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DEGENERATE EISENSTEIN SERIES FOR SYMPLECTIC GROUPS

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Abstract. In this paper we determine the poles (in the right half-plane) with their order of the degenerate Eisenstein series attached to the representations induced from a character of the Siegel maximal parabolic subgroup of a symplectic group. We explicitly determine the image of the Eisenstein series and thus determine an automorphic realization of certain irreducible global representations of $Sp_{2n}(\mathbb{A}_{\mathbb{Q}})$.

1. Introduction

The Eisenstein series are an important tool in the theory of automorphic forms, from the work of Selberg ([29, 30]) till the most recent work of Arthur ([1]). They were, beside theta-series, one of the few tools to provide explicit realization of the global representations in the spaces of automorphic forms. They were used in the construction of L–functions on classical groups ([10,18]), and they play a prominent role in Arthur's work on the trace formula in the last thirty years ([1–5]). Also, they were used to construct explicitly, without using trace formula, some representations with the prescribed Arthur parameters, or to resolve some local issues about the unitarity of representations; we mention these instances from the work of Speh and Tadić ([33, 35]) to the work of Badulescu and Renard ([7]) and Muić ([25, 26]), just to name some. The classical Eisenstein series on the real reductive groups have a great importance in number theory, so applying the results from the global to the classical setting really highlights the use of representation-theoretic methods in number theory.

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The Eisenstein series for the symplectic groups were treated in many papers from Piatetski-Shapiro and Rallis ([10]), to Kim ([16]) and Kim and Shahidi ([17]). The explicit results in these papers rely on heavy combinatorial computations regarding residues along the singular hyperplanes. In this paper, we avoid this approach by analyzing in detail all the features appearing in the constant term of Eisenstein series. We analyze the composition series of the induced representations at local places, and describe the images of the local intertwining operators. Also, we have done some very non-trivial combinatorial work to handle certain sums of normalizing factor

In our paper we treat the Siegel case (in the right half plane) of the degenerate Eisenstein series for a symplectic group, as this is the basic step toward the general case of the degenerate Eisenstein series for a symplectic group. The Siegel case for the symplectic group was treated in several occasions before; let us mention the classical work of Kudla and Rallis ([20]). The location of poles we obtained is thus not new, but the technique we employed differs, as noted above, form the techniques of Kudla and Rallis. The results in [20] are formulated in a way to fit in with the theta correspondence and depend on it. Our approach gives us a direct description of the image of the Eisenstein series, thus the automorphic realization of irreducible global representations. We believe that our method, both the combinatorial calculations and proofs of the holomorphy of the local intertwining operators, allows a direct generalization to the non-Siegel case. To treat the non-Siegel case, one needs more information about the composition series at the archimedean places. That is still unavailable and we plan to address this problem in future. We plan to apply the results of this paper in number theory, by attaching classical Eisenstein series to the automorphic one. This is not a straightforward issue (cf. the fourth section of [14]).

One of the most interesting results in our approach is a certain circularity of normalizing factors, which, we hope, can be generalized to the non-Siegel case. Namely, we look at the points in the right half plane where the normalizing factors appearing in the expression for the constant term of the Eisenstein series have a pole of very high order. The contributions of these normalizing factors, due to certain circularity, cancel each other, giving, at the end, only a pole of order at most one. Not only that, but we believe that this is a feature shared with all the classical groups. Also, we prove the holomorphy of the relevant local intertwining operators in a way that allows generalizations to the non-Siegel cases, and to the analogous situations with other classical groups as well.

We give here the main result in the most interesting and difficult case of $\chi^2 = 1$ and $\frac{n-1}{2} - s \in \mathbb{Z}_{\geq 0}$. The Eisenstein series act on the representation

$$I(s) = \mathrm{Ind}_{P_n(\mathbb{A})}^{Sp_{2n}(\mathbb{A})}(\chi|\det|^s 1_{GL_n(\mathbb{A}_\mathbb{Q})}), \tag{1.1}$$

where P_n is the Siegel parabolic subgroup of $Sp_{2n}(\mathbb{A}_\mathbb{Q})$.

Theorem 1.1. *Assume $\chi^2 = 1$ with $\chi_\infty = 1$ and $n \geq 3$ with $\frac{n-1}{2} - s \in \mathbb{Z}_{\geq 0}$. Then the Eisenstein series (2.3) with Λ_s as in (3.1) has a pole of order one on $I(s)$.*

1. *Assume $0 < s < \frac{n-1}{2}$, $f = \otimes_{p\leq\infty} f_p \in I(s)$ and let S be a finite set of finite places, and for $p \notin S$, let f_p be the normalized spherical vector. For $S_1 \subset S$, we pick $f_p \in \widehat{\sigma_{1,p}}$ and for $p \in S_2 := S \setminus S_1$, we take $f_p \in \widehat{\sigma_{2,p}}$. Then, for such f, (2.3) is holomorphic if $|S_2|$ is odd, and if $|S_2|$ is even it has a pole of order one. In the latter case (2.3) gives an automorphic realization (in the space of automorphic forms $\mathcal{A}(Sp_{2n}(\mathbb{Q}) \setminus Sp_{2n}(\mathbb{A}))$ of a global irreducible representation having a local representation $\widehat{\sigma_{2,p}}$ on the places from S_2 and $\widehat{\sigma_{1,p}}$ as a local component elsewhere on finite places ($\widehat{\sigma_{1,p}}$ is spherical for $p \notin S$, $p < \infty$).*
2. *Assume $s = \frac{n-1}{2}$. If $\chi = 1$, then for any choice S of a finite set of finite places such that if $f = \otimes_{p\leq\infty} f_p$, with f_p normalized spherical for $f_p \notin S$, the Eisenstein series has a pole of the first order. Thus, (2.3) gives an automorphic realization of the unique spherical (global) subrepresentation of $\mathrm{Ind}_{B(\mathbb{A})}^{Sp_{2n}(\mathbb{A})}(\Lambda_{-s})$, having local components isomorphic to $L(\nu_p^{n-1}, \ldots, \nu_p^1; \nu_p^0 \rtimes 1)$. If $\chi \neq 1$ we have the following. Assume S is a finite set of finite places such that f_p is normalized spherical for $f_p \notin S$. We pick a subset $S_2 \subset S$ such that for $p \in S_2$, $\chi_p \neq 1$ and f_p belongs to $\widehat{\sigma_{2,p}}$, and for $p \in S \setminus S_2$ either $\chi_p = 1$ and f_p belongs to the spherical quotient $L(\nu_p^{n-1}, \ldots, \nu_p^1; \nu_p^0 \rtimes 1)$ or $\chi_p \neq 1$ and f_p belongs to $\widehat{\sigma_{1,p}}$. Then the Eisenstein series has a pole of order one if $|S_2|$ is even, and is holomorphic if $|S_2|$ is odd, so in the former case, (2.3) gives an automorphic realization of an irreducible global representation we have just described.*

Here, for $p < \infty$, if $\chi_p = 1$ then

$$\widehat{\sigma_{1,p}} = L(\nu^{\frac{n-1}{2}+s}, \nu^{\frac{n-3}{2}+s}, \ldots, \nu^{\frac{n+1}{2}-s}, \nu^{\frac{n-1}{2}-s}, \nu^{\frac{n-1}{2}-s}, \ldots, \nu^1, \nu^1; \nu^0 \rtimes 1),$$

and

$$\widehat{\sigma_{2,p}} = L(\nu^{\frac{n-1}{2}+s}, \nu^{\frac{n-3}{2}+s}, \ldots, \nu^{\frac{n+1}{2}-s}, \nu^{\frac{n-1}{2}-s}, \nu^{\frac{n-1}{2}-s}, \ldots, \nu^2, \nu^2, \nu^1; T_2'').$$

If $\chi_p \neq 1$ then

$$\widehat{\sigma_{i,p}} = L(\nu^{\frac{n-1}{2}+s}, \nu^{\frac{n-3}{2}+s}, \ldots, \nu^{\frac{n+1}{2}-s}, \nu^{\frac{n-1}{2}-s}, \nu^{\frac{n-1}{2}-s}, \ldots, \nu^1, \nu^1; T_i^0),$$

for $i = 1, 2$. The tempered representations T_2'', T_1^0, T_2^0 are described in Lemma 4.7 and Corollary 4.8.

Note that an assumption in the theorem above is that f_∞ is the normalized spherical. This assumption is unrelated with the facts that only poles of the at most order one appear and that the relevant intertwining operators appearing in the constant term of Eisenstein series are holomorphic and non-zero. We placed this assumption only in order to be able to explicitly express the image of the Eisenstein series since there is less information about the images of intertwining operators on the archimedean places than about the ones on non-archimedean places. Also, this is partly the reason to use $\mathbb{Q}$ instead of more general number field. The other reason is a simpler use of global results in the future application in number theory, as explained above.

We now briefly describe the content of the paper. In the second section we recall the groups we work with, the Weyl group and its description in the case of a symplectic group. Then, we recall the (degenerate) Eisenstein series attached to the representation (2.2) and its constant term.

In the third section we specialize to the Siegel case and analyze normalization factors that occur in the expression for the constant term of Eisenstein series. We note the cases in which these normalization factors have possible poles of higher order. Then, in the subsection 3.1 we give an expression for certain sums of these normalizing factors, which turn out to have poles of much smaller order than each normalizing factor separately. It will become clear in the fifth section why these sums occur.

In the fourth section we give composition series of the local induced representations of type (1.1) in a form which we use later. The lengths of these composition series were known before ([12, 19]).

In the fifth section we prove that all intertwining operators appearing in the expression for the constant term of Eisenstein series are holomorphic, and prove that (normalized) intertwining operator attached to the longest element of the Weyl group is non-zero and also explicitly describe its image.

In the last section we prove that the intertwining operators belonging to the so called orbits have very similar actions, justifying grouping the normalizing factors in the sums in the third section. We also explicitly describe the image of the Eisenstein series.

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2. Preliminaries

For $n \in \mathbb{Z}_{\geq 1}$, we define J_n as a $n \times n$ matrix with $1'$s on the opposite diagonal, and zeros everywhere else. We realize the group Sp_{2n} as a matrix group in the following way:

$$Sp_{2n}(F) = \left\{ g \in GL_{2n}(F) : g^t \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix} g = \begin{bmatrix} 0 & J_n \\ -J_n & 0 \end{bmatrix} \right\}.$$

For us $F \in \{\mathbb{Q}, \mathbb{Q}_p, \mathbb{R}, \mathbb{A}\}$, where $\mathbb{A}$ is the ring of adeles of $\mathbb{Q}$. Let $K_p = Sp_{2n}(\mathbb{Z}_p)$ for $p < \infty$ and let K_∞ be the fixed point set of a Cartan involution (e.g. transposed inverse) on $Sp_{2n}(\mathbb{R})$. It is well-known that $K_\infty \cong U(n)$. We note that $K_p,\ p \leq \infty$ are maximal compact subgroups in the corresponding groups. Denote $K = \prod_{p \leq \infty} K_p$.

The upper triangular matrices in Sp_{2n} form a Borel subgroup B, which we fix. The standard parabolic subgroups are those containing this Borel subgroup. The diagonal matrices in the Borel subgroup form a maximal torus, which we denote by T. Thus

$$T(F) = \{\mathrm{diag}(t_1, t_2, \ldots, t_n, t_n^{-1}, \ldots t_2^{-1}, t_1^{-1}); t_1, \ldots, t_n \in F^*\}.$$

The unipotent matrices in B form the unipotent radical of B. Let W be the Weyl group of Sp_{2n} with respect to T. It is well-known that $W \cong S_n \ltimes \mathbb{Z}_2^n$, where S_n denotes the symmetric group of n letters. We call Weyl group elements corresponding to S_n permutations, and to $\mathbb{Z}_n$ sign changes. The action of $p \in S_n$ is given by

$$p(\mathrm{diag}(t_1, t_2, \ldots, t_n, t_n^{-1}, \ldots t_2^{-1}, t_1^{-1})) = (t_{p^{-1}(1)}, t_{p^{-1}(2)}, \ldots, t_{p^{-1}(n)}, \ldots),$$

and the action of $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \mathbb{Z}_2^n$ is given by

$$\varepsilon(\mathrm{diag}(t_1, t_2, \ldots, t_n, t_n^{-1}, \ldots t_2^{-1}, t_1^{-1})) = \mathrm{diag}(t_1^{\varepsilon_1}, t_2^{\varepsilon_2}, \ldots, t_n^{\varepsilon_n}, \ldots).$$

Note that for $p \in S_n$, we have

$$p(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)p^{-1} = (\varepsilon_{p^{-1}(1)}, \varepsilon_{p^{-1}(2)}, \ldots, \varepsilon_{p^{-1}(n)}).$$

Note that the action of the Weyl group on maximal torus T extends to characters on this torus, namely if ϕ is a character on T, we have, for $w \in W$ and $t \in T$, $(w\phi)(t) = \phi(w^{-1}t)$. In more words, if $w = p\varepsilon$ and $\phi = \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_n$ we have

$$p\varepsilon(\phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_n) = \phi_{p^{-1}(1)}^{\varepsilon_{p^{-1}(1)}} \otimes \cdots \otimes \phi_{p^{-1}(n)}^{\varepsilon_{p^{-1}(n)}}. \tag{2.1}$$

For calculation of the normalization factors of the intertwining operators, we have to introduce certain subsets of the Weyl group. Recall that, to our choice of maximal torus T and the upper triangular Borel subgroup B corresponds set Δ of simple roots, given by

$$\alpha_i(\mathrm{diag}(t_1, t_2, \ldots, t_n, t_n^{-1}, \ldots t_2^{-1}, t_1^{-1})) = t_i t_{i+1}^{-1},\ i = 1, 2, \ldots n-1,$$

$$\alpha_n(\mathrm{diag}(t_1, t_2, \ldots, t_n, t_n^{-1}, \ldots t_2^{-1}, t_1^{-1})) = t_n^2.$$

We also use $e_i - e_{i+1}$ to denote $\alpha_i,\ i = 1, \ldots, n-1$ and $2e_n$ to denote α_n. In the same way, we can describe the set of all positive roots (with respect to B) as $\Sigma^+ = \{e_i - e_j, 1 \leq i < j \leq n\} \cup \{e_i + e_j, 1 \leq i < j \leq n\} \cup \{2e_i : 1 \leq i \leq n\}$. For $\alpha_i \in \Delta$ we denote

$$W^{\alpha_i} = \{w = p\varepsilon \in W : w\alpha_i > 0\}.$$

Then, for $\Omega \subset \Delta$ we put $[W/W_\Omega] = \cap_{\alpha \in \Omega} W^\alpha$ (as is well known, this is indeed a set of representatives of left cosets of W modulo its subgroup W_Ω). The description of $[W/W_{\Delta \setminus \{\alpha_i\}}]$, where α_i is a simple root is given, e.g., in Lemma 4.4 of [36].

We use Zelevinsky notation for the parabolic induction in the case of classical groups. For such groups we know that Levi subgroups are isomorphic to $GL_{k_1} \times GL_{k_2} \times \cdots GL_{k_l} \times G'$, where G' is a group of smaller rank, but the same type as G. Thus, if π is an irreducible representation of an Levi subgroup M, then $\pi \cong \pi_1 \otimes \pi_2 \otimes \cdots \pi_l \otimes \pi'$, where π_i is an irreducible representation of GL_{k_i}, $i = 1, \dots l$ and π' is an irreducible representation of G'. We extend π trivially over the unipotent radical N of the corresponding parabolic subgroup $P = MN$ and the normalized parabolically induced representation $\mathrm{Ind}_P^G(\pi)$ is then denoted by $\pi_1 \times \pi_2 \times \cdots \times \pi_l \rtimes \pi'$.

For $p \leq \infty$, and $a, b \in \mathbb{R}$, $b - a \in \mathbb{Z}_{\geq 0}$ we denote by $\zeta(\chi\nu^a, \chi\nu^b)$ the unique irreducible subrepresentation of the representation $\chi\nu^a \times \chi\nu^{a+1} \times \cdots \times \chi\nu^b$. Here ν denotes a character $|\det(\cdot)|_F$ of $GL_n(F)$, where $|\cdot|$ denotes the usual norm on F. Note that $\zeta(\chi\nu^a, \chi\nu^b) = \nu^{\frac{b+a}{2}} 1_{GL_{b-a+1}(\mathbb{Q}_p)}$ (we again use here Zelevinsky notation for the general linear groups).

Using the notation from [24], for $p < \infty$, we denote the unique (essentially square integrable) irreducible subrepresentation of the induced representation $\nu^{l_2}\rho \times \nu^{l_2-1}\rho \times \cdots \times \nu^{-l_1}\rho$ of $GL_{(l_1+l_2+1)k}(\mathbb{Q}_p)$ by $\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho])$. Here ρ is a unitary supercuspidal representation of $GL_k(\mathbb{Q}_p)$, $l_1, l_2 \in \mathbb{R}$, $l_2 + l_1 + 1 \in \mathbb{Z}_{\geq 1}$.

2.1. *Siegel Eisenstein series and normalizing factors.* Let χ be a Grossencharacter of $\mathbb{A}^*$. It induces a character of $GL_n(\mathbb{A})$ given by $x \mapsto \chi(\det x)$. Let P_n be a maximal standard parabolic subgroup of Sp_{2n} such that its standard Levi subgroup M_n is isomorphic to GL_n. Let $P_n = M_n U_n$ be its Levi decomposition. We consider the usual induced representation for $s \in \mathbb{C}$

$$I(s) = \mathrm{Ind}_{P_n(\mathbb{A})}^{Sp_{2n}(\mathbb{A})}(\chi|\det|^s 1_{GL_n(\mathbb{A}_\mathbb{Q})}). \tag{2.2}$$

It is realized on the space of all C^∞ and right $K-$ finite functions on $Sp_{2n}(\mathbb{A})$ which satisfy

$$f(xug) = |\det x|^s \chi(\det x) \delta_{P_n}^{1/2}(x) f(g),$$

where $x \in M_n(\mathbb{A}), u \in U_n(\mathbb{A})$, $g \in Sp_{2n}(\mathbb{A})$ and δ_{P_n} is the modular character of P_n. We construct holomorphic sections $f_s \in I(s)$ using compact picture with our choice of a maximal compact subgroup K. This means that we consider f_s belonging to the space of automorphic forms denoted by $\mathcal{A}(M_n(\mathbb{Q})U_n(\mathbb{A}) \setminus Sp_{2n}(\mathbb{A}), |\det(\cdot)|^s \chi(\det(\cdot)))$ in the notation of [23], II. 1.1. This space of automorphic forms can be canonically identified with $\mathrm{Ind}_{M_n(\mathbb{A}) \cap K}^K(\chi(\det(\cdot)))$ (cf. [23]). This construction is also explained in detail in e.g., [25] (the second section). The degenerate Eisenstein series

$$E(f_s)(g) = \sum_{\gamma \in P_m(\mathbb{Q}) \setminus Sp_{2n}(\mathbb{Q})} f_s(\gamma g)$$

converges for $Re(s)$ sufficiently large, and there the convergence is absolute and uniform in (s, g) on compact sets. In the case at hand, by the result of Godement (cf. [9, 11.1 Lemma]), this Eisenstein series converges for $Re(s) >$

$\frac{n+1}{2}$. It continues to a meromorphic function in s. Outside of poles, it is an automorphic form on $Sp_{2n}(\mathbb{A})$. We usually write $E(s,f)$ instead of $E(f_s)$. We say that this Eisenstein series has a pole of finite order $l \geq 0$ for s_0 if $s \mapsto (s-s_0)^l E(s,f)$ is holomorphic near s_0 for each $f_s \in I(s)$ and non-zero for some such f_s. Then, the mapping

$$I(s_0) \xrightarrow{f_{s_0} \mapsto (s-s_0)^l E(f_s)|_{s=s_0}} \mathcal{A}(Sp_{2n}(\mathbb{Q}) \setminus Sp_{2n}(\mathbb{A})) \tag{2.3}$$

is an intertwining operator. The poles of this Eisenstein series are the same as the poles of it constant term along the Borel subgroup

$$E_{const}(s,f)(g) = \int_{U(\mathbb{Q})\setminus U(\mathbb{A})} E(s,f)(ug)\mathrm{d}u,$$

where U is the unipotent radical of the Borel subgroup we have fixed above. We can write down this constant term using standard unfolding. To do that, we introduce a character of $T(\mathbb{Q}) \setminus T(\mathbb{A})$ (depending on $s \in \mathbb{C}$):

$$\Lambda_s = \chi|\cdot|^{s-\frac{n-1}{2}} \otimes \chi|\cdot|^{s-\frac{n-1}{2}+1} \otimes \cdots \otimes \chi|\cdot|^{s+\frac{n-1}{2}}.$$

We trivially extend this character across $U(\mathbb{A})$ and obtain a character of $B(\mathbb{A})$. Then we form an induced representation $\mathrm{Ind}_{B(\mathbb{A})}^{Sp_{2n}(\mathbb{A})}(\Lambda_s)$. Note that, as an abstract representation, a representation (2.2) is a subrepresentation of $\mathrm{Ind}_{B(\mathbb{A})}^{Sp_{2n}(\mathbb{A})}(\Lambda_s)$ (this is explained in the second section of [25]). Let $w \in W$. We denote by $\overline{U}$ the lower triangular unipotent matrices in Sp_{2n}. We formally define a global intertwining operator

$$M(\Lambda_s, w) : \mathrm{Ind}_{B(\mathbb{A})}^{Sp_{2n}(\mathbb{A})}(\Lambda_s) \to \mathrm{Ind}_{B(\mathbb{A})}^{Sp_{2n}(\mathbb{A})}(w(\Lambda_s))$$

by

$$M(\Lambda_s, w)f(g) = \int_{U(\mathbb{A})\cap w\overline{U}(\mathbb{A})w^{-1}} f_s(\tilde{w}^{-1}ng)\mathrm{d}n.$$

Here $\tilde{w}$ denotes a choice of a representative of an element w of the Weyl group, but the integral does not depend on this choice. Again, this intertwining operator converges in some right half-plane, and admits a meromorphic continuation. This global intertwining operator factors as a product of the local intertwining operators

$$M(\Lambda_s, w)f = \otimes_p A(\Lambda_{s,p}, w) f_p,$$

if $f = \otimes_{p\leq\infty} f_p$, where for almost all p, f_p is the normalized spherical vector. Indeed, we use precise normalizations of Haar measures in the definition of the intertwining operators (we follow [31]), so that for f which is a pure tensor, by using Tamagawa measure, integration over $U(\mathbb{A})\cap w\overline{U}(\mathbb{A})w^{-1}$, comes down as integration over corresponding local counterparts and, formally, outside the poles, expressions for the local intertwining operators appear. The resulting vector is again in the restricted tensor product, since we get an expression which is at almost all places again normalized spherical vector multiplied by an expression containing the partial L–functions (cf. [10], pp. 21-28 or [11], Chapter I, Section II). Let ψ be a non-trivial additive character of $\mathbb{Q}\setminus\mathbb{A}$. The normalization factor for $A(\Lambda_{s,p},w)$ is given by

$$r(\Lambda_{s,p},w)=\prod_{\alpha\in\Sigma^+,w(\alpha)<0}\frac{L(1,\Lambda_{s,p}\circ\check{\alpha})\epsilon(1,\Lambda_{s,p}\circ\check{\alpha},\psi_p)}{L(0,\Lambda_{s,p}\circ\check{\alpha})},$$

where $\check{\alpha}$ is the coroot corresponding to a root α. We define the normalized (local) intertwining operator by

$$N(\Lambda_{s,p},w)=r(\Lambda_{s,p},w)A(\Lambda_{s,p},w). \tag{2.4}$$

We know that the set of roots $\Delta\setminus\{\alpha_n\}$ determines the Siegel standard parabolic subgroup P_n. We have the following expression for the constant term

$$E_{const}(s,f)(g)=\sum_{w\in W,w(\Delta\setminus\{\alpha_n\})>0}r(\Lambda_s,w)^{-1}(\otimes_{p\in S}N(\Lambda_{s,p},w)f_p)\otimes(\otimes_{p\notin S}f_{w,p}). \tag{2.5}$$

Here, $f=\otimes_{p\leq\infty}f_p$ is a pure tensor, and for all $p\notin S$ (S is a finite set of places, including the archimedean place) f_p is the normalized spherical vector with $f_p(e)=1$. Also, $f_{w,p}$ denotes the normalized spherical vector which belongs to the representation space of $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{Sp_{2n}(\mathbb{Q}_p)}(w(\Lambda_{s,p}))$, where $\Lambda_{s,p}$ denotes the local component of the character Λ_s at a place p. We use well-known property of normalization: for f_p normalized spherical, $N(\Lambda_{s,p},w)f_p=f_{w,p}$. We also denoted above

$$r(\Lambda_s,w)^{-1}=\prod_{\alpha\in\Sigma^+,w(\alpha)<0}\frac{L(0,\Lambda_s\circ\check{\alpha})}{L(1,\Lambda_s\circ\check{\alpha})\epsilon(1,\Lambda_s\circ\check{\alpha})}. \tag{2.6}$$

Note that the sum in (2.5) is over the set which we denoted by $[W/W_{\Delta\setminus\{\alpha_n\}}]$ in the previous subsection.

3. The normalizing factors

From this section till the end of this paper, we deal only with the Siegel case of the induced representation (2.2).

In this section we analyze the global normalizing factors (2.6). After that, we calculate certain sums of global normalizing factors which appear in (2.5); namely we group those ones corresponding to those elements of $[W/W_{\Delta\setminus\{\beta\}}]$ which have the same image $w(\Lambda_s)$, so that their contributions belong to the same space $\mathrm{Ind}_{B(\mathbb{A})}^{Sp_{2n}(\mathbb{A})}(w(\Lambda_s))$. These sums we can obviously use in the spherical case (i.e., when all the components f_p are spherical), but, as will turn out, also in the general case. Note that in this (the Siegel) case, a global character Λ_s is equal to

$$\Lambda_s=\chi|\cdot|^{s-\frac{n-1}{2}}\otimes\chi|\cdot|^{s-\frac{n-1}{2}+1}\otimes\cdots\otimes\chi|\cdot|^{s+\frac{n-1}{2}}. \tag{3.1}$$

Then, by Lemma 4.4 of [36], $[W/W_{\Delta\setminus\{\alpha_n\}}]$ is a union of all Y_j^n (where $0 \leq j \leq n$) such that $w = p\varepsilon \in Y_j^n$ if and only if

1. $\varepsilon_k = 1$ for $1 \leq k \leq j$;
2. $p(k_1) < p(k_2)$ for $1 \leq k_1 < k_2 \leq j$;
3. $\varepsilon_k = -1$ for $j+1 \leq k \leq n$;
4. $p(k_1) > p(k_2)$ for $j+1 \leq k_1 < k_2 \leq n$.

We can thus think of p (such that $p\varepsilon \in [W/W_{\Delta\setminus\{\alpha_n\}}]$) as having the following form:
(3.2)
$$p = \begin{bmatrix} 1 & \dots & j-1 & j & j+1 & j+2 & \dots & n \\ p(1) & < \dots < & p(j-1) < & p(j) & p(j+1) > & p(j+2) & > \dots > & p(n) \end{bmatrix}.$$

From now on, we study only the case $s \in \mathbb{R}$ (since the imaginary part of s can be absorbed in χ) and $s > 0$. For $s = 0$ the well-known result of Langlands guarantees the holomorphy of the Eisenstein series.

We now calculate (the global) normalizing factor, given in (2.6), for $w \in [W/W_{\Delta\setminus\{\alpha_n\}}]$. We determine all $\alpha \in \Sigma^+$ such that $w(\alpha) < 0$. Assume that $w \in Y_j^n$. Then, we easily get

$$\begin{aligned} &\{\alpha \in \Sigma^+,\ w(\alpha) < 0\} = \{2e_k : j+1 \leq k \leq n\} \cup \\ &\{e_k + e_l : k < l, 1 \leq k \leq j,\ j+1 \leq l \leq n,\ p(k) > p(l)\} \cup \\ &\{e_k + e_l : j+1 \leq k < l \leq n\}. \end{aligned}$$

This means that the normalizing factor becomes

$$r(\Lambda_s, w)^{-1} = \prod_{k=j+1}^{n} \frac{L(s - \frac{n-1}{2} + k - 1, \chi)}{L(s - \frac{n-1}{2} + k, \chi)\epsilon(s - \frac{n-1}{2} + k, \chi)}. \tag{3.3}$$

$$\prod_{k=1}^{j} \prod_{\substack{l=j+1, \\ p(k)>p(l)}}^{n} \frac{L(2s + k + l - n - 1, \chi^2)}{L(2s + k + l - n, \chi^2)\epsilon(2s + k + l - n, \chi^2)} \tag{3.4}$$

$$\prod_{\substack{k,l=j+1, \\ k<l}}^{n} \frac{L(2s + k + l - n - 1, \chi^2)}{L(2s + k + l - n, \chi^2)\epsilon(2s + k + l - n, \chi^2)}. \tag{3.5}$$

Note that if $j = n$, $Y_j^n = \{id\}$, so from now on we assume that $j < n$. If $j = 0$ the set Y_0^n is also a singleton, consisting of $w = p\varepsilon$, where

$$p = \begin{bmatrix} 1 & 2 & \dots & n-1 & n \\ n & n-1 & \dots & 2 & 1 \end{bmatrix}. \tag{3.6}$$

Now we analyze this expression factor by factor. When we apply the functional equation on the denominator, the factor (3.3) (which exists if $j < n$) becomes

$$\prod_{l=-(\frac{n-1}{2}-s-j)}^{\frac{n-1}{2}+s} \frac{L(l,\chi)}{L(-l,\chi)}.$$

To deal with the factor (3.4), we first fix $k \le j$. Then, we look at the set of all $l \ge j+1$ such that $p(k) > p(l)$. If the set of such l's is non-empty, it is of the form $l_k, l_k+1, \ldots, n$. Also, if set of these l's is non-empty for some k, then it is non-empty for $k+1$. So, the set of all such k's is of the form $k_p, k_p+1, \ldots, j$ (a subscript p indicates the dependence on the permutation p). There is only one $w = p\varepsilon$ from Y_j^n for which this set of l's is empty for every k; it has $p(i) = i,\ i = 1, \ldots, j$. There is no factor (3.4) if $j = 0$. Now assume $0 < j < n$. Then (3.4) becomes

$$\prod_{k=k_p}^{j} \prod_{l=l_k}^{n} \frac{L(2s+k+l-n-1,\chi^2)}{L(2s+k+l-n,\chi^2)\epsilon(2s+k+l-n,\chi^2)}. \tag{3.7}$$

There is an easy recursion for l_k, but we can use known results to calculate it. We attach to a permutation p of (3.2), a permutation p' of the following form

(3.8)

$$p' = \begin{bmatrix} 1 & \ldots & j-1 & j & j+1 & j+2 & \ldots & n \\ p(1) & < \ldots < & p(j-1) < & p(j) & p(n) < & p(n-1) & < \ldots < & p(j+1) \end{bmatrix}$$

i.e. $p'(i) = p(i)$ for $i = 1, \ldots, j$ and $p'(i) = p(n+j+1-i)$ for $i = j+1, \ldots, n$.

Then, a condition $p(l_k - 1) > p(k) > p(l_k)$ transforms into $p'(n+j+1-l_k) < p'(k) < p'(n+j+2-l_k)$. Permutations of the form (3.8) were studied before Lemma 6.9 in ([14]). In that notation, for similarly introduced j_k for the permutation p' we obtain $j_k = n+j+1-l_k$ and there an expression for $j_k = p'(k) - k + j$ is obtained, giving $l_k = k - p'(k) + n + 1$. Also, the index m_w was introduced there, we have denoted it here by k_p. After the obvious cancelations, and using expression for l_k, (3.7) becomes

$$\prod_{k=k_p}^{j} \frac{L(2s+2k-p(k),\chi^2)}{L(2s+k,\chi^2)(*)}, \tag{3.9}$$

where $(*)$ denotes the product of ϵ factors. The expression (3.5) becomes, after cancelations, equal to

$$\prod_{k=j+1}^{n-1} \frac{L(2s+2k-n,\chi^2)}{L(2s+k,\chi^2)(**)}, \tag{3.10}$$

where again $(**)$ is a product of ϵ-factors (note that this factor is trivial ($\equiv 1$) if $j = n-1$). Now we have

$$r(\Lambda_s, w)^{-1} =$$

$$\prod_{l=-(\frac{n-1}{2}-s-j)}^{\frac{n-1}{2}+s} \frac{L(l,\chi)}{L(-l,\chi)} \tag{3.11}$$

$$\prod_{k=k_p}^{j} \frac{L(2s+2k-p(k),\chi^2)}{L(2s+k,\chi^2)(*)} \prod_{k=j+1}^{n-1} \frac{L(2s+2k-n,\chi^2)}{L(2s+k,\chi^2)(**)} \tag{3.12}$$

It it not difficult to see that we can write down (3.3) as $\frac{L(s-\frac{n-1}{2}+j,\chi)}{L(s+\frac{n+1}{2},\chi)(**)}$, where $(**)$ is a product of some ε–factors. Since we assume that $s > 0$, this factor can have a pole only if $j = \frac{n-1}{2} - s + \delta$, $\delta \in \{0,1\}$ and $\chi = 1$, so in order to have a pole from the first factor we must have $\frac{n-1}{2} - s \in \{-1, 0, \ldots, [\frac{n-1}{2}]\}$ and $\chi = 1$. For such s, the pole of this factor occurs for at most two j's, so that for given w it is at most of the first order (since w belongs to Y_j^n only for one j). Note that the appearance of this pole depends only on Y_j^n and not on a particular w in it. The same conclusion follows for the third factor, where a pole appears only if $\chi^2 = 1$ and it is at most of the first order. The second factor can have a pole only if $s \leq \frac{n-1}{2}$ and $\chi^2 = 1$, (and, as for the other factors, $2s \in \mathbb{Z}$) but the order of the pole might be quite high. Analogously as in ([14]), Lemma 6-9, we get that the order of a pole is bounded by $\min(j, n-j-2s+1) - k_p + 1$. For $Re(s) > \frac{n+1}{2}$ the Eisenstein series converges so we do not have to examine L–functions for the poles.

In the next subsection we show that different contributions from normalizing factors can cancel each other to obtain a pole of a much smaller order. These calculations will become fully motivated by the analysis of the actions of the intertwining operators in the subsequent sections.

3.1. *A circular result.* Throughout this section we assume $n \geq 3$, since the case of SL_2 is well-known and for the degenerate Eisenstein series for Sp_4 we refer to [13].

COROLLARY 3.1. *Assume $\chi^2 = 1$ and $s \geq 0$. Then $w(\Lambda_s) = \Lambda_s$ for $w \in [W/W_{\Delta\setminus\{\alpha_n\}}]$ if and only if $w = id$ or $s = 0$ and $\{w\} = Y_0^n$ (cf. (3.6)).*

PROOF. Straightforward. ∎

We continue to assume $s \in \mathbb{R},\ s > 0$ (unless otherwise specifically emphasized). For given $w \in [W/W_{\Delta\setminus\{\alpha_n\}}]$, we denote $[w] = \{w' \in [W/W_{\Delta\setminus\{\alpha_n\}}] : w(\Lambda_s) = w'(\Lambda_s)\}$ and call *the orbit* of w (this notion obviously depends on s). We want to describe $[w]$ for given w. Next, we calculate $\sum_{w'\in[w]} r(\Lambda_s, w')^{-1}$ for those s for which $\prod_{k=k_p}^{j} \frac{L(2s+2k-p(k),\chi^2)}{L(2s+k,\chi^2)(*)}$ has a pole for some k and this

$w = p\varepsilon$. Namely, we show how the poles coming from the Weyl group elements in the same orbit cancel each other, up to a pole of the order one.

Proposition 3.2. *Assume that* $\chi^2 = 1$ *and* $2s \in \mathbb{Z}$ *with* $0 < 2s \leq n-1$. *Then, assume that for* $w_1 = p_1\varepsilon_1,\ w_2 = p_2\varepsilon_2 \in [W/W_{\Delta\setminus\{\alpha_n\}}], w_1 \neq w_2$ *we have* $w_1(\Lambda_s) = w_2(\Lambda_s)$. *Then, one of the following holds:*

- $w_1, w_2 \in Y_j^n$ *for some* $0 < j < n$. *Then, for every* $k \in \{1, 2, \dots, n\}$ *we have* $p_1(k) = p_2(k)$ *or* $p_1(k) = p_2(n+1-2s-k)$. *In the latter case, for* $1 \leq k \leq j$ *we must have* $k \leq \min\{n-2s-j, j\}$ *and for* $j+1 \leq k \leq n$ *we must have* $n+1-2s-j \leq k \leq n-2s$.
- $w_1 \in Y_{j_1}^n$ *and* $w_2 \in Y_{j_2}^n$ *with* $j_1 < j_2$. *Then,* $j_2 = n-2s-j_1$ *and we have*

$$p_1(k) = p_2(k) \text{ or } p_1(k) = p_2(n+1-2s-k),\ 1 \leq k \leq j_1,$$
$$p_1(k) = p_2(n+1-2s-k),\ j_1+1 \leq k \leq j_2,$$
$$p_1(k) = p_2(k) \text{ or } p_1(k) = p_2(n+1-2s-k),\ j_2+1 \leq k \leq n.$$

- *If* $w_1 = w_0$ *(where* w_0 *is described in (3.6)), then* $w_2 = p\varepsilon$, *where* $w_2 \in Y_{n-2s}^n$ *and* $p(i) = 2s+i,\ i = 1, 2, \dots, n-2s$ *and* $p(i) = n+1-i,\ i = n-2s+1, \dots, n$.

Proof. Straightforward calculation. ∎

Remark 3.3. With the assumptions of the previous proposition:

- Note that if $w_1 \in Y_j^n$, for $j \geq n-2s+1$ and $w_1(\Lambda_s) = w_2(\Lambda_s)$ for some $w_2 \in [W/W_{\Delta\setminus\{\alpha_n\}}]$, then $w_2 = w_1$. Thus, if $w \in Y_j^n$, with $j \geq n-2s+1$, then $[w] = \{w\}$.
- Note that if $j \geq n-2s+1$ and $w \in Y_j^n$, then $r(\Lambda_s, w)^{-1}$ is holomorphic (this follows form the discussion at the end of the previous section).

Now we want to examine possibilities occurring in the previous proposition more thoroughly, since this is the situation in which the poles (of the higher order) of the global normalizing factors might occur; so we continue to assume $\chi^2 = 1,\ 2s \in \mathbb{Z}$ with $0 < s \leq \frac{n-1}{2}$. Note that, in that case, the ε factors in (3.12) are trivial when they occur for χ^2 since now we assume $\chi^2 = 1$ (and we assume throughout that $F = \mathbb{Q}$). We can write $[w] = [w]' \cup [w]''$, where now, for $w \in Y_j^n$, $[w]'$ denotes part of the orbit of w in Y_j^n and $[w]''$ part of it in Y_{n-2s-j}^n. Assume now that $j < n-2s-j$ and denote $j_1 = j$ and $j_2 = n-2s-j_1$. Firstly, there is an easy-describable bijection between $[w]'$ and $[w]''$.

Lemma 3.4. *Assume* $\chi^2 = 1,\ 2s \in \mathbb{Z}$ *with* $0 < s \leq \frac{n-1}{2}$ *and* $w = p\varepsilon \in Y_{j_1}$. *Assume that* $j_1 \leq \frac{n-1}{2} - s$ *(so that* $j_2 \geq \frac{n+1}{2} - s \geq j_1 + 1$*).*

1. *Assume that* $p(j_1) < p(j_2)$. *The bijection mapping elements from* $Y_{j_1}^n \cap [w] = [w]'$ *to* $Y_{j_2}^n \cap [w] = [w]''$ *is given as follows: if* $w_1 = p_1\varepsilon_1 \in [w]'$

then $w_1 \mapsto w_2 = p_2\varepsilon_2 \in [w]''$ *with* $p_2(i) = p_1(i),\ i = 1, 2, \ldots, j_1$ *and* $i = j_2 + 1, \ldots, n$ *and* $p_2(i) = p_1(n + 1 - 2s - i)$ *for* $i = j_1 + 1, \ldots, j_2$. *Any element from* $[w]'$ *attains the same values on* $j_1 + 1, \ldots, j_2$ *as* w_1 *(i.e., as* w*) and every element in* $[w]''$ *attains the same values as* w_2 *on these places (namely* $p_1(n + 1 - 2s - i),\ i = j_1 + 1, \ldots, j_2$*). These attained values are* $n, \ldots, 2j_1 + 2s + 1$ *(in that order)*

2. *Assume that* $p(j_1) > p(j_2)$. *Then, we define* $i_t \leq j_1$ *as the largest index (*$\leq j_1$*) such that the following holds:*

$$\begin{aligned} &p_1(j_1 + j_2 + 1 - i_t) > p_1(i_t - 1), \\ &p_1(i_t) > p_1(j_1 + j_2 + 2 - i_t). \end{aligned} \tag{3.13}$$

Then, $w_1 \mapsto w_2 = p_2\varepsilon_2 \in [w]''$ *is given by* $p_2(i) = p_1(i),\ i = 1, 2, \ldots, i_t - 1$ *and* $i = n + 2 - 2s - i_t, \ldots, n$ *and* $p_2(i) = p_1(n + 1 - 2s - i)$ *for* $i = i_t, \ldots, n + 1 - 2s - i_t$. *Any element from* $[w]'$ *attains the same values as* w_1 *on* $i_t, \ldots, n + 1 - 2s - i_t$, *and every element in* $[w]''$ *attains the same values as* w_2 *on these places (namely* $p_1(n + 1 - 2s - i),\ i = j_1 + 1, \ldots, j_2$*).*

Proof. From the form of $w = p\varepsilon \in Y_{j_1}^n$ (cf. (3.2)), it follows that if $p(j_1) < p(j_2)$, then $p(j_1+1), \ldots, p(j_2)$ are the biggest elements in $\{1, 2, \ldots, n\}$, i.e., $p(j_1 + 1) = n, \ldots, p(j_2) = n + j_1 + 1 - j_2 = 2j_1 + 2s + 1$. To each p_1 such that $p_1\varepsilon_1$ is from $[w]'$, we can attach a permutation p_1' like in (3.8) (analogously for elements from $[w]''$). In this way, we can describe elements of $[w]'$ in terms of Weyl group elements for the group GL_n which turn the roots $W'(\Delta' \setminus \{e_{j_1} - e_{j_1+1}\})$ to positive roots (W' stands for the Weyl group for GL_n). We discussed similar issues for GL_n groups in [14]. Now, for this $w = p\varepsilon \in Y_{j_1}$, we can describe all the elements from $[w]'$ with using so called "intervals of change" (as defined in [14, Section 7.1]). If $p(j_1) < p(j_2)$, then intervals of change for w end with j_1, so all the elements in $[w]'$ can be described by change on the first j_1 elements, and consequently, elements $n + 1 - 2s - j_1 = j_2 + 1, \ldots, n + 1 - 2s - 1 = n - 2s$ (cf. Proposition 3.2). Thus, the values of permutations in $[w]'$ on elements $j_1 + 1, \ldots, j_2$ are all the same (and equal to $n, \ldots 2j_1 + 2s + 1$). Analogously, in the second case ($p(j_1) > p(j_2)$), our description of i_t says that the possible intervals of change end with $i_t - 1$ and the second claim follows. ∎

Remark 3.5. Note that in the case $p(j_1) > p(j_2)$ of the above lemma, the values of p_1 attained on $i_t, \ldots, n + 1 - 2s - i_t$ are the biggest possible (but not necessarily in the increasing or the decreasing order); this follows from the conditions (3.13).

LEMMA 3.6. *We retain the assumptions of Lemma 3.4. Let $w_1 \in Y^n_{j_1}$ and let $w_2 \in Y^n_{j_2}$ be its bijective image described in Lemma 3.4. We denote*

$$A_1 = \prod_{l=-(\frac{n-1}{2}-s-j_1)}^{\frac{n-1}{2}+s} \frac{L(l,\chi)}{L(-l,\chi)} \cdot \frac{1}{\prod_{k=k_{p_1}}^{n-1} L(2s+k,1)} \cdot$$

$$\prod_{k=k_{p_1}}^{j_1} L(2s+2k-p_1(k),1) \prod_{k=j_2+1}^{n-1} L(2s+2k-n,1) \prod_{k=2}^{j_2-j_1} L(k,1).$$

Then,

$$r(\Lambda_s, w_1)^{-1} = A_1 \begin{cases} \lim_{t\to 0} L(2t,1) & j_2-j_1 \text{ is even,} \\ \lim_{t\to 0} L(2t+1,1) & j_2-j_1 \text{ is odd,} \end{cases} \tag{3.14}$$

$$r(\Lambda_s, w_2)^{-1} = A_1 \lim_{t\to 0} L(2t+1,1) \cdot \begin{cases} 1 & j_2-j_1 \text{ is even,} \\ 1 & \begin{matrix} j_2-j_1 \geq 3 \text{ is odd or} \\ j_2-j_1 = 1 \text{ and } \chi \neq 1 \end{matrix}, \\ -1 & j_2-j_1 = 1 \text{ and } \chi = 1. \end{cases} \tag{3.15}$$

PROOF. We first examine (3.11). We easily obtain the following: if j_2-j_1 is even, then (3.11) for w_2 is equal to (3.11) for w_1 and holomorphic. If j_2-j_1 is odd, then, if $\chi \neq 1$, (3.11) for w_1 is equal to (3.11) for w_2 (and non-zero and holomorphic). If $j_2-j_1 \geq 3$ is odd, and $\chi = 1$, (3.11) for w_1 and w_2 are non-zero, holomorphic and the same. But if $j_2-j_1 = 1$ and $\chi = 1$, then (3.11) for w_2 and w_1 have a pole of the first order and one is negative of the other. Now we compare (3.12) for $w_1 = p_1\varepsilon_1$ and $w_2 = p_2\varepsilon_2$. First assume that $p_1(j_1) < p_1(j_2)$. Note that if k_{p_1} (introduced after (3.8)) exists, then $k_{p_1} = k_{p_2}$. Assume firstly that k_{p_1} exists. Then (3.12) for w_1 becomes

$$\frac{1}{\prod_{k=k_{p_1}}^{n-1} L(2s+k,1)} \prod_{k=k_{p_1}}^{j_1} L(2s+2k-p_1(k),1) \prod_{k=j_1+1}^{n-1} L(2s+2k-n,1). \tag{3.16}$$

Now we analyze the second factor as above, but for w_2. We have

$$\begin{aligned} &\prod_{k=k_{p_1}}^{j_2} L(2s+2k-p_2(k),1) \\ &= \prod_{k=k_{p_1}}^{j_1} L(2s+2k-p_2(k),1) \prod_{k=j_1+1}^{j_2} L(2s+2k-p_2(k),1) \\ &= \prod_{k=k_{p_1}}^{j_1} L(2s+2k-p_1(k),1) \prod_{k=j_1+1}^{j_2} L(2s+2k-p_1(n+1-2s-k),1). \end{aligned}$$

According to Lemma 3.4, case 1., the last product becomes $\prod_{k=j_1+1}^{j_2} L(k-j_1,1)$, so that (3.12) for w_2 becomes

$$\begin{aligned}\frac{1}{\prod_{k=k_{p_1}}^{n-1} L(2s+k,1)} &\prod_{k=k_{p_1}}^{j_1} L(2s+2k-p_1(k),1) \\ &\prod_{k=j_1+1}^{j_2} L(k-j_1,1) \prod_{k=j_2+1}^{n-1} L(2s+2k-n,1).\end{aligned} \tag{3.17}$$

To see how (3.16) and (3.17) are related, it is enough to compare

$$\prod_{k=j_1+1}^{j_2} L(2s+2k-n,1) \quad \text{and} \quad \prod_{k=j_1+1}^{j_2} L(k-j_1,1).$$

Note that both sides have a pole of the first order. On the right hand side it is obtained for $k=j_1+1$, and if we write $s=t+s_0$, where $n-2s_0=j_1+j_2$ and see what happens for $t\to 0$, when we recover $L(2s+2k-p_1(n+1-2s-k),1)$ we see that we actually have $\lim_{t\to 0} L(2t+k-j_1,1)$, so that the right-hand side becomes $\lim_{t\to 0} L(1+2t,1)\prod_{k=j_1+2}^{j_2} L(k-j_1,1)$. The left-hand side becomes $\lim_{t\to 0}\prod_{k=j_1+1}^{j_2} L(2t+2k-j_1-j_2,1)$. Now we apply functional equation for the negative arguments in this product. We get

1. $\lim_{t\to 0} L(2t,1)\prod_{k=2}^{j_2-j_1} L(k,1)$, if j_2-j_1 is even,
2. $\lim_{t\to 0} L(2t+1,1)\prod_{k=2}^{j_2-j_1} L(k,1)$, if j_2-j_1 is odd.

If k_{p_1} does not exist, i.e., $p_1(i)=i,\ i=1,\dots,j_1$ we have that $r(\Lambda_s,w_1)^{-1}$ consists of (3.11), and it has only the second factor from (3.12). But then it is easy to see that $k_{p_2}=j_1+1$, and we get the same results as in the previous case.

Now assume that $p_1(j_1)>p_1(j_2)$. We again have that (3.11) is the same for w_1 and w_2. Also, we can factor out $\frac{1}{\prod_{k=k_{p_1}}^{n-1} L(2s+k,1)}$ from $r(\Lambda_s,w_i)^{-1}$, $i=1,2$. We immediately see that the product

$$\prod_{k=k_{p_1}}^{i_t-1} L(2s+2k-p_1(k),1)\prod_{k=j_2+1}^{n-1} L(2s+2k-n,1) \tag{3.18}$$

is common for $r(\Lambda_s,w_1)^{-1}$ and $r(\Lambda_s,w_2)^{-1}$. Thus, we have to compare the product (belonging to $r(\Lambda_s,w_1)^{-1}$)

$$\prod_{k=i_t}^{j_1} L(2s+2k-p_1(k),1)\prod_{k=j_1+1}^{j_2} L(2s+2k-n,1) \tag{3.19}$$

and the product (belonging to $r(\Lambda_s, w_2)^{-1}$)

$$\prod_{k=i_t}^{j_2} L(2s + 2k - p_2(k), 1) \tag{3.20}$$

We now comment on poles in (3.19) and (3.20). We prove that $\prod_{k=i_t}^{j_1} L(2s + 2k - p_1(k), 1)$ from (3.19) does not have a pole, and that (3.20) has a pole of the first order. Note that we have already discussed $\prod_{k=j_1+1}^{j_2} L(2s + 2k - n, 1)$ (this expression does not depend on the permutation p_1, but just on j_1 and j_2).

Now we analyze $\prod_{k=i_t}^{j_1} L(2s + 2k - p_1(k), 1)$. We prove that it is holomorphic analogously to the proof of [14, Lemmas 7-16]. We remind the reader that $'$ means that the corresponding permutation is adjusted to apply GL-considerations from the seventh section of ([14]). Let us briefly explain. Assume that the last interval of change for p_1 is $[\beta, i_t - 1]$. If we assume that $2s + 2k - p_1(k) = 0$ for some $k \in [i_t, j_1]$, this would mean $j_k = j_1 + 2s + k$ (we remind the reader of the definition of j_k for p_1' given above (3.9)). It follows that $p_1(k-1) > p_1'(j_1 + 2s + k)$, otherwise, we could change $p_1(k) \leftrightarrow p_1(n+1-2s-k) = p_1'(j_1+2s+k)$. We inductively conclude that $p_1(k-j) > p_1'(j_1 + 2s + k - j + 1)$, when we do not have a beginning of some interval of change in $k - j + 1$. We do have the beginning of such an interval for $k - j + 1 = \beta$, but this interval of change does not have an end in k but in $i_t - 1$. Similarly, a case $2s + 2k - p_1(k) = 1$ leads to the condition $p_1(k+t+1) < p_1'(j_1+2s+k+t)$. If we would take $k+t = j_1$, then obstruction to the interval change would disappear, but we do know that cannot have the ending of the last interval of change to be at j_1.

We claim that $\prod_{k=i_t}^{j_2} L(2s + 2k - p_2(k), 1)$ has a pole of the first order, obtained for $k = i_t$. We know that $p_1(i_t) = 2s+2i_t-1$ or $p_1(n+1-2s-i_t) = 2s + 2i_t - 1$. Indeed, as we saw in Lemma 3.4, case 2. the permutation p_1 attains on the set $\{1, 2, \ldots, i_t - 1\} \cup \{n+2-2s-i_t, \ldots, n\}$ the smallest values in $\{1, 2, \ldots, n\}$, and there are exactly $2s + 2i_t - 2$ those numbers. So, the value $2s + 2i_t - 1$ is the smallest value that can be attained on the rest of the indexes and the conclusion follows. But if we would have $p_1(i_t) = 2s+2i_t-1$, the function $L(2s+2i_t-p_1(i_t), 1)$ would have a pole, and we proved that this cannot be the case by the preceding paragraph. We just have to show that there are no poles for $k > i_t$. From the fact that we cannot change interval for p_1 in i_t and then in $i_t + 1$ we get

$$p_1(n + 1 - 2s - (i_t + 1) - 1) < p_1(i_t + 1),$$

and then, inductively, as long as it makes sense, we get

$$p_1(n + 1 - 2s - (i_t + k + 1)) < p_1(i_t + k),\ k \geq 0.$$

It makes sense for $i_t + k \leq j_1$, so that the last relation becomes $p_2(i_t + k + 1) < p_2(n + 1 - 2s - (i_t + k))$, and then using j_{i_t+k} for p_2' we get that $p_2(i_t+k+1) \leq 2s+2(i_t+k+1)-2$, so that $L(2s+2l-p_2(l), 1)$ cannot have a pole for $l = i_t + k + 1$. This proves that $L(2s + 2k - p_2(k), 1)$ cannot have a pole for $k = i_t + 1, \ldots, j_1 + 1$. As for $k \geq j_1 + 2$, we simply observe that $j_k \leq n$ (here j_k is attached to p_2'). Then, as we know, $j_k = p_2'(k) - k + j_2$, which gives $2s + 2k - p_2'(k) = 2s + 2k - p_2(k) \geq 2$.

We conclude that (3.19) becomes

$$\prod_{k=i_t}^{j_1} L(2s + 2k - p_1(k), 1) \prod_{k=2}^{j_2-j_1} L(k, 1)A,$$

where $A = \lim_{t\to 0} L(2t, 1)$ if $j_2 - j_1$ is even, and $\lim_{t\to 0} L(2t + 1, 1)$ if $j_2 - j_1$ is odd. We also saw that (3.20) becomes $B \prod_{k=i_t+1}^{j_2} L(2s + 2k - p_1(n - 2s + 1 - k), 1)$, where $B = \lim_{t\to 0} L(2t + 1, 1)$. Note that all of these expressions, except A and B, are holomorphic. We now prove that these holomorphic parts are equal. To do that, we shall completely turn this into GL situation in the following way. As we saw above, $p_1(i_t), \ldots, p_1(n + 1 - 2s - i_t)$ attain values on the set $\{2s + 2i_t - 1, \ldots, n\}$. So, we denote $m' = j_1 - i_t + 1$ and $n' = j_2 - i_t + 1$ (thus, $m' + 1 \leq n'$) and introduce a new permutation on the set $1, 2, \ldots, n + 2 - 2s - 2i_t = m' + n'$ as follows:

$$p_1''(i) = p_1(i + i_t - 1) - (2s + 2i_t - 2), i = 1, 2, \ldots, m', \tag{3.21}$$

$$p_1''(i) = p_1(2m' + n' + i_t - i) - (2s + 2i_t - 2), i = m' + 1, \ldots, m' + n'. \tag{3.22}$$

We easily see that p_1'' is increasing on the first m' and on the last n' places. In terms of p_1'', the holomorphic part of (3.19) discussed above becomes

$$\prod_{i=1}^{m'} L(2i - p_1''(i), 1) \prod_{k=2}^{j_2-j_1} L(k, 1) \tag{3.23}$$

and holomorphic part of (3.20) becomes

$$\prod_{i=2}^{m'} L(2i - p_1''(m' + i), 1). \tag{3.24}$$

Note that $p_1''(m'+1) = 1$ and that the changes in p_1 $p_1(i) \leftrightarrow p_1(n+1-2s-i)$ now become $p_1''(i) \leftrightarrow p_1''(m' + i)$. Note that our conditions imply that there are no changes for p_1'' either. This means that for any $i \in \{1, \ldots, m'\}$, the interval $\{1, \ldots, i\}$ cannot be an interval of change. We easily get from this that $p''(i) > p_1''(m' + i + 1),\ i = 1, \ldots, m'$ which leads (again using j_i for p_1'') to the condition $p_1''(i) \geq 2i + 1,\ i = 1, \ldots, m'$. Thus, the arguments in $\prod_{i=1}^{m'} L(2i - p_1''(i), 1)$ from (3.23) are negative, and we use $L(2i - p_1''(i), 1) =$

$L(p_1''(i)-2i+1,1)$. We also immediately have $p_1''(m'+2)=2$. Now, to resolve (3.24) we write elements $\{2,\dots,m'+n'\}$ in the increasing order:

$$p_1''(m'+2),\dots,p_1''(j_1),p_1''(1),p_1''(j_1+1),\dots,p_1''(j_2),p_1''(2),\\ \dots,p_1''(k),p_1''(j_k+1),\dots,p_1''(j_{k+1}),p_1''(k+1),\dots. \tag{3.25}$$

This sequence ends with $\dots p_1''(m')$, or with $\dots p_1''(m'+n')$. We discuss the first case; the second is quite analogous. Note that the beginning part $p_1''(m'+2),\dots,p_1''(j_1)$ contributes to (3.24) as $\prod_{i=2}^{p_1''(1)-1} L(i,1)$. Now we use that $j_k = p_1''(k)-k+m'$ to obtain that (3.24) becomes

$$\prod_{i=2}^{p_1''(1)-1} L(i,1) \prod_{k=1}^{m'-1} \prod_{p_1''(k)-k+1}^{p_1''(k+1)-k-1} L(i-k,1),$$

since $p_1''(j_k+1)=p_1''(k)+1=p_1''(k)-k+1+k=\dots=p_1''(j_{k+1})=p_1''(k+1)-k-1+k=i+k$, for all i between j_k+1 and j_{k+1}. Then $L(2i-(i+k),1)=L(i-k,1)$. Note that the last factor of the product above corresponding to k, is the first factor in the product corresponding to $k+1$. Thus, we have some repetitions and the product above becomes $L(2,1)\dots L(n'-m'+1,1)\prod_{k=1}^{m'-1} L(p_1''(k)-2k+1,1)$. Now we use that $p_1''(m')=m'+n'$ to obtain the equality with (3.23). □

It will turn out that, when analyzing the constant term of Eisenstein series (2.5), the contributions coming from the global intertwining operators attached to the same orbits of the Weyl group elements, will contribute in a similar way, so we shall need $\sum_{w'\in[w]} r(\Lambda_s,w')^{-1}$ and $\sum_{w'\in[w]\cap Y_{j_1}^n} r(\Lambda_s,w')^{-1} - \sum_{w''\in[w]\cap Y_{j_2}^n} r(\Lambda_s,w'')^{-1}$. As we have seen in Lemma 3.6, it is enough to calculate $\sum_{w'\in[w]\cap Y_{j_1}^n} r(\Lambda_s,w')^{-1}$.

LEMMA 3.7. *We retain the assumptions from Lemma 3.6. Let* $j_1 \le \frac{n-1}{2}-s$ *and* $w\in Y_{j_1}^n$. *Then,* $\sum_{w'\in[w]\cap Y_{j_1}^n} r(\Lambda_s,w')^{-1}$ *has a pole of (at most) the first order; unless* $\chi=1$ *and* $j_2-j_1=1$; *then it has a pole of (at most) the second order.*

PROOF. Note that in the expression A_1 of Lemma 3.6, only $\prod_{k=k_{p_1}}^{j_1} L(2s+2k-p_1(k),1)$ depends on $w=p_1\varepsilon_1$; all other parts depend only on j_1. So, in the sum $\sum_{w'\in[w]\cap Y_{j_1}^n} r(\Lambda_s,w')^{-1}$ we can factor out all other factors, which are holomorphic (and non-zero), except maybe (3.11), which is holomorphic unless $\chi=1$ and $j_1=\frac{n-1}{2}-s$. So, we are interested in the sum

$$\sum_{w'=p_1\varepsilon_1\in[w]\cap Y_{j_1}^n} \prod_{k=k_{p_1}}^{j_1} L(2s+2k-p_1(k),1). \tag{3.26}$$

These kinds of sums are calculated in [14, Section 7.2]. We again make a transition $p_1 \mapsto p'_1$ (as in (3.8)). In [14, Section 7.2] the sums of products of the form $L(s' + \frac{n'-m'}{2} + 2k - p(k), 1)$ are studied, with $m' = j_1,\ n' = n - j_1$, so that we want $2s = s' + \frac{n'-m'}{2}$. We get $s' = s - \frac{j_2 - j_1}{2}$. The results in [14, Section 7.2] are given for $s' \geq 0$, so we can directly apply them if $j_1 \geq \frac{n}{2} - 2s$. The value $\alpha = \frac{m'+n'}{2} - s'$ becomes $\alpha = j_2$. Then, $n' = n - j_1 > \alpha = j_2$, and $\alpha + 1 > m' = j_1$, so [14, Lemmas 7-15] guarantee that the sum (3.26) is holomorphic. Assume for a moment that $\chi \neq 1$ or $j_1 < \frac{n-1}{2} - s$. Then, we conclude that $\sum_{w' \in [w] \cap Y^n_{j_1}} r(\Lambda_s, w')^{-1}$ has a pole of (at most) the first order.

Now we examine what happens in $s < \frac{j_2 - j_1}{2}$, i.e. $j_1 < \frac{n}{2} - 2s$. We again use the same results, but we have to look closely what's happening. Note that (recall that we are now examining only $w' \in [w]'$) the changes are $i \mapsto n + 1 - 2s - i$, so the values in $p_1(j_1 + 1), \ldots, p_1(j_2)$ cannot correspond to any changes. For example, if $p_1(j_1) < p_1(j_2)$, then only the values at the place $\{1, \ldots, j_1\} \cup \{j_2 + 1, \ldots, n\}$ might vary; on other places we have $p_1(j_1 + 1) = n, \ldots$ (cf. Lemma 3.4). We just remove indexes $j_1 + 1, \ldots, j_2$ from the considerations. So, instead of the permutation p'_1, we have the permutation $p''_1(i) = p'_1(i) = p_1(i),\ i = 1, \ldots, j_1$ and instead of $p'_1(i) = p_1(n + 1 + j_1 - i),\ i = j_1 + 1, \ldots n$, we have $p''_1(i) = p'_1(i + j_2 - j_1),\ i = j_1 + 1, \ldots, n - (j_2 - j_1)$. Now, $m' = j_1$, $n' = n - j_2$. We get $s' = s$ and $\alpha = j_1$, and we can again use Lemma 7-15 of ([14]) to conclude that the sum (3.26) is holomorphic (note that we could reason like this even in the case $s \geq \frac{j_2 - j_1}{2}$). Now, if $p_1(j_1) > p_1(j_2)$ the values on the indexes $i_t, \ldots, j_1$ are fixed for every $w' \in [w] \cap Y^n_{j_1}$ (cf. Lemma 3.4) and

$$\sum_{w' = p_1 \varepsilon_1 \in [w] \cap Y^n_{j_1}} \prod_{k = k_{p_1}}^{j_1} L(2s + 2k - p_1(k), 1) = \prod_{k = i_t}^{j_1} L(2s + 2k - p_1(k), 1) \cdot$$

$$\sum_{w' = p_1 \varepsilon_1 \in [w] \cap Y^n_{j_1}} \prod_{k = k_{p_1}}^{i_t - 1} L(2s + 2k - p_1(k), 1). \tag{3.27}$$

Note that we have proved that $\prod_{k=i_t}^{j_1} L(2s + 2k - p_1(k), 1)$ is holomorphic in the proof of Lemma 3.6. As we saw from that proof, the smallest values $\{1, 2, \ldots, 2s + 2i_t - 1\}$ are attained on the places $\{1, 2, \ldots, i_t - 1\} \cup \{n + 2 - 2s - i_t, \ldots, n\}$. Thus, we take $m' = i_t - 1,\ n' = 2s + i_t - 1$ and $s' = s$. We get $\alpha = m' = i_t - 1$, thus again

$$\sum_{w' = p_1 \varepsilon_1 \in [w] \cap Y^n_{j_1}} \prod_{k = k_{p_1}}^{i_t - 1} L(2s + 2k - p_1(k), 1)$$

is holomorphic by [14, Lemmas 7-15]. □

4. Composition series of local representations

4.1. *Non-archimedean case.* In this subsection we assume that F is a non-archimedean field of characteristic zero.

The lengths of the composition series for the degenerate principal series at a local non-archimedean place of the form we are studying were studied in the work of Gustafson ([12]) and Kudla nad Rallis ([19]). Gustafson was concerned with the unramified case and Kudla and Rallis covered other situations. We analyze this degenerate series using the Aubert involution, and then it will turn out that some of the subquotients of the degenerate principal series are Aubert duals of some discrete series representations. We do that because we will use some of the basic theory of the discrete series for classical groups in the last section, in Lemma 6.4.

In [19] these representations are analyzed in a form adjusted to fit in with the theta correspondence. We end this section by giving the explicit description (as the Langlands quotients) of the subquotients of this degenerate principal series which will show up in the description of the images of the Eisenstein series.

For an irreducible admissible representation σ of a connected algebraic group over a non-archimedean field F, let $\hat{\sigma}$ denote its Aubert dual (a genuine) representation, as defined in [6]. So, $\hat{\sigma}$ for us denotes the Aubert dual (which is defined on the level of the Grothendieck group), but multiplied with ± 1 to obtain a genuine representation. The following is a well known fact (cf. [6]).

Lemma 4.1. *Let σ be an irreducible admissible representation of $GL_k(F)$ and π an irreducible admissible representation of $Sp_{2m}(F)$. Then, in the appropriate Grothendieck group (of finite-length, smooth representations of $Sp_{2(k+m)}(F)$) the following holds*

$$\widehat{\sigma \rtimes \pi} = \hat{\sigma} \rtimes \hat{\pi}.$$

Note that the Aubert dual of a trivial representation $1_{GL_n(F)}$ is the Steinberg representation $St_{GL_n(F)}$; this extends to the twists of these representations by characters. We use Lemma 4.1 to be able to determine the length the representation $\chi\nu^s \rtimes 1$ in terms of the length of the representation $\chi\nu^s St_{GL_n(F)} \rtimes 1$ (which are the same). In the previous notation $St_{GL_n(F)} = \delta([\nu^{-\frac{n-1}{2}}, \nu^{\frac{n-1}{2}}])$. Here ρ is a trivial character of $GL_1(F) = F^*$ and we skip it in the notation. In [24], the composition series of $\chi\nu^s St_{GL_n(F)} \rtimes 1$ (and many other cases) are determined. Now we have the following basic result (cf. Theorem 9.1 of [37])

Lemma 4.2. *Assume F is of characteristic 0. In order for the representation $\delta([\nu^{s-\frac{n-1}{2}}\chi, \nu^{s+\frac{n-1}{2}}\chi]) \rtimes 1$ to be reducible, it is necessary and sufficient that there exists an index $j \in \{0, \ldots, n-1\}$ such that $\nu^{s-\frac{n-1}{2}+j}\chi \rtimes 1$ reduces (in $SL_2(F)$).*

Since reducibility for $SL_2(F)$ is well-known, and using Aubert involution, we have this simple corollary:

Corollary 4.3. *If* $\chi^2 \neq 1$ *or* $s - \frac{n-1}{2} \notin \mathbb{Z}$, *or* $|s| > \frac{n+1}{2}$ *the representation* $\chi\nu^s 1_{GL_n} \rtimes 1$ *of* $Sp_{2n}(F)$ *is irreducible.*

Now we assume that $\chi^2 = 1$ and $s - \frac{n-1}{2} \in \mathbb{Z}$, and $|s| \leq \frac{n+1}{2}$. We can fully describe the composition factors of the representation $\chi\nu^s 1_{GL_n} \rtimes 1$ of $Sp_{2n}(F)$. We further assume that $s > 0$.

Lemma 4.4. *Assume that* χ *is a trivial character of* F^* *and* $s > 0$ *such that* $s - \frac{n-1}{2} \in \mathbb{Z}$ *and* $s \leq \frac{n+1}{2}$. *Then, in the appropriate Grothendieck group, we have*

1. *Assume* $s = \frac{n+1}{2}$.
$$\nu^{\frac{n+1}{2}} 1_{GL_n} \rtimes 1 = 1_{Sp_{2n}} + L(\delta(\widehat{[\nu^1, \nu^n]}); 1),$$
where $1_{Sp_{2n}}$ *is the unique (spherical) quotient of* $\nu^{\frac{n+1}{2}} \rtimes 1$.
2. *Assume* $s = \frac{n-1}{2}$.
$$\nu^{\frac{n-1}{2}} 1_{GL_n} \rtimes 1 = L(\nu^{n-1}, \dots, \nu^1; \nu^0 \rtimes 1) + L(\delta(\widehat{[\nu^0, \nu^{n-1}]}); 1),$$
and $L(\nu^{n-1}, \dots, \nu^1; \nu^0 \rtimes 1)$ *is the unique (spherical) quotient of* $\nu^{\frac{n-1}{2}} \rtimes 1$.
3. *Assume* $s < \frac{n-1}{2}$. *Then, the representation* $\nu^s \delta([\nu^{-\frac{n-1}{2}}, \nu^{\frac{n-1}{2}}]) \rtimes 1$ *is of length 3, it has two square integrable subrepresentations, say,* σ_1 *and* σ_2 *, and the Langlands quotient* $L(\nu^s \delta([\nu^{-\frac{n-1}{2}}, \nu^{\frac{n-1}{2}}]); 1)$. *Thus,*
$$\nu^s 1_{GL_n} \rtimes 1 = \widehat{\sigma_1} + \widehat{\sigma_2} + L(\nu^s \delta(\widehat{[\nu^{-\frac{n-1}{2}}, \nu^{\frac{n-1}{2}}]}); 1).$$
Here, $L(\nu^s \delta(\widehat{[\nu^{-\frac{n-1}{2}}, \nu^{\frac{n-1}{2}}]}); 1)$ *is the unique subrepresentation of* $\nu^s \rtimes 1$ *and* $\widehat{\sigma_1}$ *and* $\widehat{\sigma_2}$ *are irreducible quotients. One of* $\widehat{\sigma_i}$, $i = 1, 2$ *is the spherical subquotient.*

Proof. We use the Aubert involution, and switch to discrete series subquotients, because it will be handy later. So we use Proposition 3.1 (i), Theorem 4.1.(ii) and Theorem 2.1 of [24] and Aubert involution to get all the irreducible subquotients. To get claims on quotients and subrepresentations, we proceed as follows. Assume $s = \frac{n+1}{2}$. We then have an epimorphism
$$\nu^n \times \nu^{n-1} \times \dots \times \nu^1 \rtimes 1 \twoheadrightarrow \nu^{\frac{n+1}{2}} \rtimes 1,$$
and we know that $\nu^n \times \nu^{n-1} \times \dots \times \nu^1 \rtimes 1$ has $L(\nu^n, \nu^{n-1}, \dots, \nu^1; 1) = 1_{Sp_{2n}(F)}$ as the unique quotient, so the claim of the first part follows. Analogously, we have for $s = \frac{n-1}{2}$ an epimorphism
$$\nu^{n-1} \times \nu^{n-2} \times \dots \times \nu^0 \rtimes 1 \twoheadrightarrow \nu^{\frac{n-1}{2}} \rtimes 1.$$

The representation $L(\nu^{n-1}, \nu^{n-2}, \ldots, \nu^0 \rtimes 1)$ is the unique quotient of $\nu^{n-1} \times \nu^{n-2} \times \cdots \times \nu^0 \rtimes 1$, and the second claim follows. As for the claims in the third part, we use the description of the Jacquet modules of the Aubert dual of a representation. Indeed, by [6], we have

$$r_{M,G}(\hat{\pi}) = w \circ D_{w^{-1}(M)} \circ r_{w^{-1}(M),G}(\pi). \tag{4.1}$$

Here, for a reductive group G and its Levi subgroup M (where $P = MN$ is the corresponding parabolic subgroup), $r_{M,G}$ denotes the Jacquet module with respect to M (more precisely, to P). $D_{w^{-1}(M)}$ denotes Aubert involution on the representations of the group $w^{-1}(M)$; here w denotes a certain element of the Weyl group of G, but for our choice of maximal Levi subgroup $GL_n(F)$ of $Sp_{2n}(F)$, we have $w^{-1}(M) = M$. The above relation is meant on the level of the Grothendieck group. Using the Frobenius reciprocity, to prove that $\widehat{L(\nu^s\delta([\nu^{-\frac{n-1}{2}}, \nu^{\frac{n-1}{2}}]); 1)}$ is the unique irreducible subrepresentation of $\nu^s 1_{GL_n} \rtimes 1$ it is enough to show that $\nu^s \otimes 1$ appears with the multiplicity one in $r_{M,Sp_{2n}(F)}(\nu^s 1_{GL_n} \rtimes 1)$ and that it appears (as a subquotient) in $r_{M,Sp_{2n}(F)}(\widehat{L(\nu^s\delta([\nu^{-\frac{n-1}{2}}, \nu^{\frac{n-1}{2}}]); 1)})$. But, as a basic property of Langlands quotients, we know that $\nu^{-s}\delta([\nu^{-\frac{n-1}{2}}, \nu^{\frac{n-1}{2}}]) \otimes 1$ appears in $r_{M,Sp_{2n}(F)}(\nu^s\delta([\nu^{-\frac{n-1}{2}}, \nu^{\frac{n-1}{2}}]) \rtimes 1)$ with multiplicity one, and that it appears in $r_{M,Sp_{2n}(F)}(L(\nu^s\delta([\nu^{-\frac{n-1}{2}}, \nu^{\frac{n-1}{2}}]); 1)$. Now the claim follows immediately from (4.1). To see that both of $\widehat{\sigma_1}$ and $\widehat{\sigma_2}$ are irreducible quotients, we first note that this is equivalent to showing that $\sigma_i \hookrightarrow \nu^{-s} 1_{GL_n} \rtimes 1$ (this easily follows from a result of Waldspurger, cf. Theorem 2-6 of [15]). Now we use the proof of Lemma 4.6 of [22], but applied to Aubert duals of the representations treated in that Lemma. Let $1_{GL_{n-2s}(F)} \rtimes 1 = \widehat{T_1} \oplus \widehat{T_2}$, where T_1 and T_2 are irreducible and tempered representations. Then, using the same Jacquet module calculation from Lemma 4.6 of [22] and using (4.1), we show that $\widehat{\sigma_i}$ is the unique irreducible quotient of $\nu^{n/2} 1_{GL_{2s}} \rtimes \widehat{T_j}$, for one of the $\widehat{T_j}$. We can take that

$$\widehat{\sigma_i} \hookrightarrow \nu^{-n/2} 1_{GL_{2s}} \rtimes \hat{T}_i, \ i = 1, 2. \tag{4.2}$$

We comment on this choice below. On the other hand

$$\nu^{-n/2} 1_{GL_{2s}} \rtimes \hat{T}_1 \oplus \nu^{-n/2} 1_{GL_{2s}} \rtimes \hat{T}_2 = \nu^{-n/2} 1_{GL_{2s}} \times 1_{GL_{n-2s}} \rtimes 1,$$

and $\nu^{-s} 1_{GL_n} \rtimes 1 \hookrightarrow \nu^{-n/2} 1_{GL_{2s}} \times 1_{GL_{n-2s}} \rtimes 1$, and since $\widehat{\sigma_i}, \ i = 1, 2$ occur with the multiplicity one in $\nu^{-n/2} 1_{GL_{2s}} \times 1_{GL_{n-2s}} \rtimes 1$, we are done. We note that in this case (of unramified principal series), the Iwahori-Matsumoto involution (which in this case coincides with the Aubert involution) takes generic to spherical representations, and the last claim follows. $\square$

LEMMA 4.5. *Assume that* χ *is a quadratic character of* F^*, $\chi \neq 1$, *and* $s > 0$ *such that* $s - \frac{n-1}{2} \in \mathbb{Z}$ *and* $s \leq \frac{n+1}{2}$. *Then, in the appropriate Grothendieck group, we have*

1. *Assume* $s = \frac{n+1}{2}$. *Then the representation* $\nu^s \chi 1_{GL_n} \rtimes 1$ *is irreducible.*
2. *Assume* $s \leq \frac{n-1}{2}$. *Then, analogously as in the case of trivial character, we have:*

$$\nu^s \chi 1_{GL_n} \rtimes 1 = \widehat{\sigma_1} + \widehat{\sigma_2} + L(\nu^s \delta(\widehat{[\nu^{-\frac{n-1}{2}}, \nu^{\frac{n-1}{2}}]}); 1).$$

Here, $L(\nu^s \delta(\widehat{[\chi\nu^{-\frac{n-1}{2}}, \chi\nu^{\frac{n-1}{2}}]}); 1)$ *is the unique subrepresentation of* $\nu^s \chi 1_{GL_n} \rtimes 1$ *and* $\widehat{\sigma_1}$ *and* $\widehat{\sigma_2}$ *are irreducible quotients.*

PROOF. Straightforward from Theorem 2.1 and Proposition 3.1(ii) of [24]. $\square$

Now, we use some results of [22] to describe more thoroughly the discrete series whose Aubert duals appear as subquotients in our degenerate principal series. The results in [22], at the time of publishing, were dependent on so-called basic assumption, which describes the cuspidal reducibility in the generalized rank-one case. But, the results we use are independent of these assumptions, because we are, essentially, in the principal series case, where the cuspidal reducibilities boil down to the SL_2–case, which is very well known, without any assumptions. Meanwhile, the work of Arthur confirmed the basic assumption from [22]. The discrete series of classical groups are, by [22], described by their partial cuspidal support (in our case, the trivial representation of the trivial group)-this is the cuspidal representation of the smaller classical group from which the discrete series (along some representation on the GL-part of a Levi subgroup) is induced; then with so called Jordan block (this, roughly, describes the GL-part of this induced representation), and some function on this Jordan block. For our σ_i, $i = 1, 2$ the Jordan block is $\{(1_{GL_1}, 1), (\chi, n - 2s), (\chi, n + 2s)\}$. The information in the parametrization of the discrete series which distinguishes σ_1 from σ_2 is a ± 1–valued function, so called ε-function, on the Jordan block. Then, for one of these discrete series $\varepsilon(\chi, n - 2s) = \varepsilon(\chi, n + 2s) = 1$, and for the other $\varepsilon(\chi, n - 2s) = \varepsilon(\chi, n + 2s) = -1$ (actually for $\chi = 1$ the function ε is defined only on pair $\{(\chi, n-2s), (\chi, n+2s)\}$ but again the same notation is used as if it were defined on the particular element). We know that the longest normalized intertwining operator $N(w_0', \Lambda_{s,p}') : \chi 1_{GL_{n-2s}} \rtimes 1 = \widehat{T_1} \oplus \widehat{T_2} \to \chi 1_{GL_{n-2s}} \rtimes 1$ acts as identity on one the summands, and as minus identity on the other (cf. Lemma 4.6). We take, from now on, that the one on which this intertwining operator acts as the identity, is $\widehat{T_1}$; the other one is $\widehat{T_2}$. With that convention, note that if χ is unramified, $\widehat{T_1}$ is spherical (because $N(w_0', \Lambda_{s,p}')$ must act as the identity on the spherical vector).

Lemma 4.6. *Assume F is a non-archimedean local field of characteristic zero and let $\chi^2 = 1$ be a character of F^*. For an odd positive integer m, greater or equal to 3, we define $\pi = \zeta(\chi\nu^{-\frac{m-1}{2}}, \chi\nu^{\frac{m-1}{2}}) \rtimes 1$. Let T be normalized intertwining operator (normalized as in (2.4)) of $Sp_{2m}(F)$ attached to the longest element of the Weyl group, modulo the longest element in the Weyl group of the Siegel Levi subgroup. Then, the representation π is a sum of two irreducible representations, and T is holomorphic, acting as the identity on one subrepresentation, and minus identity on the other.*

Proof. The operator T is holomorphic and unitary because of the properties of the normalization which is used. This is discussed in detail, for example, in [26], before Theorem 6-21. Also, the basic properties of this normalization guarantee that $T^2 = 1$. These observations are valid also for archimedean case. Now, the representation π is a sum of mutually non-isomorphic irreducible representations in both archimedean case ([21], Theorems 5.5. and 5.6) and non-archimedean case ([12], Theorem 10.16, [19]); in the latter case the length of the representation is two. This means that T can act as plus or minus identity on each subrepresentation.

In the non-archimedean case, for the Aubert dual $\widehat{\pi}$ we have

$$\widehat{\pi} = \delta([\chi\nu^{-\frac{m-1}{2}}, \chi\nu^{\frac{m-1}{2}}]) \rtimes 1,$$

and Harish-Chandra commuting algebra theorem says that the analogous operator $\widehat{T}$ has to act as identity on one subrepresentation, and as minus identity on the other. But this claim transfers to our case in question (cf. [8], Theorem 5.1). ∎

Later on, we will see that the representations $\widehat{\sigma_i}$ appear as images on the local places of the global representations which are the automorphic images of Eisenstein series. It is important to be able to describe these representations explicitly; i.e., as Langlands quotients. As we saw from (4.2), we have

$$\chi\zeta(\nu^{\frac{n+1}{2}-s}, \nu^{\frac{n-1}{2}+s}) \rtimes \widehat{T_i} \twoheadrightarrow \widehat{\sigma_i},$$

and $\widehat{\sigma_i}$ is the unique quotient in this representation. Since $\widehat{T_i}$ has strictly smaller exponents from those appearing in $\chi\zeta(\nu^{\frac{n+1}{2}-s}, \nu^{\frac{n-1}{2}+s})$ in its cuspidal support, the main problem is to determine $\widehat{T_i}$. We do it by induction.

Lemma 4.7. *Let $St_{SL_2(F)}$ denote the Steinberg representation of $SL_2(F)$. Then $\nu^0 \rtimes St_{SL_2(F)}$ is reducible, $\nu^0 \rtimes St_{SL_2(F)} = T_1'' \oplus T_2''$, where T_2'' is the unique common tempered subquotient of $\nu^0 \rtimes St_{SL_2(F)}$ and $\zeta(\nu^0, \nu^1) \rtimes 1$. Let $m \in \mathbb{Z}_{\geq 1}$. Then*

$$\zeta(\nu^{-m}, \nu^m) \rtimes 1 = \widehat{T_1} \oplus \widehat{T_2},$$

where $\widehat{T_1}$ *is a spherical subquotient. Then,*

$$\widehat{T_1} = L(\nu^m, \nu^m, \nu^{m-1}, \nu^{m-1}, \dots, \nu^2, \nu^2, \nu^1, \nu^1; \nu^0 \rtimes 1),$$
$$\widehat{T_2} = L(\nu^m, \nu^m, \nu^{m-1}, \nu^{m-1}, \dots, \nu^2, \nu^2, \nu^1; T_2'').$$

Assume now that $\chi^2 = 1$ *but* $\chi \neq 1$. *Then* $\chi \rtimes 1 = T_1^0 \oplus T_2^0$ *is a reducible representation of* $SL_2(F)$. *Then, if* $m \in \mathbb{Z}_{\geq 1}$ *and* $\chi\zeta(\nu^{-m}, \nu^m) \rtimes 1 = \widehat{T_1} \oplus \widehat{T_2}$, *we have*

$$\widehat{T}_i = L(\chi\nu^m, \chi\nu^m, \chi\nu^{m-1}, \chi\nu^{m-1}, \dots, \chi\nu^2, \chi\nu^2, \chi\nu^1, \chi\nu^1; T_i^0),\ i = 1, 2.$$

Proof. In the first part of the proof the main step is the case $m = 1$. Indeed, we have

$$\widehat{T_1} \hookrightarrow \nu^{-1} \times \nu^0 \times \nu^1 \rtimes 1 \xrightarrow{A} \nu^{-1} \times \nu^0 \times \nu^{-1} \rtimes 1$$
$$\xrightarrow{B} \nu^{-1} \times \nu^{-1} \times \nu^0 \rtimes 1.$$

Here A and B are normalized intertwining operators. The kernel of the operator A is $\nu^{-1} \times \nu^0 \rtimes St_{SL_2(F)}$, so that $\widehat{T_1}$, as a spherical subquotient, cannot be a subquotient of that kernel, so we have

$$\widehat{T_1} \hookrightarrow \nu^{-1} \times \nu^0 \times \nu^{-1} \rtimes 1.$$

Analogously, the kernel of the operator B is $\nu^{-1} \times \nu^{-\frac{1}{2}} St_{GL_2(F)} \rtimes 1$, so that we have

$$\widehat{T_1} \hookrightarrow \nu^{-1} \times \nu^{-1} \times \nu^0 \rtimes 1,$$

but the representation on the right-hand side has a unique subrepresentation, namely $L(\nu^1, \nu^1; \nu^0 \rtimes 1)$.

On the other hand

$$\widehat{T_2} \hookrightarrow \zeta(\nu^{-1}, \nu^0) \times \nu^1 \rtimes 1.$$

So we have that, in the appropriate Grothendieck group

$$\widehat{T_2} \leq \zeta(\nu^{-1}, \nu^0) \rtimes 1_{SL_2(F)} \text{ or } \widehat{T_2} \leq \zeta(\nu^{-1}, \nu^0) \rtimes St_{SL_2(F)}.$$

On the other hand, we know that $\widehat{T_1} \leq \zeta(\nu^{-1}, \nu^0) \rtimes 1_{SL_2(F)}$ since $\widehat{T_1}$ is spherical. When we calculate the multiplicity of $\zeta(\nu^{-1}, \nu^1) \otimes 1$ in the appropriate Jacquet module of $\zeta(\nu^{-1}, \nu^0) \rtimes 1_{SL_2(F)}$, it is one (and each of $\widehat{T}_i,\ i = 1, 2$ has that part in its Jacquet module). We conclude that we must have

$$\widehat{T_2} \leq \zeta(\nu^{-1}, \nu^0) \rtimes St_{SL_2(F)}.$$

Also, we have $\nu^0 \rtimes St_{SL(2)} = T_1'' \oplus T_2''$, for some tempered representations T_1'' and T_2''. Let T_2'' be the unique common subquotient of $\nu^0 \rtimes St_{SL(2)}$ and $\zeta(\nu^0, \nu^1) \rtimes 1$. We then have

$$0 \to T_2'' \hookrightarrow \zeta(\nu^0, \nu^1) \rtimes 1 \twoheadrightarrow L(\nu^1; \nu^0 \rtimes 1) \to 0.$$

Thus, if $\widehat{T_2}$ is not in the kernel of the intertwining operator C below

$$\widehat{T_2} \hookrightarrow \nu^{-1} \times \zeta(\nu^0, \nu^1) \rtimes 1 \xrightarrow{C} \nu^{-1} \rtimes L(\nu^1; \nu^0 \rtimes 1),$$

we would have $\widehat{T_2} \hookrightarrow \nu^{-1} \rtimes L(\nu^1; \nu^0 \rtimes 1)$, but then $\nu^{-1} \otimes L(\nu^1; \nu^0 \rtimes 1)$ would have to appear in the appropriate Jacquet module of $\widehat{T_2}$, which is impossible (we actually check the appropriate Jacquet module of $\zeta(\nu^{-1}, \nu^0) \rtimes St_{SL_2(F)}$ but it does not appear there either). The kernel of C is $\nu^{-1} \rtimes T_2''$, so we must have $\widehat{T_2} \hookrightarrow \nu^{-1} \rtimes T_2''$ and the claim follows.

Now, for general $m \geq 2$, we have

$$\begin{aligned}\zeta(\nu^{-m}, \nu^m) \rtimes 1 &\hookrightarrow \zeta(\nu^{-m}, \nu^{m-1}) \times \nu^m \rtimes 1 \cong \zeta(\nu^{-m}, \nu^{m-1}) \times \nu^{-m} \rtimes 1 \\ &\cong \nu^{-m} \times \zeta(\nu^{-m}, \nu^{m-1}) \rtimes 1 \hookrightarrow \nu^{-m} \times \nu^{-m} \times \zeta(\nu^{-(m-1)}, \nu^{m-1}) \rtimes 1\end{aligned}$$

and we just apply the induction on the representation $\zeta(\nu^{-(m-1)}, \nu^{m-1}) \rtimes 1$. The case of $\chi \neq 1$ is similar but easier. ∎

Corollary 4.8. *Assuming the notation from the previous lemmas form this section, we let* $\frac{n-1}{2} - s \in \mathbb{Z}_{>0}$. *If* $\chi = 1$ *then*

$$\widehat{\sigma_1} = L(\nu^{\frac{n-1}{2}+s}, \nu^{\frac{n-3}{2}+s}, \ldots, \nu^{\frac{n+1}{2}-s}, \nu^{\frac{n-1}{2}-s}, \nu^{\frac{n-1}{2}-s}, \ldots, \nu^1, \nu^1; \nu^0 \rtimes 1),$$

and

$$\widehat{\sigma_2} = L(\nu^{\frac{n-1}{2}+s}, \nu^{\frac{n-3}{2}+s}, \ldots, \nu^{\frac{n+1}{2}-s}, \nu^{\frac{n-1}{2}-s}, \nu^{\frac{n-1}{2}-s}, \ldots, \nu^2, \nu^2, \nu^1; T_2'').$$

If $\chi \neq 1$ *we get analogously*

$$\widehat{\sigma_i} = L(\nu^{\frac{n-1}{2}+s}, \nu^{\frac{n-3}{2}+s}, \ldots, \nu^{\frac{n+1}{2}-s}, \nu^{\frac{n-1}{2}-s}, \nu^{\frac{n-1}{2}-s}, \ldots, \nu^1, \nu^1; T_i^0),\ i = 1, 2.$$

4.2. *The archimedean (real) case.* Throughout this subsection, we use the results of [21], where the author studies the Siegel case of the degenerate series for $Sp_{2n}(\mathbb{R})$. Here $\varepsilon \in \{0, 1\}$.

Thus, we have

Corollary 4.9. *Assume* $n \geq 3$. *The representation* $\chi_\infty \nu^s 1_{GL_n} \rtimes 1$ *is reducible if and only if* $\chi_\infty = sgn^\varepsilon$ *and* $s + \frac{n+1}{2} \in \mathbb{Z}$.

Later, we need the following

Lemma 4.10. *Assume* $\chi_\infty = 1$ *and* $\frac{n-1}{2} - s \in \mathbb{Z}_{\geq 0}$. *Then, the spherical vector in the representation* $\nu_\infty^{-s} 1_{GL_n} \rtimes 1$ *of* $Sp(2n, \mathbb{R})$ *spans an irreducible subrepresentation.*

Proof. This easily follows from Theorems 5.2, 5.4, and 5.6 of [21]. There the socle series of the degenerate representation $\nu_\infty^{-s} 1_{GL_n} \rtimes 1$ are studied, and the subquotients are described by their K-types (and not as Langlands quotients). It is a matter of a straightforward check to see that the socle of representation $\nu_\infty^{-s} 1_{GL_n} \rtimes 1$ contains a subrepresentation which contains the trivial K-type. ∎

REMARK 4.11. Note that from the proof of Lemma 4.6, it follows that, in the archimedean case, we can take $sgn^{\varepsilon}1_{GL_m} \rtimes 1 = \widehat{T_{1,\infty}} \oplus \widehat{T_{2,\infty}}$ (for m odd greater than 3), where we define $\widehat{T_{1,\infty}} = \{f \in sgn^{\varepsilon}1_{GL_m} \rtimes 1 : Tf = f\}$ and $\widehat{T_{2,\infty}} = \{f \in sgn^{\varepsilon}1_{GL_m} \rtimes 1 : Tf = -f\}$, where T is the normalized intertwining operator, defined analogously as in Lemma 4.6. The more difficult situation in the archimedean case comes from the possibility that $\widehat{T_{i,\infty}}, i = 1, 2$ are reducible (or zero).

5. The local intertwining operators

In this section we analyze the action of normalized intertwining operators occurring in (2.5), where $w \in [W/W_{\Delta\setminus\{\alpha_n\}}]$. We recall (3.1) for the form of the inducing character of the maximal global torus. When we untangle action of the Weyl group on Λ_s (cf.(2.1)), we get that $w = p\varepsilon$ consists of shuffles (cf. [19] for the definition) of $(\Lambda_s)_1, \dots, (\Lambda_s)_j$ among $(\Lambda_s)_n^{-1}, \dots, (\Lambda_s)_{j+1}^{-1}$ (in that order), so that $w(\Lambda_s)$ looks something like this

$$(\Lambda_s)_n^{-1} \otimes (\Lambda_s)_1 \otimes (\Lambda_s)_2 \otimes \cdots \otimes (\Lambda_s)_j \otimes (\Lambda_s)_{j+1}^{-1}, \tag{5.1}$$

where $(\Lambda_s)_i = \chi\nu^{s-\frac{n-1}{2}+i-1}$. So, each of the (normalized) local intertwining operators is composed from the rank-one intertwining operators, starting from the one (in $SL(2)$) from $(\Lambda_{s,p})_n \rtimes 1$ to $(\Lambda_{s,p})_n^{-1} \rtimes 1$, then $GL(2)$-operators which jump over $(\Lambda_{s,p})_j, (\Lambda_{s,p})_{j-1} \dots$ until $(\Lambda_{s,p})_n^{-1}$ is positioned on its final place, e.g. like in (5.1). Then we repeat the same procedure with $(\Lambda_{s,p})_{n-1}$ and so on until $(\Lambda_{s,p})_{j+1}$. Thus, our normalized intertwining operators are composed from the rank-one operators ($p \leq \infty$)

- $\chi_p\nu^{s-\frac{n-1}{2}+i_1-1} \rtimes 1 \to \chi_p^{-1}\nu^{-(s-\frac{n-1}{2}+i_1-1)} \rtimes 1$, where $i_1 = j+1, \dots, n$
- $$\chi_p\nu^{s-\frac{n-1}{2}+i_2-1} \times \chi_p^{-1}\nu^{-(s-\frac{n-1}{2}+i_1-1)} \to \chi_p^{-1}\nu^{-(s-\frac{n-1}{2}+i_1-1)} \times \chi_p\nu^{s-\frac{n-1}{2}+i_2-1},$$

  where $i_1 = j+1, \dots, n$ and $i_2 \in \{1, \dots, j\}$.

The normalized intertwining operators are holomorphic in "standard" cases (i.e. $s - \frac{n-1}{2} + i_2 \geq -(s - \frac{n-1}{2} + i_1)$ for $GL(2)$-cases and $s - \frac{n-1}{2} + i_1 \geq 0$ for the $SL(2)$-cases) and in non-standard cases if the inducing representations are irreducible. Reducibilities for $SL(2, \mathbb{Q}_p)$ and $GL(2, \mathbb{Q}_p)$ are well–known; we first have (cf. [27], Theorem 2.4 and Theorem 2.5)

LEMMA 5.1. *The representation $\chi_p\nu^{s-\frac{n-1}{2}+i_1-1} \rtimes 1$ of $SL(2, \mathbb{Q}_p)$ is reducible if and only if*

1. $p < \infty : \chi_p = 1$ *and* $s - \frac{n-1}{2} + i_1 - 1 = \pm 1$ *or* $\chi_p^2 = 1$, $\chi_p \neq 1$ *and* $s - \frac{n-1}{2} + i_1 - 1 = 0$;
2. $p = \infty : s - \frac{n-1}{2} + i_1 - 1 \in \mathbb{Z}$ *and* $\chi_\infty = sgn^{\varepsilon}$, *where* $\varepsilon \equiv (s - \frac{n-1}{2} + i_1)(\mathrm{mod} 2)$.

LEMMA 5.2. *The representation* $\chi_p\nu^{s-\frac{n-1}{2}+i_2-1} \times \chi_p^{-1}\nu^{-(s-\frac{n-1}{2}+i_1-1)}$ *of* $GL(2,\mathbb{Q}_p)$ *is reducible if and only if*

1. $p<\infty : \chi_p^2 = 1$ *and* $s-\frac{n-1}{2}+i_2-1+s-\frac{n-1}{2}+i_1-1 = 2s-n-1+i_1+i_2 = \pm 1$;
2. $p=\infty : \chi_\infty^2 = sgn^{k+1}$ *and* $2s-n-1+i_1+i_2 = k$, *for some* $k \in \mathbb{Z}\setminus\{0\}$; *i.e.,* $2s-n+1+i_1+i_2$ *is an odd integer, and* $\chi_\infty^2 = 1$.

We immediately conclude

COROLLARY 5.3. *Assume* $s \geq \frac{n-1}{2}$ *or* $\chi_p^2 \neq 1$, $p \leq \infty$. *Then all the local intertwining operators in (2.5) are holomorphic.*

PROOF. If $s \geq \frac{n-1}{2}$ we are in, aforementioned, standard cases. If $\chi_p^2 \neq 1$ all the local representations mentioned in Lemmas 5.2 and 5.1 are irreducible. □

Now we assume $\chi_p^2 = 1$. We can be more precise with the occurrence of the poles of the intertwining operators that come from the GL-poles. Since we are interested in the action of normalized intertwining operators not on the whole induced principal series of $Sp(2n,\mathbb{Q}_p)$, but on $\chi_p\nu^s 1_{GL_n} \rtimes 1$, from the discussion about the action of the intertwining operators described at the beginning of this section, it follows that we are really interested in the poles of the intertwining operator

$$\begin{aligned}(5.2)\qquad &\zeta(\chi_p\nu^{s-\frac{n-1}{2}+t_{i_1}-1}, \chi_p\nu^{s-\frac{n-1}{2}+i_1-2}) \times \chi_p\nu^{-(s-\frac{n-1}{2}+i_1-1)}\\ &\quad \to \chi_p\nu^{-(s-\frac{n-1}{2}+i_1-1)} \times \zeta(\chi_p\nu^{s-\frac{n-1}{2}+t_{i_1}}, \chi_p\nu^{s-\frac{n-1}{2}+i_1-2}).\end{aligned}$$

Here t_{i_1} denotes the index of the last one among $(\Lambda_{s,p})_1, \cdots (\Lambda_{s,p})_j$ where, at the end, $(\Lambda_{s,p})_{i_1}^{-1}$ will be situated. We have three cases. If $-(s-\frac{n-1}{2}+i_1-1) \leq s-\frac{n-1}{2}+t_{i_1}-1$ then we are in the standard case (i.e., this operator factorizes in rank-one operators action on standard representations), thus the operator is holomorphic. If, on the other hand, $-(s-\frac{n-1}{2}+i_1-1) \in [s-\frac{n-1}{2}+t_{i_1}-1, s-\frac{n-1}{2}+i_1-2]$, the representation $\zeta(\chi_p\nu^{s-\frac{n-1}{2}+t_{i_1}}, \chi_p\nu^{s-\frac{n-1}{2}+i_1-2}) \times \chi_p\nu^{-(s-\frac{n-1}{2}+i_1-1)}$ is irreducible, (for all $p \leq \infty$) so the holomorphy follows. Assume that $-(s-\frac{n-1}{2}+i_1-1) > s-\frac{n-1}{2}+i_1-2$ and that the representation $\zeta(\chi_p\nu^{s-\frac{n-1}{2}+t_{i_1}-1}, \chi_p\nu^{s-\frac{n-1}{2}+i_1-2}) \times \chi_p\nu^{-(s-\frac{n-1}{2}+i_1-1)}$ is reducible. This means that $\frac{n+2-2s}{2} > i_1 \geq j+1$, so $j < \frac{n-2s}{2}$.

Now we analyze more thoroughly the case when a possible pole comes from the $SL(2)$ situations (cf. Lemma 5.1). For $p \leq \infty$, to have a possible pole, we need to have $s-\frac{n-1}{2}+i_1 \in \mathbb{Z}$ and $\chi_p^2 = 1$. If $p<\infty$, the only possibility of a pole occurs if we have $\chi_p = 1$ and the intertwining operator $\nu^{s-\frac{n-1}{2}+i_1-1} \rtimes 1 \to \nu^{-(s-\frac{n-1}{2}+i_1-1)} \rtimes 1$ with $s-\frac{n-1}{2}+i_1-1 = -1$ occurring in the factorizations of the intertwining operator $N(w,\Lambda_{s,p})$. Note that this implies $w \in Y_j^n$, with $i_1 \geq j+1$, i.e. $i_1 = \frac{n-1}{2}-s \geq j+1$. Analogously, for

$p=\infty$, to have a pole we must have $s-\frac{n-1}{2}+i_1-1\leq -1$ with $i_1\geq j+1$, so again $j+1\leq \frac{n-1}{2}-s$.

We thus have, form the discussion above on GL– and SL–situations, the following

Lemma 5.4. *Assume $j>\frac{n-1}{2}-s$. Then, all the normalized intertwining operators $N(w,\Lambda_{s,p})$ for $w\in Y_j^n$, are holomorphic.*

To deal with $j\leq \frac{n-1}{2}-s$, we first resolve the case of $N(w_0,\Lambda_{s,p})$ (we assume that $\frac{n-1}{2}-s>0$). Recall that an element w_0 is described by (3.6) (cf. Proposition 3.2).

Lemma 5.5. *Assume $p<\infty$. Then, if $\frac{n-1}{2}-s\in\mathbb{Z}_{\geq 0}$, $r(\Lambda_{s,p},w_0)$ is holomorphic and non-zero for $s<\frac{n-1}{2}$ and has a zero of the first order if $s=\frac{n-1}{2}$. If $\frac{n-1}{2}-s\in\frac{1}{2}+\mathbb{Z}_{\geq 0}$ then, $r(\Lambda_{s,p},w_0)$ has a zero of the first order.*

Proof. When we cancel out the factors in the numerator and the denominator in the local versions of the expressions (3.3) and (3.5) with $\chi_p=1$, $j=0$, we get

$$r(\Lambda_{s,p},w_0)^{-1}=\frac{L_p(s-\frac{n-1}{2},1)}{L_p(s+\frac{n+1}{2},1)(*)}\prod_{k=1}^{n-1}\frac{L_p(2s+2k-n,1)}{L_p(2s+k,1)(**)},$$

where $(*)$ and $(**)$ are products of ϵ factors. Now the conclusion follows. □

Lemma 5.6. *Assume $p<\infty$, $\chi_p=1$. Then, if $\frac{n-1}{2}-s\in\mathbb{Z}_{>0}$, the intertwining operator $A(w_0,\Lambda_{s,p})$ is holomorphic (and non-zero). It has a pole of the first order for $s=\frac{n-1}{2}$ and if $\frac{n-1}{2}-s\in\frac{1}{2}+\mathbb{Z}_{\geq 0}$.*

Proof. We denote by $\overline{N}$ the opposite unipotent radical to N, which is the unipotent radical of the standard Siegel parabolic subgroup. To investigate the poles, it is enough, by a result of Rallis (cf. [32]), to investigate the poles of the expression $A(w_0,\Lambda_{s,p})f(e)$, where $f\in\chi_p\nu^s 1_{GL_n}\rtimes 1$ is such that it has a compact support modulo P inside $P\overline{N}$. So, for s big enough, we have

$$A(w_0,\Lambda_{s,p})f(e)=\int_N f(w_0^{-1}u)\mathrm{d}u=\int_{\mathcal{C}} f\left(\begin{pmatrix}0 & -I_n\\ I_n & X\end{pmatrix}\right)\mathrm{d}u.$$

Since the support of f lies in $P\overline{N}$, we get that X must be regular, so the last integral is over the set $\mathcal{C}=\{X\in GL(\mathbb{Q}_p): X^t=J_nXJ_n\}$. In that case $\begin{pmatrix}0 & -I_n\\ I_n & X\end{pmatrix}=\begin{pmatrix}X^{-1} & -I_n\\ 0 & X\end{pmatrix}\begin{pmatrix}I_n & 0\\ X^{-1} & I_n\end{pmatrix}$, so that

$$A(w_0,\Lambda_{s,p})f(e)=\int_{\mathcal{C}}|\mathrm{det}X|^{-s-\frac{n+1}{2}} f\left(\begin{pmatrix}I_n & 0\\ X^{-1} & I_n\end{pmatrix}\right)\mathrm{d}X.$$

Since f restricted on $\left\{\begin{pmatrix} I_n & 0 \\ X^{-1} & I_n \end{pmatrix} : X \in M_n(\mathbb{Q}_p),\ X^t = J_n X J_n\right\}$ is a Schwartz function on that space, we can apply the results of Igusa, and Piatetski-Shapiro and Rallis ([28], Theorem in Appendix 1), because this integral is zeta-integral related to a prehomogeneous vector space formed by symmetric matrices. On the regular part of this space, $GL_n(\mathbb{Q}_p)$ acts $X \mapsto gXg^t$ with a finite number of orbits. Note that $GL_n(\mathbb{Q}_p)$–invariant measure on $\mathcal{C}$ is given by $\mathrm{d}^*X = \frac{\mathrm{d}X}{|\det|^{\frac{n+1}{2}}}$ (cf. [34]). We use that for the change of variables $X \mapsto X^{-1}$ in the previous relation, so that we obtain exactly $Z(f,s)$ as in ([28], Appendix 1). We get that the possible poles of this integral are the poles of the expression

$$\left(\prod_{l=l_1}^{n} L_p(2s+n-2l+2,1)\right) L_p(s-\frac{n-1}{2},1), \tag{5.3}$$

where $l_1 = [\frac{n}{2}] + \varepsilon_n$, where ε_n is 1 if n is even, and equal to 2 if n is odd. Now, the claim of lemma follows. □

Corollary 5.7. *Assume $2s \in \mathbb{Z}$ and $0 < 2s < n-1$. Then, for $p < \infty$, $N(w_0, \Lambda_{s,p})$ is holomorphic and non-zero.*

Note that the result of ([28], Theorem in Appendix 1) holds also in the archimedean case. Also, to check the normalization factors, we need to check both the case $\chi_\infty = 1$ and the case $\chi_\infty = sgn$, but the same kind of cancelations between the zeros of the normalization factors and the poles of the intertwining operators occur. Here $L_\infty(s,1) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$ and $L_\infty(s,sgn) = \pi^{-\frac{s+1}{2}}\Gamma(\frac{s+1}{2})$. We conclude

Corollary 5.8. *Assume $p = \infty$ and $2s \in \mathbb{Z}$ and $0 < 2s \leq n-1$. Then, the normalized intertwining operator $N(w_0, \Lambda_{s,p})$ is holomorphic and non-zero.*

Lemma 5.9. *Assume $j \leq \frac{n-1}{2} - s$. Then, the normalized intertwining operator $N(w, \Lambda_{s,p})$ with $w \in Y_j^n$ is holomorphic for $p \leq \infty$.*

Proof. We construct one special element in Y_j^n. Let $\overline{w_j} \in Y_j^n$ with $\overline{w_j} = p\varepsilon$ be defined in the following fashion: $p(i) = n-j+i,\ i = 1,\ldots,j$ and $p(i) = n+1-i;\ i = j+1,\ldots,n$. Then we can describe the action of $\overline{w_j}$ in the following way.

(5.4)

$$\begin{aligned}
\chi_p\nu^s \rtimes 1 &\hookrightarrow \chi_p\nu^s\zeta(-\frac{n-1}{2}, -\frac{n-1}{2}+j-1) \times \chi_p\nu^s\zeta(-\frac{n-1}{2}+j, \frac{n-1}{2}) \rtimes 1 \\
&\xrightarrow{N(w_0')} \chi_p\nu^s\zeta(-\frac{n-1}{2}, -\frac{n-1}{2}+j-1) \times \chi_p\nu^{-s}\zeta(-\frac{n-1}{2}, \frac{n-1}{2}-j) \rtimes 1 \\
&\xrightarrow{C} \chi_p\nu^{-s}\zeta(-\frac{n-1}{2}, \frac{n-1}{2}-j) \times \chi_p\nu^s\zeta(-\frac{n-1}{2}, -\frac{n-1}{2}+j-1) \rtimes 1.
\end{aligned}$$

Note that the operator $N(\overline{w_j}, \Lambda_{s,p})$ is holomorphic: the operator $N(w_0')$ is induced from $Sp_{2(n-j)}$–operator and is attached to the longest element in the appropriate Weyl group there, so we can apply Corollaries 5.7 and 5.8 for $Sp_{2(n-j)}(\mathbb{Q}_p)$ and $s+\frac{j}{2}$ instead of s. The operator C is holomorphic, induced from the GL–case. Its holomorphy follows from the fact that in the case $j \leq \frac{n-1}{2}-s$, the representation $\chi_p\nu^{-s}\zeta(-\frac{n-1}{2}, \frac{n-1}{2}-j)\times\chi_p\nu^s\zeta(-\frac{n-1}{2}, -\frac{n-1}{2}+j-1)$ is irreducible (for $p \leq \infty$). Then the normalized intertwining operator maps (a χ_p twist of–) the normalized spherical vector to (a χ_p twist of–) the normalized spherical vectors, but these vectors span these representations. Now we can write $N(w, \Lambda_{s,p}) = N(w\overline{w_j}^{-1}, \overline{w_j}(\Lambda_{s,p}))N(\overline{w_j}, \Lambda_{s,p})$. Note that $N(w\overline{w_j}^{-1}, \overline{w_j}(\Lambda_{s,p}))$ is induced from the $GL(n)$–intertwining operator, i.e., we can write it as $p_1\varepsilon_1$, where $\varepsilon_1 = (1, 1, \ldots, 1)$ and $p_1(i) = p(n+1-i)$, $i = 1, \ldots, n-j$ and $p_1(i) = p(i-(n-j))$, $i = n-j+1, \ldots, n$, thus it is an increasing permutation on the first $n-j$ and on the last j elements. So, essentially, $N(w\overline{w_j}^{-1}, \overline{w_j}(\Lambda_{s,p}))$ acts on $\chi_p\nu^{-s}\zeta(-\frac{n-1}{2}, \frac{n-1}{2}-j)\times\chi_p\nu^s\zeta(-\frac{n-1}{2}, -\frac{n-1}{2}+j-1)$, which is irreducible, so again $N(w\overline{w_j}^{-1}, \overline{w_j}(\Lambda_{s,p}))$ is holomorphic. $\square$

We conclude:

Corollary 5.10. *Assume* $2s \in \mathbb{Z}$ *such that* $0 < 2s \leq n-1$ *and* $\chi_p^2 = 1$. *Then, all the intertwining operators* $N(w, \Lambda_{s,p})$ *for* $w \in [W \setminus W_{\Delta\setminus\{\alpha_n\}}]$ *and* $p \leq \infty$, *are holomorphic.*

6. The image of Eisenstein series

In our situation, the image of (the normalized) Eisenstein series (2.3) is isomorphic to its (normalized) constant term (e.g. Lemma 2-8 of [14]). Thus, we have to determine the image of the local intertwining operators appearing in the expansion of the constant term of the Eisenstein series. We have seen that it is important to determine the image of the local intertwining operator $N(w_0, \Lambda_{s,p})$. Namely, the sums of normalizing factors coming from w_0 have the greatest order of a pole, cf. Lemma 3.7, and, as we will see shortly, the other intertwining operators have kernels at least as big as $N(w_0, \Lambda_{s,p})$.

Lemma 6.1. *Assume that* $p < \infty$. *Assume* $\chi_p^2 = 1$, $\frac{n-1}{2} - s \in \mathbb{Z}_{>0}$. *Then, following the notation of Lemma 4.4 and Lemma 4.5, we have*

$$N(w_0, \Lambda_{s,p})(\chi_p\nu^s 1_{GL_n} \rtimes 1) = \widehat{\sigma_1} \oplus \widehat{\sigma_2}.$$

The representation $L(\nu^s\delta(\widehat{[\chi_p\nu^{-\frac{n-1}{2}}, \chi_p\nu^{\frac{n-1}{2}}]}); 1)$ *is in the kernel of all intertwining operators* $N(w, \Lambda_{s,p})$, *where* $w \in Y_j^n$ *with* $j \leq \frac{n-1}{2} - s - 1$. *If* $s = \frac{n-1}{2}$, *then, if* $\chi_p \neq 1$, *the discussion is the same as in the case when* $s < \frac{n-1}{2}$. *If* $s = \frac{n-1}{2}$ *and* $\chi_p = 1$, *then* $N(w_0, \Lambda_{s,p})(\chi_p\nu^s 1_{GL_n} \rtimes 1) = L(\nu^{n-1}, \ldots, \nu^1; \nu^0 \rtimes 1)$.

Proof. As we saw in the proofs of Lemma 4.4 and 4.5,

$$\nu^{n/2}\chi_p 1_{GL_{2s}} \times \chi_p 1_{GL_{n-2s}} \rtimes 1 = \nu^{n/2}\chi_p \rtimes \widehat{T_1} \oplus \nu^{n/2}\chi_p \rtimes \widehat{T_2}.$$

We take $j = 2s$ and define an element $\overline{w_j} = p\varepsilon \in Y_j^n$ as in the proof of Lemma 5.9 for this specific j. The corresponding intertwining operator acting on $\nu^{n/2}\chi_p 1_{GL_{2s}} \times \chi_p 1_{GL_{n-2s}} \rtimes 1$ acts like this

(6.1)

$$\nu^{n/2}\chi_p 1_{GL_{2s}} \times \chi_p 1_{GL_{n-2s}} \rtimes 1 \xrightarrow{N(w_0')} \nu^{n/2}\chi_p 1_{GL_{2s}} \times \chi_p 1_{GL_{n-2s}} \rtimes 1 \xrightarrow{C} \chi_p 1_{GL_{n-2s}} \times \nu^{n/2}\chi_p 1_{GL_{2s}} \rtimes 1.$$

The operator $N(w_0')$ is induced from the $Sp_{2(n-j)}$- operator, acting as the identity on $\widehat{T_1}$ and as minus identity on $\widehat{T_2}$; the image of the operator C is precisely $\chi_p\nu^s 1_{GL_n} \rtimes 1$ which is a subrepresentation of $\chi_p 1_{GL_{n-2s}} \times \nu^{n/2}\chi_p 1_{GL_{2s}} \rtimes 1$. Note that $N(\overline{w_j}, \Lambda'_{s,p})$ is holomorphic on $\nu^{n/2}\chi_p 1_{GL_{2s}} \times \chi_p 1_{GL_{n-2s}} \rtimes 1$ (as $N(w_0')$ and C are holomorphic and non-zero). We then apply $N(w_0, \Lambda_{s,p})$ on $\chi_p\nu^s 1_{GL_n} \rtimes 1$. Let us denote by $\Lambda'_{s,p}$ a character of the maximal torus from which a principal series representation $\mathrm{Ind}_{B_n}^{Sp_{2n}}(\Lambda'_{s,p})$ is induced such that the representation $\nu^{n/2}\chi_p 1_{GL_{2s}} \times \chi_p 1_{GL_{n-2s}} \rtimes 1$ is naturally embedded in $\mathrm{Ind}_{B_n}^{Sp_{2n}}(\Lambda'_{s,p})$. Thus, we have

$$\begin{aligned} &N(w_0, \Lambda_{s,p})N(\overline{w_j}, \Lambda'_{s,p})(\nu^{n/2}\chi_p 1_{GL_{2s}} \times \chi_p 1_{GL_{n-2s}} \rtimes 1) \\ &\quad = N(w_0, \Lambda_{s,p})(\chi_p\nu^s 1_{GL_n} \rtimes 1). \end{aligned}$$

When we calculate $w_0\overline{w_j}$ directly we get $w_0\overline{w_j} = p_1\varepsilon_1$, where $p_1(i) = j + 1 - i,\ i = 1, \dots, j$ and $p_1(i) = i,\ i = j + 1, \dots, n$. The corresponding intertwining operator $N(w_0\overline{w_j}, \Lambda'_{s,p})$ acts as

$$\begin{aligned} &\nu^{n/2}\chi_p 1_{GL_{2s}} \times \chi_p 1_{GL_{n-2s}} \rtimes 1 = \nu^{n/2}\chi_p \rtimes \widehat{T_1} \oplus \nu^{n/2}\chi_p \rtimes \widehat{T_2} \\ &\quad \to \nu^{-n/2}\chi_p \rtimes \widehat{T_1} \oplus \nu^{-n/2}\chi_p \rtimes \widehat{T_2}. \end{aligned}$$

On the other hand,

$$N(w_0\overline{w_j}, \Lambda'_{s,p})|_{\nu^{n/2}\chi_p 1_{GL_{2s}} \rtimes \widehat{T_i}}(\nu^{n/2}\chi_p 1_{GL_{2s}} \rtimes \widehat{T_i}) = \widehat{\sigma_i},\ i = 1, 2.$$

Indeed, if this operator is non-zero and holomorphic, than the image has to be $\widehat{\sigma_i},\ i = 1, 2$ since $\widehat{\sigma_i},\ i = 1, 2$ is the unique irreducible subrepresentation of $\nu^{-n/2}\chi_p 1_{GL_{2s}} \rtimes \widehat{T_i}$ and the unique irreducible quotient of $\nu^{n/2}\chi_p 1_{GL_{2s}} \rtimes \widehat{T_i},\ i = 1, 2$ and appears with the multiplicity one in $\nu^{n/2}\chi_p 1_{GL_{2s}} \rtimes \widehat{T_i},\ i = 1, 2$ (cf. Lemma 4.6 of [22]). The holomorphy and non-vanishing of this operator on $\nu^{n/2}\chi_p 1_{GL_{2s}} \rtimes \widehat{T_i}$ is the content of Lemma 3.5 of [25].

The claim in the case $s = \frac{n-1}{2}$ follows from the fact that representation $L(\nu^{n-1}, \dots, \nu^1; \nu^0 \rtimes 1)$ is the unique irreducible quotient of $\nu^{\frac{n-1}{2}} 1_{GL_n} \rtimes 1$ and the unique irreducible subrepresentation of $\nu^{-\frac{n-1}{2}} 1_{GL_n} \rtimes 1$ and the normalized intertwining operator $N(w_0, \Lambda_{s,p})$ is holomorphic (and non-zero) on the spherical vector which generates the whole representation $\nu^{\frac{n-1}{2}} 1_{GL_n} \rtimes 1$. $\square$

Now we describe more thoroughly the superposition of the images of $N(w,\Lambda_{s,p})$ over all w belonging to the same orbit, as described in Proposition 3.2. First, we deal with the case of the longest element of $[W\setminus W_{\Delta\setminus\{\alpha_n\}}]$. Let $w_0\in Y_0^n$ and $w'\in Y_{n-2s}$ such that $w_0(\Lambda_s)=w'(\Lambda_s)$ (w' is explicitly described in Proposition 3.2). Denote, for a moment, $j=n-2s$ (j is odd). Note that the representation $\chi_p 1_{GL_j}\rtimes 1$ is an reducible (unitary) representation of $Sp_{2j}(\mathbb{Q}_p)$, $p\le\infty$. Let w_2 be the longest element of the Weyl group $Sp_{2j}(\mathbb{Q}_p)$, modulo the longest one in the Weyl group of its Siegel subgroup. Note that we can view w_2 as an element of $[W\setminus W_{\Delta\setminus\{\alpha_n\}}]$, by prescribing $w_2(i)=i$, $i=1,\dots,n-j$ and $w_2(i)=2n-j+1-i$, $i=n-j+1,\dots,n$. As before, we denote $\chi_p 1_{GL_j}\rtimes 1=\widehat{T_1}\oplus\widehat{T_2}$ where $N(w_2,j,\chi_p)$ acts as the identity on $\widehat{T_1}$, and as minus identity on $\widehat{T_2}$ (similarly for the archimedean case; cf. remark after Lemma 4.10). In the next lemma we can be more specific in the case $p<\infty$ than in the case of $p=\infty$. The problem with the latter is that we do not know which irreducible subquotients of $\chi_\infty\nu^s\rtimes 1$ belong to each of $\chi_\infty\nu^{-s}\zeta(-\frac{n-1}{2},\frac{n-1}{2}-j)\rtimes\widehat{T_{i,\infty}}$, $i=1,2$. The problem of classifying the subquotients (in terms of Langlands quotients) of the degenerate principal series for $Sp(2n,\mathbb{R})$ like these is still unsolved (cf. [21]) and we plan to address it in some other occasion.

Lemma 6.2. *Assume $\chi^2=1$ and $\frac{n-1}{2}-s\in\mathbb{Z}_{\ge 0}$ with $s>0$. Let $f_s=\otimes_{p\le\infty}f_{p,s}\in I(s)$. We denote by $f_{p,-s}$ the normalized spherical vector in $\chi_p\nu^{-s}1_{GL_n}\rtimes 1$. Then,*

1. *Assume $p<\infty$. If $s<\frac{n-1}{2}$, if $f_{p,s}$ belongs to the subquotient $\widehat{\sigma_1}$ we have $N(w_0,\Lambda_{s,p})f_{p,s}=N(w',\Lambda_{s,p})f_{p,s}$ and if $f_{p,s}$ belongs to the subquotient $\widehat{\sigma_2}$ we have $N(w_0,\Lambda_{s,p})f_{p,s}=-N(w',\Lambda_{s,p})f_{p,s}$; the same is true if $s=\frac{n-1}{2}$ and $\chi_p\ne 1$. If $s=\frac{n-1}{2}$ and $\chi_p=1$ then $N(w_0,\Lambda_{s,p})f_{p,s}=N(w',\Lambda_{s,p})f_{p,s}$, for every $f_{p,s}\in\chi_p\nu^s 1_{GL_n}\rtimes 1$.*
2. *Assume $p=\infty$. If $s<\frac{n-1}{2}$ or $s=\frac{n-1}{2}$ and $\chi_p=sgn$, so that we have $\chi_\infty\zeta(-(\frac{n-1}{2}-s),\frac{n-1}{2}-s)\rtimes 1=\widehat{T_{1,\infty}}\oplus\widehat{T_{2,\infty}}$, then $N(w_0,\Lambda_{s,\infty})f_{\infty,s}=N(w',\Lambda_{s,\infty})f_{\infty,s}$ if $f_{\infty,s}$ belongs to $\chi_\infty\nu^{-s}\zeta(-\frac{n-1}{2},\frac{n-1}{2}-j)\rtimes\widehat{T_{1,\infty}}$ and $N(w_0,\Lambda_{s,\infty})f_{\infty,s}=-N(w',\Lambda_{s,\infty})f_{\infty,s}$ if $f_{\infty,s}$ belongs to*

$$\chi_\infty\nu^{-s}\zeta(-\frac{n-1}{2},\frac{n-1}{2}-j)\rtimes\widehat{T_{2,\infty}}.$$

Now assume additionally that $\chi_\infty=1$ and $f_{\infty,s}$ is normalized spherical. Assume that S is a finite set of finite places and for $p\notin S$, let $f_{p,s}$ be the normalized spherical vector.

1. *Assume* $s < \frac{n-1}{2}$. *For* $S_1 \subset S$ *we pick* $f_{p,s} \in \widehat{\sigma_1}$ *and for* $p \in S_2 := S \setminus S_1$, *we take* $f_{p,s} \in \widehat{\sigma_2}$. *Then, the expression*

$$\begin{aligned} & r(\Lambda_s, w_0)^{-1}(\otimes_{p \in S_1} N(w_0, \Lambda_{s,p}) f_{p,s}) \\ & \quad \otimes (\otimes_{p \in S_2} N(w_0, \Lambda_{s,p}) f_{p,s}) \otimes (\otimes_{p \notin S} f_{p,-s}) \\ & \quad + r(\Lambda_s, w')^{-1}(\otimes_{p \in S_1} N(w', \Lambda_{s,p}) f_{p,s}) \\ & \quad \otimes (\otimes_{p \in S_2} N(w', \Lambda_{s,p}) f_{p,s}) \otimes (\otimes_{p \notin S} f_{p,-s}) \end{aligned} \tag{6.2}$$

is holomorphic if $|S_2|$ *is odd, and has a pole of the first order if* $|S_2|$ *is even.*

2. *Assume* $s = \frac{n-1}{2}$. *Then, if* $\chi = 1$, *the expression*

$$\begin{aligned} & r(\Lambda_s, w_0)^{-1} (\otimes_{p \in S} N(w_0, \Lambda_{s,p}) f_{p,s}) \otimes (\otimes_{p \notin S} f_{p,-s}) \\ & \quad + r(\Lambda_s, w')^{-1} (\otimes_{p \in S} N(w', \Lambda_{s,p}) f_{p,s}) \otimes (\otimes_{p \notin S} f_{p,-s}) \end{aligned}$$

has a pole of the first order.
If $\chi \neq 1$, *let* $S_1 \subset S$ *be a finite set of places such that* $\chi_p = 1$ *for* $p \in S_1$ *and* $S_2 = S \setminus S_1$ *such that* $\chi_p \neq 1$ *for* $p \in S_2$. *We take* $S_2 = S_2' \cup S_2''$ *such that for* $p \in S_2'$, $f_{p,s} \in \widehat{\sigma_1}$, *and for* $p \in S_2''$ $f_{p,s} \in \widehat{\sigma_2}$. *Then*

$$\begin{aligned} & r(\Lambda_s, w_0)^{-1} (\otimes_{p \in S_1} N(w_0, \Lambda_{s,p}) f_{p,s}) \otimes \big(\otimes_{p \in S_2'} N(w_0, \Lambda_{s,p}) f_{p,s}\big) \\ & \quad \otimes \big(\otimes_{p \in S_2''} N(w_0, \Lambda_{s,p}) f_{p,s}\big) \otimes (\otimes_{p \notin S} f_{p,-s}) \\ & \quad + r(\Lambda_s, w')^{-1} (\otimes_{p \in S_1} N(w', \Lambda_{s,p}) f_{p,s}) \otimes \big(\otimes_{p \in S_2'} N(w', \Lambda_{s,p}) f_{p,s}\big) \\ & \quad \otimes \big(\otimes_{p \in S_2''} N(w', \Lambda_{s,p}) f_{p,s}\big) \otimes (\otimes_{p \notin S} f_{p,-s}) \end{aligned} \tag{6.3}$$

is holomorphic if $|S_2''|$ *is odd, and has a pole of the first order if* $|S_2''|$ *is even.*

Proof. In the proof we omit a subscript s in $f_{p,s}$ if it is understood that $f_p = f_{p,s}$ belongs to $I(s.)$ We examine the actions of the intertwining operators $N(w_0, \Lambda_{s,p})$ and $N(w', \Lambda_{s,p})$. We have $w_0 = w_2 w'$, where w_2 is a Weyl group element introduced in the discussion before this Lemma. We note that

$$w'(\Lambda_s) = \Lambda_n^{-1} \otimes \Lambda_{n-1}^{-1} \otimes \cdots \otimes \Lambda_{j+1}^{-1} \otimes \Lambda_1 \otimes \Lambda_2 \otimes \cdots \otimes \Lambda_j,$$

where $j = n - 2s$. We can now describe the action of $N(w', \Lambda_{s,p})$ on $\chi_p \nu^s \rtimes 1$:

$$\begin{aligned} \chi_p \nu^s \rtimes 1 & \hookrightarrow \chi_p \nu^s \zeta(-\frac{n-1}{2}, -\frac{n-1}{2} + j - 1) \times \chi_p \nu^s \zeta(-\frac{n-1}{2} + j, \frac{n-1}{2}) \rtimes 1 \\ & \to \chi_p \nu^s \zeta(-\frac{n-1}{2}, -\frac{n-1}{2} + j - 1) \times \chi_p \nu^{-s} \zeta(-\frac{n-1}{2}, \frac{n-1}{2} - j) \rtimes 1 \\ & \to \chi_p \nu^{-s} \zeta(-\frac{n-1}{2}, \frac{n-1}{2} - j) \times \chi_p \nu^s \zeta(-\frac{n-1}{2}, -\frac{n-1}{2} + j - 1) \rtimes 1. \end{aligned}$$

Since $\chi_p \nu^s \zeta(-\frac{n-1}{2}, -\frac{n-1}{2} + j - 1) \rtimes 1 = \widehat{T_1} \oplus \widehat{T_2}$ (always reducible if $s < \frac{n-1}{2}$ or $s = \frac{n-1}{2}$ and $\chi_p \neq 1$, analogously if $p = \infty$), the representation in the last line of the expression above decomposes as

$$(6.4)\quad \chi_p\nu^{-s}\zeta(-\frac{n-1}{2},\frac{n-1}{2}-j)\rtimes\widehat{T_1}\oplus\chi_p\nu^{-s}\zeta(-\frac{n-1}{2},\frac{n-1}{2}-j)\rtimes\widehat{T_2}.$$

Since (the intertwining operator related to) w_2 acts as plus/minus identity on $\widehat{T_1}$ and $\widehat{T_2}$ this action induces in the same way to the action on the two summands in the relation (6.4) and the first part of the lemma follows. This means that (6.2) becomes

$$(r(\Lambda_s,w_0)^{-1}+(-1)^{|S_2|}r(\Lambda_s,w')^{-1})\,(\otimes_{p\in S_1}N(w_0,\Lambda_{s,p})f_p)$$
$$\otimes(\otimes_{p\in S_2}N(w_0,\Lambda_{s,p})f_p)\otimes(\otimes_{p\notin S}f_{p,-s}).$$

We now calculate $(r(\Lambda_s,w_0)^{-1}+(-1)^{|S_2|}r(\Lambda_s,w')^{-1})$. This is essentially done in Lemma 3.6, but note that $j_2-j_1=n-2s$ is odd, by our assumption ($\frac{n-1}{2}-s\in\mathbb{Z}_{\geq 0}$). Also, we do not even need Lemma 3.7, since $[w_0]\cap Y_0^n$ is a singleton. Thus, analyzing the quantity A_1 from Lemma 3.6, we see that, if $j_2-j_1\geq 3$, $r(\Lambda_s,w_0)^{-1}=r(\Lambda_s,w')^{-1}$ and this expression has precisely the pole of the first order, which is the same we obtain if $n-2s=1$, but $\chi\neq 1$. If $n-2s=1$ and $\chi=1$ we get that $r(\Lambda_s,w_0)^{-1}=-r(\Lambda_s,w_0)^{-1}$, and it has a pole of the second order. On the other hand, if $s=\frac{n-1}{2}$, the representation $\chi_p\nu^s\zeta(-\frac{n-1}{2},-\frac{n-1}{2}+j-1)\rtimes 1$ becomes $\chi_p\rtimes 1$ (representation of $SL(2)$), and this is reducible if $\chi_p\neq 1$ and irreducible if $\chi_p=1$. Thus, if $\chi_p=1$ and $s=\frac{n-1}{2}$, $N(w_0,\Lambda_{s,p})f_p=N(w',\Lambda_{s,p})f_p$ for every $f_p\in\chi_p\nu^s 1_{GL_n}\rtimes 1$. ☐

Now we examine what happens in the situation of orbits attached to elements $w\in Y_j^n$, where now j is any integer form the set $\{1,2,\ldots,n-2s-1\}$. Note that, by a remark after Proposition 3.2, the orbits for $w\in Y_j^n$, where $j\geq n-2s+1$, are singletons.

Lemma 6.3. *We keep the assumptions of Lemma 6.2. Assume $w'\in[w]$, where $w,w'\in Y_j^n$, for some $j\in\{1,2,\ldots,n-2s-1\}$. Then,*

$$N(w,\Lambda_{s,p})f_{p,s}=N(w',\Lambda_{s,p})f_{p,s},$$

for all $f_{p,s}\in\chi_p\nu^s 1_{GL_n}\rtimes 1$ for $p\leq\infty$. Further, the representation $L(\nu^s\delta(\widehat{[\chi_p\nu^{-\frac{n-1}{2}},\chi_p\nu^{\frac{n-1}{2}}]});1)$ is in the kernel of all intertwining operators $N(w,\Lambda_{s,p})$, $p<\infty$, where $w\in Y_j^n$ with $j\leq n-2s$.

Proof. We recall of the element $\overline{w_j}\in Y_j^n$ introduced in the proof of Lemma 5.9. For $w\in Y_j^n$ we have

$$N(w,\Lambda_{s,p})=N(w\overline{w_j}^{-1},\overline{w_j}(\Lambda_{s,p})N(\overline{w_j},\Lambda_{s,p}).$$

If $w'\in Y_j^n$ is in the orbit of w, i.e. $w(\Lambda_s)=w'(\Lambda_s)$, we can look at action of $N(w'\overline{w_j}^{-1},\overline{w_j}(\Lambda_{s,p}))$ on $\chi_p\nu^{-s}\zeta(-\frac{n-1}{2},\frac{n-1}{2}-j)\times\chi_p\nu^s\zeta(-\frac{n-1}{2},-\frac{n-1}{2}+j-1)$. If we prove that the actions of $N(w'\overline{w_j}^{-1},\overline{w_j}(\Lambda_{s,p}))$ and $N(w\overline{w_j}^{-1},\overline{w_j}(\Lambda_{s,p}))$ coincide on $\pi:=\chi_p\nu^{-s}\zeta(-\frac{n-1}{2},\frac{n-1}{2}-j)\times\chi_p\nu^s\zeta(-\frac{n-1}{2},-\frac{n-1}{2}+j-1)$, the actions of $N(w,\Lambda_{s,p})$ and $N(w',\Lambda_{s,p})$ will coincide on $\chi_p\nu^s \rtimes 1$, but for $j \le \frac{n-1}{2}$, π is irreducible, and, for such j, the claim follows.

Now we examine what happens if $\frac{n+1}{2}-s \le j \le n-2s$ (actually, it is enough to see what happens for $\frac{n+1}{2}-s \le j \le n-2s-1$, since orbit for Y^n_{n-2s} and Y^n_0 were discussed previously). Note that in that situation, $n-2s-j \le \frac{n-1}{2}-s$, and we are going to exploit that, using the cases we have just studied. We again assume $1 \le j \le \frac{n-1}{2}-s$, so that $n-2s-j \ge \frac{n+1}{2}-s$, and denote $j_1 = j$ and $j_2 = n-2s-j$ as we recall our considerations about orbits in $Y^n_{j_1}$ and $Y^n_{j_2}$ from the second section. Let $w = p_1\varepsilon_1 \in Y_{j_1}$. Assume firstly that $p_1(j_1) < p_1(j_2)$. Then, we have $p'(j_1) < p'(j_2)$, for every $w' = p'\varepsilon' \in Y^n_{j_1}$ from $[w]$. We recall the bijection between $Y^n_{j_1}\cap[w]$ and $Y^n_{j_2}\cap[w]$ from Lemma 3.4. Now we can explain the action of $N(w_2,\Lambda_{s,p})$ as follows (of course now Λ_i actually denotes $\Lambda_{i,p}$).

$$
\begin{aligned}
&\chi_p\nu^s \rtimes 1 \hookrightarrow \zeta(\Lambda_1,\Lambda_{j_2})\times\zeta(\Lambda_{j_2+1},\Lambda_n)\rtimes 1\\
&\to \zeta(\Lambda_1,\Lambda_{j_2})\times\zeta(\Lambda_n^{-1},\Lambda_{j_2+1}^{-1})\rtimes 1 \to \zeta(\Lambda_n^{-1},\Lambda_{j_2+1}^{-1})\times\zeta(\Lambda_1,\Lambda_{j_2})\rtimes 1\\
&\hookrightarrow \zeta(\Lambda_n^{-1},\Lambda_{j_2+1}^{-1})\times\zeta(\Lambda_1,\Lambda_{j_1})\times\zeta(\Lambda_{j_1+1},\Lambda_{j_2})\rtimes 1\\
&\to \underbrace{\zeta(\Lambda_n^{-1},\Lambda_{j_2+1}^{-1})\times\zeta(\Lambda_1,\Lambda_{j_1})}\times\zeta(\Lambda_{j_1+1},\Lambda_{j_2})\rtimes 1
\end{aligned}
\tag{6.5}
$$

where $\underbrace{*}$ part denotes a representation on which GL–intertwining operators act, as in the beginning of the proof of this lemma. Analogously, we see that the action of $w_1 \in Y^n_{j_1}\cap[w]$, the bijective image of w_2 above, will acts similarly, but, composed with the intertwining operator induced from $\zeta(\Lambda_{j_1+1},\Lambda_{j_2})\rtimes 1 \to \zeta(\Lambda_{j_2}^{-1},\Lambda_{j_1+1}^{-1})\rtimes 1$ to obtain

$$\underbrace{\zeta(\Lambda_n^{-1},\Lambda_{j_2+1}^{-1})\times\zeta(\Lambda_1,\Lambda_{j_1})}\times\zeta(\Lambda_{j_2}^{-1},\Lambda_{j_1+1}^{-1})\rtimes 1.$$

Note that the representation $\zeta(\Lambda_{j_1+1},\Lambda_{j_2})$ (so $\zeta(\Lambda_{j_1+1},\Lambda_{j_2})\rtimes 1$ also) is unitary. Moreover, $\zeta(\Lambda_{j_1+1},\Lambda_{j_2})\rtimes 1 = \chi_p\zeta(-(\frac{n-1}{2}-s-j_1),\frac{n-1}{2}-s-j_1)\rtimes 1$ is, for $p \le \infty$, reducible, unless $\chi_p = 1$ and $j_1 = \frac{n-1}{2}-s$. So, let $\chi_p\zeta(-(\frac{n-1}{2}-s-j_1),\frac{n-1}{2}-s-j_1)\rtimes 1 = \pi_1\oplus\pi_2$ be such that the intertwining operator above acts on π_1 as the identity, and on π_2 as the minus identity (for $p=\infty, \pi_1$ and π_2 can be reducible or zero). Then, if we denote by π_3 the image of the (GL–induced) intertwining operators acting on $\underbrace{\zeta(\Lambda_n^{-1},\Lambda_{j_2+1}^{-1})\times\zeta(\Lambda_1,\Lambda_{j_1})}$, we see that if $N(w_2,\Lambda_{s,p})f_p = v \in \pi_3\rtimes\pi_1$, then $N(w_1,\Lambda_{s,p})f_p = N(w_2,\Lambda_{s,p})f_p$ and if $N(w_2,\Lambda_{s,p})f_p = v \in \pi_3\rtimes\pi_2$, then $N(w_1,\Lambda_{s,p})f_p = -N(w_2,\Lambda_{s,p})f_p$.

From this reasoning, we also conclude the following: since we know that for $w_1,w_1' \in Y^n_{j_1}\cap[w]$ we have $N(w_1,\Lambda_{s,p})f_p = N(w_1',\Lambda_{s,p})f_p$ (since $j_1 \le \frac{n-1}{2}-s$), from the bijection of $Y^n_{j_1}$– and $Y^n_{j_2}$–parts of the orbit $[w]$, we get that for any $w_2,w_2' \in Y^n_{j_2}\cap[w]$ we have $N(w_2,\Lambda_{s,p})f_p = N(w_2',\Lambda_{s,p})f_p$ for every $f_p \in \chi_p\nu^s\rtimes 1$. If $\chi_p = 1$ and $j_1 = \frac{n-1}{2}-s$ then $\chi_p\zeta(-(\frac{n-1}{2}-s-j_1),\frac{n-1}{2}-s-$

$j_1) \rtimes 1 = 1_p \rtimes 1$ is a spherical irreducible representation of $SL(2,\mathbb{Q}_p)$ and the corresponding intertwining operator is just the identity, so that we trivially have $N(w_2,\Lambda_{s,p})f_p = N(w_2',\Lambda_{s,p})f_p = N(w_1,\Lambda_{s,p})f_p$ for every $f_p \in \chi_p\nu^s \rtimes 1$.

We have to examine the second possibility, namely when for $w = p_1\varepsilon_1 \in Y^n_{j_1}$ $p_1(j_1) > p_1(j_2)$ holds. According to our discussion in the second section, there exists $i_t \le j_1$ such that the intervals of change for w end with $i_t - 1$. Again, we use the bijection from Lemma 3.4 case 2: let $w' = p'\varepsilon' \in Y_{j_1} \cap [w]$ and let $w'' = p''\varepsilon'' \in Y^n_{j_2} \cap [w]$ be its bijective image. We have the following description of the action of $N(w'',\Lambda_{s,p})$ for $w'' \in Y^n_{j_2} \cap [w]$ on $\chi_p\nu^s 1_{GL_n} \rtimes 1$:

$$\begin{aligned}
&\chi_p\nu^s 1_{GL_n} \rtimes 1\\
&\quad\hookrightarrow \zeta(\Lambda_1,\Lambda_{i_t-1}) \times \zeta(\Lambda_{i_t},\Lambda_{j_1}) \times \zeta(\Lambda_{j_1+1},\Lambda_{j_2})\\
&\qquad \times \zeta(\Lambda_{j_2+1},\Lambda_{n+1-2s-i_t}) \times \zeta(\Lambda_{n+2-2s-i_t},\Lambda_n) \rtimes 1\\
&\quad\to \zeta(\Lambda_1,\Lambda_{i_t-1}) \times \zeta(\Lambda_{i_t},\Lambda_{j_1}) \times \zeta(\Lambda_{j_1+1},\Lambda_{j_2})\\
&\qquad \times \zeta(\Lambda_{j_2+1},\Lambda_{n+1-2s-i_t}) \times \zeta(\Lambda_n^{-1},\Lambda^{-1}_{n+2-2s-i_t}) \rtimes 1\\
&\quad\to \underbrace{\zeta(\Lambda_n^{-1},\Lambda^{-1}_{n+2-2s-i_t}) \times \zeta(\Lambda_1,\Lambda_{i_t-1})} \times \zeta(\Lambda_{i_t},\Lambda_{j_1})\\
&\qquad \times \zeta(\Lambda_{j_1+1},\Lambda_{j_2}) \times \zeta(\Lambda_{j_2+1},\Lambda_{n+1-2s-i_t}) \rtimes 1\\
&\quad\to \underbrace{\ *\ } \times \zeta(\Lambda^{-1}_{n+1-2s-i_t},\Lambda^{-1}_{j_2+1}) \times \zeta(\Lambda_{i_t},\Lambda_{j_1}) \times \zeta(\Lambda_{j_1+1},\Lambda_{j_2}) \rtimes 1,
\end{aligned}$$

where $\underbrace{\ *\ }$ denotes $\underbrace{\zeta(\Lambda_n^{-1},\Lambda^{-1}_{n+2-2s-i_t}) \times \zeta(\Lambda_1,\Lambda_{i_t-1})}$ and is precisely the part of the representation on which both w'' and its bijective image $w' \in Y^n_{j_1} \cap [w]$ have the same action. On that part, the same GL-intertwining operators act. For w'-action, on the obtained representation, we have further action as follows:

$$\begin{aligned}
&\underbrace{\ *\ } \times \zeta(\Lambda^{-1}_{n+1-2s-i_t},\Lambda^{-1}_{j_2+1}) \times \zeta(\Lambda_{i_t},\Lambda_{j_1}) \times \zeta(\Lambda_{j_1+1},\Lambda_{j_2}) \rtimes 1\\
&\xrightarrow{T_1} \underbrace{\ *\ } \times \zeta(\Lambda_{i_t},\Lambda_{j_1}) \times \zeta(\Lambda^{-1}_{n+1-2s-i_t},\Lambda^{-1}_{j_2+1}) \times \zeta(\Lambda_{j_1+1},\Lambda_{j_2}) \rtimes 1\\
&\xrightarrow{T_2} \underbrace{\ *\ } \times \zeta(\Lambda_{i_t},\Lambda_{j_1}) \times \zeta(\Lambda^{-1}_{n+1-2s-i_t},\Lambda^{-1}_{j_2+1}) \times \zeta(\Lambda^{-1}_{j_2},\Lambda^{-1}_{j_1+1}) \rtimes 1.
\end{aligned}$$

The operator T_1 is induced by GL-action

$$\zeta(\Lambda^{-1}_{n+1-2s-i_t},\Lambda^{-1}_{j_2+1}) \times \zeta(\Lambda_{i_t},\Lambda_{j_1}) \to \zeta(\Lambda_{i_t},\Lambda_{j_1}) \times \zeta(\Lambda^{-1}_{n+1-2s-i_t},\Lambda^{-1}_{j_2+1}),$$

and, when we resolve the indexes, we see that both of these factors are the same, thus T_1 is the identity. The operator T_2 acts on the unitary representation $\zeta(\Lambda_{j_1+1},\Lambda_{j_2}) \rtimes 1$ as a sum of identity and -identity action, as explained in the previous case. We can, thus, describe the action of $N(w'',\Lambda_{s,p})$ by $N(w'',\Lambda_{s,p}) = BA$, where A denotes the action in the first two arrows of (6.6), and further action GL–action on $\underbrace{\ *\ }$, and B denotes the GL-action that will occur on places occupied by $\zeta(\Lambda_{n+1-2s-i_t}^{-1},\Lambda_{j_2+1}^{-1})\times\zeta(\Lambda_{i_t},\Lambda_{j_1})\times\zeta(\Lambda_{j_1+1},\Lambda_{j_2})$ in the case of $N(w'',\Lambda_{s,p})$ and on

$$\zeta(\Lambda_{i_t},\Lambda_{j_1})\times\zeta(\Lambda_{n+1-2s-i_t}^{-1},\Lambda_{j_2+1}^{-1})\times\zeta(\Lambda_{j_2}^{-1},\Lambda_{j_1+1}^{-1})$$

in the case of $N(w',\Lambda_{s,p})$. Thus, $N(w',\Lambda_{s,p})=BT_2T_1A=BT_2A$. Thus again, if $Af_p\in\pi_3\rtimes\pi_1$, then $N(w',\Lambda_{s,p})f_p=N(w'',\Lambda_{s,p})f_p$, and if $Af_p\in\pi_3\rtimes\pi_2$, then $N(w',\Lambda_{s,p})f_p=-N(w'',\Lambda_{s,p})f_p$, where $\pi_i,\ i=1,2$ has the same meaning as in the previous case. In this way, we proved the first part of the lemma. Note that B (if non-zero) acts as an isomorphism on the image of an operator A described for the action of the operator $N(w'',\Lambda_{s,p})=BA$: indeed, in (6.6), we used the embedding

$$\begin{aligned}\chi_p\nu^s 1_{GL_n}\rtimes 1\hookrightarrow&\zeta(\Lambda_1,\Lambda_{i_t-1})\times\zeta(\Lambda_{i_t},\Lambda_{j_1})\times\zeta(\Lambda_{j_1+1},\Lambda_{j_2})\\&\times\zeta(\Lambda_{j_2+1},\Lambda_{n+1-2s-i_t})\times\zeta(\Lambda_{n+2-2s-i_t},\Lambda_n)\rtimes 1,\end{aligned}$$

but we only needed $\zeta(\Lambda_{i_t},\Lambda_{j_2})$ instead of $\zeta(\Lambda_{i_t},\Lambda_{j_1})\times\zeta(\Lambda_{j_1+1},\Lambda_{j_2})$. This means that B is induced from the GL-operator acting on

$$\zeta(\Lambda_{n+1-2s-i_t}^{-1},\Lambda_{j_2+1}^{-1})\times\zeta(\Lambda_{i_t},\Lambda_{j_2}),$$

but this representation is irreducible (and B is non-zero on it). We conclude that the kernel of the operator $N(w'',\Lambda_{s,p})$ is contained in the kernel of the operator $N(w',\Lambda_{s,p})$ (of course, both of these operators we view as operators on $\chi_p\nu^s\rtimes 1$). In general, from the fact that B acts as an isomorphism on the image of A we cannot conclude that it acts as an isomorphism on the image of T_2A, but in this case we reason as follows. We saw that B is induced from the GL–operator, say B', acting on the principal series in which the representation $\zeta(\Lambda_{n+1-2s-i_t}^{-1},\Lambda_{j_2+1}^{-1})\times\zeta(\Lambda_{i_t},\Lambda_{j_2})$ is naturally embedded. The image of the operator T_2 (an isomorphism) is

$$\underbrace{*}\times\zeta(\Lambda_{i_t},\Lambda_{j_1})\times\zeta(\Lambda_{n+1-2s-i_t}^{-1},\Lambda_{j_2+1}^{-1})\times\zeta(\Lambda_{j_2}^{-1},\ldots,\Lambda_{j_1+1}^{-1})\rtimes 1,$$

so that again

$$\begin{aligned}&\ldots\times\zeta(\Lambda_{n+1-2s-i_t}^{-1},\Lambda_{j_2+1}^{-1})\times\zeta(\Lambda_{i_t},\Lambda_{j_2})\rtimes 1\\&\hookrightarrow T_2(\underbrace{*}\times\zeta(\Lambda_{i_t},\Lambda_{j_1})\times\zeta(\Lambda_{n+1-2s-i_t}^{-1},\Lambda_{j_2+1}^{-1})\times\zeta(\Lambda_{j_2}^{-1},\Lambda_{j_1+1}^{-1})\rtimes 1)\\&=\underbrace{*}\times\zeta(\Lambda_{i_t},\Lambda_{j_1})\times\zeta(\Lambda_{n+1-2s-i_t}^{-1},\Lambda_{j_2+1}^{-1})\times\zeta(\Lambda_{j_2}^{-1},\Lambda_{j_1+1}^{-1})\rtimes 1,\end{aligned}$$

so that GL-part on which B' acts is actually the same (irreducible)

$$\zeta(\Lambda_{n+1-2s-i_t}^{-1},\Lambda_{j_2+1}^{-1})\times\zeta(\Lambda_{i_t},\Lambda_{j_2}).$$

We conclude that B is also an isomorphism on the image of T_2A. We conclude that the kernels of $N(w'',\Lambda_{s,p})$ and $N(w',\Lambda_{s,p})$ coincide. □

In the course of the proof of Lemma 6.3 we obtained that we can write $N(w'',\Lambda_{s,p})=BA$ and $N(w',\Lambda_{s,p})=BT_2A$ (where $B=id$ if $p_1(j_1)<p_1(j_2)$; here we assume $j_1<\frac{n-1}{2}-s$). It is important to examine when $Af_p\in\pi_3\rtimes\pi_1$.

This we can resolve for $p < \infty$. As expected, it turns out that for f_p from the subquotient $\widehat{\sigma_1}$ we have $Af_p \in \pi_3 \rtimes \pi_1$; analogously for $\widehat{\sigma_2}$ and $\pi_3 \rtimes \pi_2$.

LEMMA 6.4. *Retaining the notation from above, for $p < \infty$ we have: if $Af_p \neq 0$, then $Af_p \in \pi_3 \rtimes \pi_i,\ i = 1, 2$ for $f_p \in \widehat{\sigma_i},\ i = 1, 2$.*

PROOF. Assume that $j \leq \frac{n-1}{2} - s - 1$, and let $w' \in Y_j^n$. We recall the element $\overline{w_j}$, introduced in the proof of Lemma 5.9. In (5.4), the first operator $N(w_0')$ is induced from the longest operator, say, $N(w_0')'$, acting as

$$\chi_p\nu^s\zeta(-\frac{n-1}{2}+j, \frac{n-1}{2}) \rtimes 1 \xrightarrow{N(w_0')'} \chi_p\nu^{-s}\zeta(-\frac{n-1}{2}, \frac{n-1}{2}-j) \rtimes 1.$$

Because of our assumption on j, this is, essentially, a situation from Lemma 6.1, so the operator $N(w_0')'$ is holomorphic and the image of $N(w_0')$ is isomorphic to

$$\chi_p\nu^s\zeta(-\frac{n-1}{2}, -\frac{n-1}{2}+j-1) \rtimes \widehat{\sigma_1'} \oplus \chi_p\nu^s\zeta(-\frac{n-1}{2}, -\frac{n-1}{2}+j-1) \rtimes \widehat{\sigma_2'},$$

where σ_i' are discrete series representations, analogous to one described in Lemma 4.4 and Lemma 4.5. The Jordan block of σ_i'; $i = 1, 2$ is equal to $\{(1,1), (\chi_p, n-2s-2j), (\chi_p, n+2s)\}$. Note that the operator C from (5.4), with our assumption on j, is an isomorphism. Note that then the operator $N(w\overline{w_j}^{-1}, \overline{w_j}(\Lambda_{s,p}))$ is also an isomorphism, as discussed after (5.4). Since we know that $L(\nu^s\delta(\widehat{[\chi_p\nu^{-\frac{n-1}{2}}, \chi_p\nu^{\frac{n-1}{2}}]}); 1)$ is in the kernel of $N(w, \Lambda_{s,p})$ (Lemma 6.1), we get that $\widehat{\sigma_i},\ i = 1, 2$ are indeed subrepresentations in $\chi_p\nu^s\zeta(-\frac{n-1}{2}, -\frac{n-1}{2}+j-1) \rtimes (\widehat{\sigma_1'} \oplus \widehat{\sigma_2'})$ if they are not in the kernel of the intertwining operator $N(w_0')$. Now, by looking at the cuspidal support of $\chi_p\nu^{-s}\delta(\frac{n-1}{2}-j+1, \frac{n-1}{2}) \rtimes \sigma_1'$ we see that this representation can have at most one square–integrable subrepresentation, with the Jordan block equal to $\{(1,1), (\chi_p, n-2s), (\chi_p, n+2s)\}$), and the value of the ε–function on a element $(\chi_p, n+2s)$ of the Jordan block of both of these representations must coincide (cf. [22], Proposition 2.1). This means, if $\widehat{\sigma_i}$ appears in

$$\chi_p\nu^s\zeta(-\frac{n-1}{2}, -\frac{n-1}{2}+j-1) \rtimes \widehat{\sigma_1'} \oplus \chi_p\nu^s\zeta(-\frac{n-1}{2}, -\frac{n-1}{2}+j-1) \rtimes \widehat{\sigma_2'},$$

then we must have $\widehat{\sigma_i} \hookrightarrow \chi_p\nu^s\zeta(-\frac{n-1}{2}, -\frac{n-1}{2}+j-1) \rtimes \widehat{\sigma_i'},\ i = 1, 2$. If $\chi_p = 1$, we can get this more directly-namely, then $\widehat{\sigma_1}$ is spherical so it has to be a subquotient of $\chi_p\nu^s\zeta(-\frac{n-1}{2}, -\frac{n-1}{2}+j-1) \rtimes \widehat{\sigma_i'}$, for $i = 1$, because $\widehat{\sigma_1'}$ is spherical. Recall that $\chi_p\zeta(-(\frac{n-1}{2}-s-j_1), \frac{n-1}{2}-s-j_1) \rtimes 1 = \pi_1 \oplus \pi_2$, for our $j = j_1$. Then, by the same considerations as in Lemma 4.5 and 4.4, we have

$$\widehat{\sigma_i'} \hookrightarrow \chi_p\nu^{-\frac{n-j_1}{2}} 1_{GL_{2s+j_1}} \rtimes \pi_i,$$

and the Lemma is proved. ∎

We retain the notation from previous two lemmas.

COROLLARY 6.5. *Assume* $\chi^2 = 1$, $\frac{n-1}{2} - s \in \mathbb{Z}_{>0}$. *Let* $w' \in [w]$, *where* $w \in Y^n_{j_1}$, *and let* w'' *the bijective image of* w', *with* $w'' \in Y^n_{j_2}$.

1. *Assume* $p < \infty$. *Let* $j_1 < \frac{n-1}{2} - s$ *or* $j_1 = \frac{n-1}{2} - s$ *and* $\chi_p \neq 1$. *Then, for* $f_p \in \widehat{\sigma_1}$, *we have* $N(w', \Lambda_{s,p})f_p = N(w'', \Lambda_{s,p})f_p$ *and if* $f_p \in \widehat{\sigma_2}$, *we have* $N(w', \Lambda_{s,p})f_p = -N(w'', \Lambda_{s,p})f_p$.
2. *Assume* $j_1 = \frac{n-1}{2} - s$ *and* $\chi_p = 1$. *Then, for* $p \leq \infty$ *and for every* $f_p \in \chi_p \nu^s 1_{GL_n} \rtimes 1$, *we have* $N(w', \Lambda_{s,p})f_p = N(w'', \Lambda_{s,p})f_p$.

PROOF. Immediately from Lemma 6.3 and Lemma 6.4. ∎

Now we can group different contributions in (2.5) according to orbits, since the images of the intertwining operators in the same orbit are in the same principal series $(\mathrm{Ind}_{B(\mathbb{A})}^{Sp_{2n}(\mathbb{A})}(w(\Lambda_s)))$. Assume our constant term acts on a pure tensor $\otimes_{p\leq\infty} f_p$ and assume that $\frac{n-1}{2} - s \in Z_{\geq 0}$, which is the most interesting and involved case. By $f_{w',p}, f_{w'',p}$ etc. we denote the normalized spherical vector in $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{Sp_{2n}(\mathbb{Q}_p)}(w'(\Lambda_{p,s}))$ (we suppress from the notation the dependence on s). Thus, we can divide the expression in (2.5) in several sums:

1. identity (i.e. $\otimes_{p\leq\infty} f_p$),
2. $r(\Lambda_s, w)^{-1}(\otimes_{p\in S} N(\Lambda_{s,p}, w)f_p) \otimes (\otimes_{p\notin S} f_{w,p})$, for $w \in Y^n_j$, with $j \geq n - 2s + 1$. Note that the orbits here are singletons, and normalization factors and operators are holomorphic (cf. remark after Proposition 3.2).
3. Assume $0 < j_1 < \frac{n-1}{2} - s$ and $w \in Y^n_{j_1}$. Then,

$$\begin{aligned}\sum_{w'\in[w]} & r(\Lambda_s, w')^{-1}(\otimes_{p\in S} N(\Lambda_{s,p}, w')f_p) \otimes (\otimes_{p\notin S} f_{w',p}) \\ &= \sum_{w'\in[w]\cap Y_{j_1}} r(\Lambda_s, w')^{-1}(\otimes_{p\in S} N(\Lambda_{s,p}, w')f_p) \otimes (\otimes_{p\notin S} f_{w',p}) \\ &\quad + \sum_{w''\in[w]\cap Y_{j_2}} r(\Lambda_s, w'')^{-1}(\otimes_{p\in S} N(\Lambda_{s,p}, w'')f_p) \otimes (\otimes_{p\notin S} f_{w'',p}).\end{aligned}$$

4.

$$\begin{aligned}& r(\Lambda_s, w_0)^{-1}(\otimes_{p\in S} N(\Lambda_{s,p}, w_0)f_p) \otimes (\otimes_{p\notin S} f_{w,p}) \\ &\qquad + r(\Lambda_s, w')^{-1}(\otimes_{p\in S} N(\Lambda_{s,p}, w_0)f_p) \otimes (\otimes_{p\notin S} f_{w,p}),\end{aligned}$$

   where this expression was analyzed in detail in Lemma 6.2.

Note that Corollary 6.5 together with Lemma 3.7 and Lemma 3.6 enables us to do the same reasoning in the above third case, as in the fourth case (i.e. case of Lemma 6.2). But in Lemma 3.7 the poles of the sums of normalization factors were at most of the order which appears in the fourth case; also the intertwining operators from the third case might have bigger kernels than $N(\Lambda_{s,p}, w_0)$ (cf. Lemma 6.3 and Lemma 6.1). Thus, the meromorphic properties of (2.5) are governed by the contribution of w_0 (and w').

REMARK 6.6. Note that the meromorphic properties of Eisenstein series (poles of order at most one) and holomorphy of the local intertwining operators appearing in (2.5) for all $p \leq \infty$ are studied without any assumptions about the archimedean place. We only introduced this extra condition ($\chi_\infty = 1$ and f_∞ is the normalized spherical vector) in Lemma 6.2 to be able to describe explicitly the image of the intertwining operators at the archimedean place and, thus, find some irreducible global representations which are automorphic (i.e. appear in the space of automorphic forms on $Sp(2n, \mathbb{A}_\mathbb{Q})$). Note that, by Lemma 4.10, the image of this spherical vector under $N(w_0, \Lambda_{s,\infty})$ spans an irreducible $(\mathfrak{g}, K_\infty)$-module.

REMARK 6.7. As we saw above, if we pick local representations such that the contributions in (2.5) corresponding to non-identity elements of Weyl group are holomorphic, they cannot cancel the identity contribution (with $\frac{n-1}{2} - s \in \mathbb{Z}_{\geq 0}$), meaning we get an automorphic realization of a reducible global representation. In order to realize global irreducible representation as an automorphic representation, we look for the data which produce a pole of order one appearing in the contribution corresponding to w_0, since we have shown that with the appropriate choice of data, the image of $N(w_0, \Lambda_{s,p})$ is irreducible.

We have proved the following

THEOREM 6.8. *Assume $\chi^2 = 1$ with $\chi_\infty = 1$, and $n \geq 3$ with $\frac{n-1}{2} - s \in \mathbb{Z}_{\geq 0}$. Let $\widehat{\sigma_{i,p}}, i = 1, 2$ be the representations described in Lemmas 4.4 ,4.5 and Corollary 4.8 at the place $p < \infty$. Then, the Eisenstein series* (2.3) *with* Λ_s *as in* (3.1) *has a pole of order one on* $I(s)$.

1. *Assume $0 < s < \frac{n-1}{2}$, $f = \otimes_{p \leq \infty} f_p \in I(s)$ and let S be a finite set of finite places, and for $p \notin S$, let f_p be the normalized spherical vector. For $S_1 \subset S$, we pick $f_p \in \widehat{\sigma_{1,p}}$ and for $p \in S_2 := S \setminus S_1$, we take $f_p \in \widehat{\sigma_{2,p}}$. Then, for such f, (2.3) is holomorphic if $|S_2|$ is odd, and if $|S_2|$ is even it has a pole of order one. In the latter case (2.3) gives an automorphic realization (in the space of automorphic forms $\mathcal{A}(Sp_{2n}(\mathbb{Q}) \setminus Sp_{2n}(\mathbb{A})$) of a global irreducible representation having a local representation $\widehat{\sigma_{2,p}}$ on the places from S_2 and $\widehat{\sigma_{1,p}}$ as a local component elsewhere on finite places ($\widehat{\sigma_{1,p}}$ is spherical for $p \notin S$, $p < \infty$).*
2. *Assume $s = \frac{n-1}{2}$. If $\chi = 1$, then for any choice S of a finite set of finite places such that if $f = \otimes_{p \leq \infty} f_p$, with f_p normalized spherical for $f_p \notin S$, the Eisenstein series has a pole of the first order. Thus, (2.3) gives an automorphic realization of the unique spherical (global) subrepresentation of $\mathrm{Ind}_{B(\mathbb{A})}^{Sp_{2n}(\mathbb{A})}(\Lambda_{-s})$, having local components isomorphic to $L(\nu_p^{n-1}, \ldots, \nu_p^1; \nu_p^0 \rtimes 1)$. If $\chi \neq 1$ we have the following. Assume S*

is a finite set of finite places such that f_p is normalized spherical for $f_p \notin S$. We pick a subset $S_2 \subset S$ such that for $p \in S_2$, $\chi_p \neq 1$ and f_p belongs to $\widehat{\sigma_{2,p}}$, and for $p \in S \setminus S_2$ either $\chi_p = 1$ and f_p belongs to the spherical quotient $L(\nu_p^{n-1}, \ldots, \nu_p^1; \nu_p^0 \rtimes 1)$ or $\chi_p \neq 1$ and f_p belongs to $\widehat{\sigma_{1,p}}$. Then the Eisenstein series has a pole of order one if $|S_2|$ is even, and is holomorphic if $|S_2|$ is odd, so in the former case, (2.3) gives an automorphic realization of an irreducible global representations we have just described.

Now we cover the remaining straightforward cases.

THEOREM 6.9. *Assume $s > 0$.*

1. *If $\chi^2 \neq 1$ or $2s \notin \mathbb{Z}$ or $s > \frac{n+1}{2}$ the attached Eisenstein series is holomorphic and* (2.3) *gives an automorphic realization of the whole induced representation* (2.2)*.*
2. *Assume $s = \frac{n+1}{2}$ and $\chi = 1$. Then, the attached Eisenstein series have a pole of order one, and the image is an automorphic realization of the global trivial representation of $Sp_{2n}(\mathbb{A})$.*

PROOF. We note that the inverses of global normalization factors are holomorphic in the cases of the first part of the theorem by the discussion in the third section. Also, the local intertwining operators appearing in (2.5) are all holomorphic by the fifth section. By Corollary 3.1 contributions form the non-trivial elements of the Weyl group cannot cancel the identity contribution, and the result follows. The second claim follows from the discussion in the third section, from which is obvious that $r(\Lambda_s, w_0)^{-1}$ has a pole of order one and all the other inverses of the normalization factors are holomorphic. Here w_0 is given by (3.6). All the relevant local intertwining operators in (2.5) are holomorphic and, by Langlands classification, it is straightforward that the image of $N(w_0, \Lambda_{\frac{n+1}{2},p})$ acting on $\nu_p^{\frac{n+1}{2}} 1_{GL_n} \rtimes 1$ for all $p \leq \infty$ is the trivial representation. $\square$

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