ly$* , PhD Thesis, University of KwaZulu-Natal, Pietermaitzburg, 2012.
- [17] N. S. WHITELEY, *Fischer Matrices and Character Tables of Group Extensions*, MSc Thesis, University of Natal, Pietermaritzburg, 1993.
- [18] R. A. WILSON ET AL., *Atlas of finite group representations*, available at <http://web.mat.bham.ac.uk/atlas/v2.0>.