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Multiplicity of solutions for p-Laplacian equation in $\mathbb{R}^N$ with indefinite weight*

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Abstract. In this article, we study the existence of infinitely many nontrivial solutions for a class of superlinear p-Laplacian equations

$$-\Delta_p u + V(x)|u|^{p-2}u = f(x,u),$$

where the primitive of the nonlinearity f is of subcritical growth near ∞ in u and the weight function V is allowed to be sign-changing. Our results extend the recent results of Zhang and Xu [Q. Y. Zhang, B. Xu, *Multiplicity of solutions for a class of semilinear Schrödinger equations with sign-changing potential*, J. Math. Anal. Appl **377**(2011), 834–840].

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1. Introduction

In this paper, we are concerned with the study of the p-Laplacian equation

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = f(x,u), \\ u \in W^{1,p}(\mathbb{R}^N), \end{cases} \tag{P}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator with $p > 1$. With the aid of variational methods, the existence and multiplicity of nontrivial solutions for problem (P) have been extensively investigated in the literature over the past several decades. (P) with a constant sign potential $V(x)$ was considered in [7]. More precisely, if the potential is periodic or bounded and of constant sign, the author proved the existence of ground states of (P). In [13], the authors considered (P) with a potential $V(x)$ that may change the sign.

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For $p = 2$, (P) turns into a kind of Schrödinger equation of the form

$$\begin{cases} -\Delta u + V(x)u = f(x,u) \\ u \in H^1(\mathbb{R}^N) \end{cases}, \tag{P_1}$$

which has been studied extensively. For more results, we refer the reader to [2, 9, 10, 11, 12, 14] (for constant sign potential) and [3, 15, 16, 17] (for sign-changing potential).

The quasilinear case $1 < p < N$ appears in a variety of applications, such as non-Newtonian fluids, image processing, nonlinear elasticity and reaction -diffusion; for more details see [4].

In the present paper, we will study the existence of infinitely many nontrivial solutions of (P) under suitable conditions. Precisely, we assume that the potential $V(x)$ and the nonlinearity $f(x,u)$ satisfy the following conditions:

H(V) :

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ is bounded from below.

(V_2) There exists $r > 0$ such that for any $b > 0$

$$\lim_{|y|\to\infty} \mu(\{x \in \mathbb{R}^N : V(x) \le b\} \cap B_r(y))) = 0,$$

where μ is the Lebesgue measure on $\mathbb{R}^N$.

H(f) :

(f_1) $f \in C(\mathbb{R}^N, \mathbb{R})$ and there exist constants $c_1 > 0$ and $p < \alpha < p^*$ such that

$$|f(x,u)| \le c_1(|u|^{p-1} + |u|^{\alpha-1}), \ \forall\, (x,u) \in \mathbb{R}^N \times \mathbb{R},$$

where p^* denotes the critical Sobolev exponent, i.e., $p^* = \frac{Np}{N-p}$ for $p < N$ and $p^* = +\infty$ for $p \ge N$.

(f_2) $F(x,0) = 0$, $F(x,u) \ge 0$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$, and

$$\lim_{|u|\to\infty} \frac{F(x,u)}{|u|^p} = +\infty$$

uniformly in $\mathbb{R}^N$, where $F(x,t) := \int_0^t f(x,s)ds$.

(f_3) There exists a constant $\theta \ge 1$ such that

$$\theta\mathcal{F}(x,u) \ge \mathcal{F}(x,su), \ \forall\, (x,u) \in \mathbb{R}^N \times \mathbb{R} \ \ s \in [0,1],$$

where $\mathcal{F}(x,t) := uf(x,u) - pF(x,u)$.

(f_4) $f(x,-u) = -f(x,u)$, $\forall\, (x,u) \in \mathbb{R}^N \times \mathbb{R}$.

Our main result reads as follows.

Theorem 1. *Suppose that* **H(V)** *and* **H(f)** *are satisfied. Then problem* (P) *possesses infinitely many nontrivial solutions.*

Remark 1. *Condition* $\mathbf{H(V)}$ *is due to [17]. The condition* (f_3) *was introduced in [8], and then was used by many authors, for example, [7]. In addition, we note that the usual condition* $f(x,u) = o(|u|^{p-1})$ *as* $|u| \to 0$ *is not needed in our Theorem 1.1. Let* V *be a zig-zag function with respect to* $|x|$ *defined by*

$$V(x) = n \sin\Big[\big(|x| - (n-1)\big)\pi\Big] - 1,\ n-1 \le |x| \le n, n \in \mathbb{N}.$$

It is easy to check that V *satisfy* $\mathbf{H(V)}$ *in our Theorem 1.1.*

Remark 2. *The following condition used in [1] is even stronger than* (V_2)*:* (V_3) $\mu(\{x \in \mathbb{R}^N : V(x) \le M\}) < +\infty$, $\forall M > 0$. *Note that for any* $M > 0$, (V_3) *implies* $\{x \in \mathbb{R}^N : V(x) \le M\}$ *is a bounded set. Thus for* $|y|$ *large,* $\{x \in \mathbb{R}^N : V(x) \le M\} \cap B_r(y) = \emptyset$. *Hence, Theorem 1 generalizes the main results of [13]. On the other hand, our main result extends a result for* (P) *when* $p = 2$ *given by Zhang and Xu in [17] to the general problem* (P) *where* $p > 1$. *The approach of this paper is similar to that of [17], but our proof is much more complex and more delicate than that of the main result in [17].*

2. Preliminaries

Firstly, by (V_1), there exists a constant $V_0 > 0$ such that $\overline{V}(x) := V(x) + V_0 \ge 1$ for all $x \in \mathbb{R}^N$. Let $\overline{f}(x,u) := f(x,u) + V_0|u|^{p-2}u$. Then it is easy to verify the following lemma.

Lemma 1. *Problem* (P) *is equivalent to the following problem*

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \overline{V}(x)|u|^{p-2}u = \overline{f}(x,u), \\ u \in W^{1,p}(\mathbb{R}^N). \end{cases} \tag{P_2}$$

Proof. Obviously, $\overline{V}$ satisfies $\mathbf{H(V)}$, and $\overline{f}(x,-u) = -\overline{f}(x,u)$. Furthermore, we choose $c_1 = c_0 + V_0$. It results in

$$\begin{aligned} |\overline{f}(x,u)| \le& |f(x,u)| + V_0|u|^{p-1} \\ \le& c_0(|u|^{p-1} + |u|^{\alpha-1}) + V_0|u|^{p-1} \\ \le& c_1(|u|^{p-1} + |u|^{\alpha-1}), \end{aligned}$$

so condition (f_1) holds.

We clearly have $\overline{F}(x,u) = F(x,u) + \frac{V_0}{p}|u|^p$, $\overline{F}(x,0) = 0$, $\overline{F}(x,u) \ge 0$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$, and

$$\lim_{|u|\to\infty} \frac{\overline{F}(x,u)}{|u|^p} = \lim_{|u|\to\infty} \frac{F(x,u)}{|u|^p} + \frac{V_0}{p} = +\infty.$$

So, condition (f_2) holds.

Meanwhile, we have

$$\begin{aligned}\overline{\mathcal{F}}(x,u) =&u\overline{f}(x,u)-p\overline{F}(x,u)\\ =&u(f(x,u)+V_0|u|^{p-1}u)-p(F(x,u)+\frac{V_0}{p}|u|^p)\\ =&uf(x,u)-pF(x,u)\\ =&\mathcal{F}(x,u).\end{aligned}$$

Thus, we have

$$\theta\overline{\mathcal{F}}(x,u) =\theta\mathcal{F}(x,u)\geq\mathcal{F}(x,su)=\overline{\mathcal{F}}(x,su).$$

So, condition (f_3) holds. □

Therefore, all the assumptions of Theorem 1 have been verified. Hence, we can assume without loss of generality that $V(x)\geq 1$ for all $x\in\mathbb{R}^N$. Hence, we consider the following function space

$$E=\Big\{u\in W^{1,p}(\mathbb{R}^N):\int_{\mathbb{R}}(|\nabla u|^p+V(x)|u|^p)dx<+\infty\Big\}$$

endowed with the norm

$$\|u\|=\Big[\int_{\mathbb{R}^N}(|\nabla u|^p+V(x)|u|^p)dx\Big]^{\frac{1}{p}}.$$

Clearly, E is a reflexive, separable Banach space. Evidently, E is continuously embedded into $W^{1,p}(\mathbb{R}^N)$ and hence continuously embedded into $L^q(\mathbb{R}^N)$ for $p\leq q\leq p^*$, i.e., there exists $c>0$ such that

$$|u|_p\leq c\|u\|,\ \forall\, u\in E, \tag{1}$$

where $|u|_q$ denotes the usual norm in $L^q(\mathbb{R}^N)$ for $p\leq q<p^*$. In fact, we further have the following lemma due to [6].

Lemma 2 (see [6], Theorem 2.1)**.** *The embedding from E into $L^q(\mathbb{R}^N)$ is compact for $p\leq q<p^*$.*

Lemma 3 (see [15], Lemma 2.2)**.** *Assume that $p_1,p_2>1$, $r\geq 1$ and $\Omega\subseteq\mathbb{R}^N$. Let $g(x,t)$ be a Carathéodory function on $\Omega\times\mathbb{R}$ and satisfy*

$$|g(x,t)|\leq a_1|t|^{\frac{p_1-1}{r}}+a_2|t|^{\frac{p_2-1}{r}},\ \forall\,(x,t)\in\Omega\times\mathbb{R},$$

where $a_1,a_2\geq 0$. If $u_n\to u$ in $L^{p_1}(\Omega)\cap L^{p_2}(\Omega)$, and $u_n\to u$ a.e. $x\in\Omega$, then

$$\lim_{n\to\infty}\int_\Omega|g(x,u_n)-g(x,u)|^r|u_n-u|dx.$$

To prove Theorem 1, we consider the C^1 functional $\varphi:E\to\mathbb{R}$ defined by

$$\varphi(u)=\frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p+V(x)|u|^p)dx-\lambda\int_{\mathbb{R}^N}F(x,u)dx. \tag{2}$$

It is clear that critical points of φ are weak solutions of (P). In order to find a critical point of this functional, we will use the following variant fountain theorem.

Let X be a Banach space with the norm $\|\cdot\|$ and $X = \overline{\oplus_{j\in\mathbb{N}} X_j}$ with $\dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \oplus_{j=1}^{k} X_j$ and $Z_k = \overline{\oplus_{j=k}^{\infty} X_j}$. Consider the following C^1 -functional $\varphi_\lambda(u) := A(u) - \lambda B(u), \lambda \in [1,2]$. Then we have

Theorem 2 (see [18], Theorem 2.1)**.** *Assume that the above functional φ_λ satisfies:*

(1) φ_λ maps bounded sets to bounded sets for $\lambda \in [1,2]$, and $\varphi_\lambda(-u) = \varphi_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times E$.

(2) $B(u) \geq 0$ for all $u \in E$, and $A(u) \to \infty$ as $\|u\| \to \infty$.

(3) There exist $r_k > \rho_k > 0$ such that

$$a_k(\lambda) := \inf_{u\in Z_k, \|u\|=\rho_k} \varphi_\lambda(u) > b_k(\lambda) := \inf_{u\in Y_k, \|u\|=r_k} \varphi_\lambda(u),\ \forall\, \lambda \in [1,2].$$

Then

$$a_k(\lambda) \leq \zeta_k(\lambda) := \inf_{\gamma\in\Gamma_k} \max_{u\in B_k} \varphi_\lambda(\gamma(u)),\ \forall\, \lambda \in [1,2],$$

where $B_k := \{u \in Y_k : \|u\| \leq r_k\}$ and $\Gamma_k := \{\gamma \in C(B_k, X) : \gamma \text{ is odd},\ \gamma|_{\partial B_k} = \mathrm{id}\}$. Moreover, for a.e. $\lambda \in [1,2]$, there exists a sequence $\{u_m^k(\lambda)\}_k^\infty$ such that

$$\sup_m \|u_m^k(\lambda)\| < \infty,\ \varphi_\lambda'(u_m^k(\lambda)) \to 0 \text{ and } \varphi_\lambda(u_m^k(\lambda)) \to \zeta_k(\lambda) \text{ as } m \to \infty.$$

3. Proof of Theorem 1.1

In order to apply Theorem 2 to prove our main result, we define the functionals A, B and φ_λ on our working space E by

$$\begin{aligned} A(u) &= \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + V(x)|u|^p)dx, \\ B(u) &= \int_{\mathbb{R}^N} F(x,u)dx, \\ \varphi_\lambda(u) &= A(u) - \lambda B(u), \end{aligned} \tag{3}$$

for all $u \in E$ and $\lambda \in [1,2]$. By condition (f_1), we have

$$|F(x,u)| \leq \frac{c_1}{p}|u|^p + \frac{c_1}{\alpha}|u|^\alpha \leq c_1(|u|^p + |u|^\alpha),\ \forall\,(x,u) \in \mathbb{R}^N \times \mathbb{R}. \tag{4}$$

Consequently, from $\mathbf{H(V)}$,(f_1) and proposition 2.3 in [13], we know that $\varphi_\lambda \in C^1(E, \mathbb{R})$ for all $\lambda \in [1,2]$. Moreover, we have

$$\begin{aligned} \langle A'(u), v\rangle &= \int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \nabla v + V(x)|u|^{p-2}uv)dx \\ \langle B'(u), v\rangle &= \int_{\mathbb{R}^N} f(x,u)v dx, \end{aligned} \tag{5}$$

for all $u, v \in E$. Furthermore, we know that (see [5]), $A' : E \to E^*$ is a mapping of (S_+)-type, i.e., if $u_n \rightharpoonup u$ in E and $\lim\sup_{n\to\infty} \langle A'(u_n), u_n - u\rangle \le 0$, then $u_n \to u$ in E.

As E is a separable and reflexive Banach space, there exist $\{e_n\}_{n=1}^{\infty} \subset E$ and $\{f_n\}_{n=1}^{\infty} \subset E^*$ such that

$$f_n(e_m) = \begin{cases} 1, \text{ if } n = m, \\ 0, \text{ if } n \neq m, \end{cases}$$

$E = \overline{\operatorname{span}\{e_n : n = 1, 2, \cdots, \}}$ and $E^* = \overline{\operatorname{span}\{f_n : n = 1, 2, \cdots, \}}$. For $k = 1, 2, \cdots$, denote $X_k = \operatorname{span}\{e_k\}$, $Y_k = \oplus_{j=1}^{k} X_j$ and $Z_k = \overline{\oplus_{j=k}^{\infty} X_j}$.

Claim 3.1. Let $l_p(k) := \sup_{u\in Z_k, \|u\|=1} |u|_p \to 0$ and $l_\alpha(k) := \sup_{u\in Z_k, \|u\|=1} |u|_\alpha \to 0$. Then $l_p(k) \to 0$ and $l_\alpha(k) \to 0$, as $k \to \infty$.

It is obvious that $l_p(k) \ge l_p(k+1) \ge 0$ and $l_\alpha(k) \ge l_\alpha(k+1) \ge 0$, so $l_p(k) \to l_p \ge 0$ and $l_\alpha(k) \to l_\alpha \ge 0$ as $k \to \infty$. For each $k = 1, 2, \cdots$, we take $u_k \in Z_k$, $\|u_k\| = 1$ such that

$$0 \le l_p(k) - |u_k|_p < \frac{1}{k} \text{ and } 0 \le l_\alpha(k) - |u_k|_\alpha < \frac{1}{k}.$$

As E is reflexive, $\{u_k\}$ has a weakly convergent subsequence, and without loss of generality, we can suppose that $u_k \rightharpoonup u$. We claim that $u = 0$. In fact, for any $f_m \in \{f_n : n = 1, 2, \cdots\}$, we have $f_m(u_k) = 0$ when $k > m$, so $f_m(u_k) \to 0$ as $k \to \infty$, this concludes $f_m(u) = 0$ for any $f_m \in \{f_n : n = 1, 2, \cdots\}$, therefore $u = 0$. Since E is compactly embedded to both $L^p(\mathbb{R}^N)$ and $L^\alpha(\mathbb{R}^N)$, we have that $u_k \to 0$ in $L^p(\mathbb{R}^N)$ and $u_k \to 0$ in $L^\alpha(\mathbb{R}^N)$. This implies $l_p(k) \to 0$ and $l_\alpha(k) \to 0$ as $k \to \infty$. This proves Claim 3.1.

Claim 3.2. There exists a positive integer k_1 and a sequence $\rho_k \to \infty$ as $k \to \infty$ such that

$$a_k(\lambda) := \inf_{u\in Z_k, \|u\|=\rho_k} \varphi_\lambda(u) > 0, \ \forall\, k \ge k_1. \tag{6}$$

By (3) and (4), we have

$$\begin{aligned} \varphi_\lambda(u) =& A(u) - \lambda B(u) \\ \ge& \frac{1}{p}\|u\|^p - 2\int_{\mathbb{R}^N} F(x, u)dx \\ \ge& \frac{1}{p}\|u\|^p - 2c_1(|u|_p^p + |u|_\alpha^\alpha) \\ \ge& \frac{1}{p}\|u\|^p - 2c_1(l_p^p(k)\|u\|^p + l_\alpha^\alpha(k)\|u\|^\alpha), \ \forall\, (\lambda, u) \in [1, 2] \times Z_k. \end{aligned} \tag{7}$$

By Claim 3.1, there exists a positive integer k_1 such that

$$2c_1 l_p^p(k) \le \frac{1}{2p}, \ \forall\, k \ge k_1. \tag{8}$$

For each $k \ge k_1$, choose

$$\rho_k := [8pc_1 l_\alpha^\alpha(k)]^{\frac{1}{p-\alpha}}. \tag{9}$$

Then $\rho_k \to +\infty$ as $k \to \infty$, since $\alpha > p$. Using (8) and (9) in (7), for each $k \geq k_1$, we have

$$a_k(\lambda) = \inf_{u\in Z_k, \|u\|=\rho_k} \varphi_\lambda(u) \geq \frac{1}{4p}\rho_k^p > 0,$$

which concludes the proof of Claim 3.2.

Claim 3.3. For each $k \geq k_1$, there exist $r_k > \rho_k$ such that

$$b_k(\lambda) := \inf_{u\in Y_k, \|u\|=r_k} \varphi_\lambda(u) < 0, \ \forall\, k \geq k_1. \tag{10}$$

We first prove that for any finite dimensional subspace $F \subset E$ there exists a constant $\varepsilon > 0$ such that

$$\text{meas}(\{x \in \mathbb{R}^N : |u(x)| \geq \varepsilon\|u\|\}) \geq \varepsilon, \ \forall\, u \in F\backslash\{0\}. \tag{11}$$

Suppose that this is not true. Then for any $n \in \mathbb{N}$ there exists $u_n \in F\backslash\{0\}$ such that

$$\text{meas}(\{x \in \mathbb{R}^N : |u_n(x)| \geq \frac{\|u_n\|}{n}\}) < \frac{1}{n}.$$

Set $v_n = \frac{u_n}{\|u_n\|} \in F$ for all $n \in \mathbb{N}$. Then $\|v_n\| = 1$ for all $n \in \mathbb{N}$, and

$$\text{meas}\Big(\Big\{x \in \mathbb{R}^N : |v_n(x)| \geq \frac{1}{n}\Big\}\Big) < \frac{1}{n}, \ \forall\, n \in \mathbb{N}. \tag{12}$$

Passing to a subsequence if necessary, we may assume $v_n \to v_0$ in E for some $v_0 \in F$, since F is of finite dimension. Evidently, $\|v_0\| = 1$. In view of Lemma 2 and the equivalence of any two norms on F, we have

$$\int_{\mathbb{R}^N} |v_n - v_0|^p dx \to 0, \text{ as } n \to \infty. \tag{13}$$

Since $v_0 \neq 0$, there exists a constant $\delta_0 > 0$ such that

$$\text{meas}(\Omega_0) := \text{meas}(\{x \in \mathbb{R}^N : |v_0(x)| \geq \delta_0\}) \geq \delta_0. \tag{14}$$

For $n \in \mathbb{N}$, let

$$\Omega_n := \Big\{x \in \mathbb{R}^N : |v_n(x)| < \frac{1}{n}\Big\} \text{ and } \Omega_n^c := \mathbb{R}^N\backslash\Omega_n = \Big\{x \in \mathbb{R}^N : |v_n(x)| \geq \frac{1}{n}\Big\}.$$

Then for n large enough, by (12) and (14), we have

$$\text{meas}(\Omega_0 \cap \Omega_n) \geq \text{meas}(\Omega_0) - \text{meas}(\Omega_n^c) \geq \delta_0 - \frac{1}{n} \geq \frac{\delta_0}{2}.$$

Consequently, for n large enough, there holds

$$\begin{aligned}
\int_{\mathbb{R}^N} |v_n - v_0|^p dx &\geq \int_{\Omega_0\cap\Omega_n} |v_n - v_0|^p dx \\
&\geq \int_{\Omega_0\cap\Omega_n} (|v_0| - |v_n|)^p dx \\
&\geq \Big(\delta_0 - \frac{1}{n}\Big)^p \text{meas}(\Omega_0 \cap \Omega_n) \\
&\geq \Big[\frac{\delta_0}{2}\Big]^{p+1}.
\end{aligned}$$

This contradicts (13). Therefore, (11) holds. Note that Y_k is finite dimensional for each $k \in \mathbb{N}$. Then by (11), for each $k \in \mathbb{N}$, there exists a constant $\varepsilon_k > 0$ such that

$$\operatorname{meas}(\{x \in \mathbb{R}^N : |u(x)| \geq \varepsilon_k \|u\|\}) \geq \varepsilon_k, \ \forall\, u \in Y_k \backslash \{0\}. \tag{15}$$

By (f_2), for each $k \in \mathbb{N}$, there exists a constant $\eta_k > 0$ such that

$$F(x,u) \geq \frac{|u|^p}{\varepsilon_k^{p+1}}, \ \forall\, x \in \mathbb{R}^N \text{ and } |u| \geq \eta_k. \tag{16}$$

Combining (3), (15), (16) and (f_2), for any $k \in \mathbb{N}$ and $\lambda \in [1,2]$, we have

$$\begin{aligned}
\varphi_\lambda(u) =& A(u) - \lambda B(u) \\
\leq & \frac{1}{p}\|u\|^p - \int_{\mathbb{R}^N} F(x,u)dx \\
\leq & \frac{1}{p}\|u\|^p - \int_{\{x \in \mathbb{R}^N : |u(x)| \geq \varepsilon_k \|u\|\}} F(x,u)dx \\
\leq & \frac{1}{p}\|u\|^p - \int_{\{x \in \mathbb{R}^N : |u(x)| \geq \varepsilon_k \|u\|\}} \frac{|u|^p}{\varepsilon_k^{p+1}}dx \\
\leq & \frac{1}{p}\|u\|^p - \varepsilon_k^p \|u\|^p \frac{\operatorname{meas}(\{x \in \mathbb{R}^N : |u(x)| \geq \varepsilon_k \|u\|\})}{\varepsilon_k^{p+1}} \\
\leq & \frac{1}{p}\|u\|^p - \|u\|^p \\
= & -\frac{p-1}{p}\|u\|^p,
\end{aligned} \tag{17}$$

for all $u \in Y_k$ with $\|u\| \geq \frac{\eta_k}{\varepsilon_k}$. Choose $r_k > \max\{\rho_k, \frac{\eta_k}{\varepsilon_k}\}$ for all $k \geq k_1$. Then (17) implies

$$b_k(\lambda) := \inf_{u \in Y_k, \|u\| = r_k} \varphi_\lambda(u) \leq -\frac{p-1}{p}\|r_k\|^p < 0, \ \forall\, k \geq k_1.$$

This proves Claim 3.3.

From (1), (3) and (4), it follows that φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. By (f_4), $\varphi_\lambda(-u) = \varphi_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times E$. Therefore, condition (1) of Theorem 2 holds. On the other hand, by (3) and (f_2), condition (2) of Theorem 2 holds. In addition, Claim 3.2 and Claim 3.3 imply that condition (3) of Theorem 2 holds. Therefore, by Theorem 2, we have that for each $k \geq k_1$ and $\lambda \in [1,2]$, there exists a sequence $\{u_m^k(\lambda)\}_{m=1}^\infty \subset E$ such that

$$\sup_m \|u_m^k(\lambda)\| < \infty, \ \varphi_\lambda'(u_m^k(\lambda)) \to 0 \text{ and } \varphi_\lambda(u_m^k(\lambda)) \to \zeta_k(\lambda), \text{ as } m \to \infty, \tag{18}$$

where

$$\zeta_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \varphi_\lambda(\gamma(u)), \ \forall\, \lambda \in [1,2],$$

with $B_k := \{u \in Y_k : \|u\| \leq r_k\}$ and $\Gamma_k := \{\gamma \in C(B_k, X) : \gamma \text{ is odd}, \ \gamma|_{\partial B_k} = \mathrm{id}\}$.

Furthermore, it follows from the proof of Claim 3.2 that $\zeta_k(\lambda) \in [c_k, d_k]$ for all $k \geq k_1$ and $\lambda \in [1,2]$, where $d_k := \max_{u \in B_k} \varphi_1(u)$ and $c_k := \frac{1}{4p}\rho_k^p$.

In view of (18), for each $k \geq k_1$, we can choose $\lambda_n \to 1$ (depending on k) and get the corresponding sequences satisfying

$$\sup_m \|u_m^k(\lambda_n)\| < \infty \text{ and } \varphi'_{\lambda_n}(u_m^k(\lambda_n)) \to 0, \text{ as } m \to \infty. \tag{19}$$

Claim 3.4. For each λ_n given above, the sequence $\{u_m^k(\lambda_n)\}_{m=1}^\infty$ has a strongly convergent subsequence. Furthermore, if we assume that $\lim_{m\to\infty} u_m^k(\lambda_n) = u_n^k \in E$ for all $n \in \mathbb{N}$ and $k \geq k_1$, then the sequence $\{u_n^k\}_{n=1}^\infty$ is bounded.

In fact, by (19), without loss of generality, we may assume

$$u_m^k(\lambda_n) \rightharpoonup u_n^k \text{ in } E, \text{ as } m \to \infty \tag{20}$$

for some $u_n^k \in E$. By Lemma 2, we have

$$\begin{aligned} u_m^k(\lambda_n) &\to u_n^k \text{ in } L^p(\mathbb{R}^N), \text{ as } m \to \infty; \\ u_m^k(\lambda_n) &\to u_n^k \text{ in } L^\alpha(\mathbb{R}^N), \text{ as } m \to \infty. \end{aligned} \tag{21}$$

From the choice of the sequence $\{u_m^k(\lambda_n)\}_{m=1}^\infty$, we have

$$|\langle \varphi'_{\lambda_n}(u_m^k(\lambda_n)), u_m^k(\lambda_n)\rangle| \leq \varepsilon_m, \ \varepsilon_m \downarrow 0, \tag{22}$$

which implies that

$$|\langle A'(u_m^k(\lambda_n)), u_m^k(\lambda_n) - u_n^k\rangle| - \lambda_n \int_{\mathbb{R}^N} f(x, u_m^k(\lambda_n))(u_m^k(\lambda_n) - u_n^k)dx \leq \varepsilon_m, \tag{23}$$

for all $m \geq 1$.

From Lemma 3, hypothesis $\mathbf{H(f)}(f_1)$ and (21), we obtain

$$\lim_{m\to\infty} \int_{\mathbb{R}^N} f(x, u_m^k(\lambda_n))(u_m^k(\lambda_n) - u_n^k)dx = 0. \tag{24}$$

Using (22) (23) and (24), we obtain

$$\limsup_{m\to\infty} \langle A'(u_m^k(\lambda_n)), u_m^k(\lambda_n) - u_n^k\rangle \leq 0. \tag{25}$$

Because A' is a mapping of (S_+)-type, from (25), it follows that

$$u_m^k(\lambda_n) \to u_n^k \text{ in } E, \text{ as } m \to \infty. \tag{26}$$

It remains to prove that the sequence $\{u_n^k\}_{n=1}^\infty$ is bounded. Suppose that this is not true. Then passing to a subsequence is necessary, we can assume that $\|u_n^k\| \to \infty$ as $n \to \infty$. Set $w_n^k = \frac{u_n^k}{\|u_n^k\|}$, $n \geq 1$.

We may assume that

$$\begin{aligned} &w_n^k \rightharpoonup w_0^k \text{ in } E; \\ &w_n^k \to w_0^k \text{ in } L^p(\mathbb{R}^N); \\ &w_n^k \to w_0^k \text{ in } L^\alpha(\mathbb{R}^N); \\ &w_n^k(x) \to w_0^k(x) \text{ a.e. on } \mathbb{R}^N. \end{aligned} \tag{27}$$

If $w_0^k = 0$, we choose a sequence $\{t_n\}_{n=1}^{\infty} \subset [0,1]$ such that

$$\varphi_{\lambda_n}(t_n w_n^k) = \max_{t\in[0,1]} \varphi_{\lambda_n}(t w_n^k). \tag{28}$$

For any $R > 0$, we set $\overline{w}_n^k = \sqrt[p]{pR}\, w_n^k$. Combining (3), (4), (27) and (f_2), we have

$$0 \le \int_{\mathbb{R}^N} F(x, \overline{w}_n^k)dx \le c_1 \int_{\mathbb{R}^N} (|\overline{w}_n^k|^p + |\overline{w}_n^k|^{\alpha})dx \to 0 \text{ as } n \to \infty. \tag{29}$$

Thus, for sufficiently large n, we have

$$\begin{aligned}
\varphi_{\lambda_n}(t_n u_n^k) \ge& \varphi_{\lambda_n}(\overline{w}_n^k) \\
=& \frac{1}{p}\|\overline{w}_n^k\|^p - \lambda_n \int_{\mathbb{R}^N} F(x, \overline{w}_n^k)dx \\
=& R - \lambda_n \int_{\mathbb{R}^N} F(x, \overline{w}_n^k)dx \\
\ge& \frac{R}{2},
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \varphi_{\lambda_n}(t_n u_n^k) = +\infty.$$

On the one hand, it is clear from (18) (19) and (26) that

$$\varphi'_{\lambda_n}(u_n^k) = 0,\ \varphi_{\lambda_n}(u_n^k) \in [c_k, d_k],\ \forall\, n \in \mathbb{N} \text{ and } k \ge k_1. \tag{30}$$

On the other hand, from (28) it follows that

$$0 = t_n \frac{d}{dt}\Big|_{t=t_n} \varphi_{\lambda_n}(t u_n^k) = \langle \varphi'_{\lambda_n}(t_n u_n^k), t_n u_n^k\rangle. \tag{31}$$

So from (3), (5), (28) and condition (f_3), we conclude that

$$\begin{aligned}
\varphi_{\lambda_n}(u_n^k) =& \varphi_{\lambda_n}(u_n^k) - \frac{1}{p}\langle \varphi'_{\lambda_n}(u_n^k), u_n^k\rangle \\
=& \frac{\lambda_n}{p} \int_{\mathbb{R}^N} \mathcal{F}(x, u_n^k)dx \\
\ge& \frac{\lambda_n}{\theta p} \int_{\mathbb{R}^N} \mathcal{F}(x, t_n u_n^k)dx \\
=& \frac{1}{\theta}\varphi_{\lambda_n}(t_n u_n^k) - \frac{1}{\theta p}\langle \varphi'_{\lambda_n}(t_n u_n^k), t_n u_n^k\rangle \\
=& \frac{1}{\theta}\varphi_{\lambda_n}(t_n u_n^k) \to +\infty \text{ as } n \to \infty.
\end{aligned}$$

This provides a contradiction to (30).

If $w_0^k \neq 0$, then $\Theta := \{x \in \mathbb{R}^N : w_0^k(x) \neq 0\}$ has a positive Lebesgue measure. Thus for $x \in \Theta$, we have $u_n^k(x) \to \infty$. This, together with (3), (27) and (f_2), implies

$$\begin{aligned}\frac{1}{p} - \frac{\varphi_{\lambda_n}(u_n^k)}{\|u_n^k\|^p} =& \lambda_n \int_{\mathbb{R}^N} \frac{F(x, u_n^k)}{\|u_n^k\|^p} dx \\ \geq& \lambda_n \int_{\Theta} |w_n^k|^p \frac{F(x, u_n^k)}{|u_n^k|^p} dx \to \infty, \text{ as } n \to \infty,\end{aligned}$$

which is a contradiction to (30) again. Therefore, the sequence $\{u_n^k\}_{n=1}^{\infty}$ is bounded. This proves that Claim 3.4 is true.

In view of Claim 3.4 and (30), for each $k \geq k_1$, using similar arguments in the proof of Claim 3.4, we can also prove that the sequence $\{u_n^k\}_{n=1}^{\infty}$ has a strong convergent subsequence with the limit u^k being just a critical point of $\varphi_1 = \varphi$. We already know that $\varphi(u^k) \in [c_k, d_k]$ for all $k \geq k_1$. Recalling that $c_k \to +\infty$ as $k \to \infty$, we deduce the existence of infinitely many nontrivial critical points of φ. Therefore, problem (P) possesses infinitely many nontrivial solutions. The proof of Theorem 1 is complete. □

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