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A REMARK ON CENTRALIZERS IN SEMIPRIME RINGS

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Abstract. The purpose of this paper is to prove the following result: Let $m \geq 1$, $n \geq 1$ be fixed integers and let R be a $(m+n+2)!$-torsion free semiprime ring with the identity element. Suppose there exists an additive mapping $T : R \to R$, such that $T(x^{m+n+1}) = x^m T(x) x^n$ holds for all $x \in R$. In this case T is a centralizer.

This research has been motivated by the work of Vukman [7]. Throughout, R will represent an associative ring with center $Z(R)$. A ring R is n-torsion free, where n is an integer, in case $nx = 0, x \in R$ implies $x = 0$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$. We shall use basis commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]\, z + y\, [x, z]$. Recall that R is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$, and is semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping $D : R \to R$, where R is an arbitrary ring, is called a derivation in case $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$. Obviously, any derivation is a Jordan derivation. The converse is in general not true. Herstein [5] proved that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein theorem can be found in [3]. Cusack [4] has extended Herstein theorem to 2-torsion free semiprime rings (see also [1] for an alternative proof). An additive mapping $T : R \to R$ is called a left (right) centralizer in case $T(xy) = T(x)y$ ($T(xy) = xT(y)$) holds for all $x, y \in R$. We follow Zalar [8] and call T a centralizer in case T is both left and right centralizer. In case R has the identity element $T : R \to R$ is left (right) centralizer iff T is of the form $T(x) = ax$ ($T(x) = xa$) for some fixed element $a \in R$. An additive mapping $T : R \to R$ is called a left (right) Jordan centralizer in case $T(x^2) = T(x)x$ ($T(x^2) = xT(x)$)

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holds for all $x \in R$. Following ideas from [1] Zalar [8] has proved that any left (right) Jordan centralizer on a 2-torsion free semiprime ring is a left (right) centralizer. Recently Vukman [6] has proved that in case $T : R \to R$ is an additive mapping, where R is a 2-torsion free semiprime ring, which satisfies the identity $2T(x^2) = T(x)x + xT(x)$ for all $x \in R$, then T is a centralizer. In [7] Vukman set the following conjecture:

Let R be a semiprime ring with suitable torsion restrictions. Suppose there exists an additive mapping $T : R \to R$, such that $T(x^{m+n+1}) = x^m T(x) x^n$ holds for all $x \in R$, where $m \geq 1$, $n \geq 1$ are some integers. In this case T is a centralizer.

The result below gives an affirmative answer to the question above in case R has the identity element.

THEOREM 1. *Let $m \geq 1$, $n \geq 1$ be fixed integers and let R be a $(m+n+2)!$-torsion free semiprime ring with the identity element. Suppose there exists an additive mapping $T : R \to R$, such that $T(x^{m+n+1}) = x^m T(x) x^n$ holds for all $x \in R$. In this case T is a centralizer.*

Let us see the background of the conjecture and the theorem above. An additive mapping $D : R \to R$, where R is an arbitrary ring, is called Jordan triple derivation, in case $D(xyx) = D(x)yx + xD(y)x + xyD(x)$ holds for all pairs $x, y \in R$. One can easily prove that any Jordan derivation on an arbitrary ring is a Jordan triple derivation (see [3]). Brešar [2] has proved that any Jordan triple derivation on a 2-torsion free semiprime ring is a derivation. This result inspired Vukman [7] to prove the following result: Let $T : R \to R$ be an additive mapping, where R is a 2-torsion free semiprime ring. Suppose that

$$T(xyx) = xT(y)x \tag{1}$$

holds for all $x \in R$. In this case T is a centralizer. For $y = x$ the identity (1) reduces to

$$T(x^3) = xT(x)x, \qquad x \in R. \tag{2}$$

The question arises whether the identity (2) on a 2-torsion free semiprime ring implies that T is a centralizer. Vukman [7] proved that the answer to this question is affirmative in case R has the identity element. The identity (2) leads to the conjecture above.

PROOF OF THEOREM 1. We have the relation

$$T(x^{m+n+1}) = x^m T(x) x^n, \qquad x \in R. \tag{3}$$

Replacing in the above relation $x+e$ for x, where e denotes the identity element, we obtain

$$\sum_{i=0}^{m+n+1}\binom{m+n+1}{i}T\left(x^{m+n+1-i}\right)=$$

$$(4)\qquad =\left(\sum_{i=0}^{m}\binom{m}{i}x^{m-i}\right)\left(T\left(x\right)+a\right)\left(\sum_{i=0}^{n}\binom{n}{i}x^{n-i}\right),\qquad x\in R,$$

where a stands for $T\left(e\right)$. Using (3) and rearranging the equation (4) in sense of collecting together terms involving equal number of factors of e we obtain:

$$\begin{aligned}
&\binom{m+n+1}{1}T\left(x^{m+n}e\right)-\binom{m}{0}\binom{n}{1}x^{m}T\left(x\right)x^{n-1}e-\binom{m}{0}\binom{n}{0}x^{m}ax^{n}\\
&-\binom{m}{1}\binom{n}{0}x^{m-1}T\left(x\right)x^{n}e\\
&+\binom{m+n+1}{2}T\left(x^{m+n-1}e^{2}\right)-\binom{m}{0}\binom{n}{2}x^{m}T\left(x\right)x^{n-2}e^{2}\\
&-\binom{m}{0}\binom{n}{1}x^{m}ax^{n-1}e-\binom{m}{1}\binom{n}{1}x^{m-1}T\left(x\right)x^{n-1}e^{2}\\
&-\binom{m}{1}\binom{n}{0}x^{m-1}ax^{n}e-\binom{m}{2}\binom{n}{0}x^{m-2}T\left(x\right)x^{n}e^{2}\\
&+\cdots+\\
&+\binom{m+n+1}{m+n}T\left(xe^{m+n}\right)-\binom{m}{m-1}\binom{n}{n}xae^{m+n-1}\\
&-\binom{m}{m}\binom{n}{n}T\left(x\right)e^{m+n}-\binom{m}{m}\binom{n}{n-1}axe^{m+n-1}=0,\qquad x\in R,
\end{aligned}$$

or shortly

$$(5)\qquad \sum_{i=1}^{m+n}f_{i}\left(x,e\right)=0,\qquad x\in R,$$

where $f_i\left(x,e\right)$ stands for the expression of terms involving i factors of e.

Replacing e by $2e,3e,\ldots,(m+n)\,e$ in turn in the equation (5), and expressing the resulting system of $m+n$ homogeneous equations, we see that the coefficient matrix of the system is a van der Monde matrix

$$\begin{bmatrix}1 & 1 & \cdots & 1\\ 2 & 2^{2} & \cdots & 2^{m+n}\\ \vdots & \vdots & \vdots & \vdots\\ m+n & (m+n)^{2} & \cdots & (m+n)^{m+n}\end{bmatrix}.$$

Since the determinant of the matrix is different from zero, it follows that the system has only a trivial solution.

In particular,

$$\begin{aligned} f_{m+n-1}(x,e) &= \binom{m+n+1}{m+n-1}T\left(x^2\right) - \binom{m}{m-2}x^2a - \binom{n}{n-2}ax^2 \\ &\quad - \binom{m}{m-1}xT(x) - \binom{n}{n-1}T(x)x \\ &\quad - \binom{m}{m-1}\binom{n}{n-1}xax \\ &= 0, \qquad x \in R \end{aligned}$$

and

$$\begin{aligned} f_{m+n}(x,e) &= \binom{m+n+1}{m+n}T(x) - \binom{m}{m-1}xa - \binom{n}{n-1}ax \\ &\quad - \binom{m}{m}\binom{n}{n}T(x) \\ &= 0, \qquad x \in R. \end{aligned}$$

Since R is a $(m+n+2)!$-torsion free ring, the above equations reduce to

$$\begin{aligned} (m+n+1)(m+n)T\left(x^2\right) &= m(m-1)x^2a + n(n-1)ax^2 + 2mnxax \\ &\quad +2mxT(x) + 2nT(x)x, \qquad x \in R \end{aligned} \tag{6}$$

and

$$(m+n)T(x) = mxa + nax, \qquad x \in R, \tag{7}$$

respectively.

According to (7) one obtains the relation

$$(m+n)T\left(x^2\right) = mx^2a + nax^2, \qquad x \in R.$$

Using the above connection one can replace the expression $(m+n)T\left(x^2\right)$ with $mx^2a + nax^2$ in the relation (6). Thus we have

$$\begin{aligned} (m+n+1)\left(mx^2a + nax^2\right) &= m(m-1)x^2a + n(n-1)ax^2 \\ &\quad +2mnxax + 2mxT(x) + 2nT(x)x \end{aligned}$$

From the above relation we obtain

$$\begin{aligned} &(2m+mn)x^2a + (2n+mn)ax^2 - 2mnxax \\ &-2mxT(x) - 2nT(x)x = 0, \qquad x \in R. \end{aligned} \tag{8}$$

Rearranging the above relation gives

$$\begin{aligned} &2m\left(x^2a - xT(x)\right) + 2n\left(ax^2 - T(x)x\right) \\ &+mn\left(x^2a + ax^2 - 2xax\right) = 0, \qquad x \in R. \end{aligned} \tag{9}$$

Left and right multiplication of the relation (7) by x gives

$$(m+n)\,xT(x) = mx^2a + nxax, \qquad x \in R \tag{10}$$

and

$$(m+n)\,T(x)\,x = mxax + nax^2, \qquad x \in R, \tag{11}$$

respectively.

Multiplication of the relation (9) by $(m+n)\,e$ gives

$$\begin{aligned} &2m\big((m+n)\,x^2a - (m+n)\,xT(x)\big) + 2n\big((m+n)\,ax^2 - (m+n)\,T(x)\,x\big) \\ &+mn\,(m+n)\big(x^2a + ax^2 - 2xax\big) = 0, \qquad x \in R. \end{aligned} \tag{12}$$

Using (10) and (11) in the relation (12) one obtains

$$\begin{aligned} 0 &= 2m\big((m+n)\,x^2a - mx^2a - nxax\big) + 2n\big((m+n)\,ax^2 - mxax - nax^2\big) \\ &\quad +mn\,(m+n)\big(x^2a + ax^2 - 2xax\big) \\ &= \big(2m\,(m+n) - 2m^2 + mn\,(m+n)\big)\,x^2a \\ &\quad + \big(2n\,(m+n) - 2n^2 + mn\,(m+n)\big)\,ax^2 - (4mn + 2mn\,(m+n))\,xax \\ &= mn\,(m+n+2)\,x^2a + mn\,(m+n+2)\,ax^2 \\ &\quad -2mn\,(m+n+2)\,xax, \qquad x \in R. \end{aligned}$$

Since R is a $(m+n+2)!$-torsion free ring we obtain

$$x^2a + ax^2 - 2xax = 0, \qquad x \in R. \tag{13}$$

The above relation can be written in the form

$$[[a,x],x] = 0, \qquad x \in R. \tag{14}$$

The rest of the proof goes through in the same way as in the end of the proof of Theorem 2 in [7], but we proceed for the sake of completeness. Putting $x+y$ for x in the above relation we obtain

$$[[a,x],y] + [[a,y],x] = 0, \qquad x, y \in R. \tag{15}$$

Putting xy for y in relation (15) we obtain because of (14) and (15):

$$\begin{aligned} 0 &= [[a,x],xy] + [[a,xy],x] \\ &= [[a,x],x]\,y + x\,[[a,x],y] + [[a,x]\,y + x\,[a,y],x] \\ &= x\,[[a,x],y] + [[a,x],x]\,y + [a,x]\,[y,x] + x\,[[a,y],x] \\ &= [a,x]\,[y,x], \qquad x, y \in R. \end{aligned}$$

Thus we have

$$[a,x]\,[y,x] = 0, \qquad x, y \in R.$$

The substitution ya for y in the above relation gives

$$[a,x]\,y\,[a,x] = 0, \qquad x, y \in R. \tag{16}$$

Let us point out that so far we have not used the assumption that R is semiprime. Since R is semiprime, it follows from the relation (16) that $[a, x] = 0$, $x \in R$. In other words, $a \in Z(R)$, which reduces the relation (7) to $T(x) = ax$, $x \in R$. The proof of the theorem is complete. □

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