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ON CERTAIN IDENTITY RELATED TO JORDAN *-DERIVATIONS

Irena Kosi-Ulbl and Joso Vukman
University of Maribor, Slovenia

Abstract. In this paper we prove the following result. Let H be a real or complex Hilbert space, let $\mathcal{L}(H)$ be the algebra of all bounded linear operators on H and let $A(H) \subseteq \mathcal{L}(H)$ be a standard operator algebra. Suppose we have an additive mapping $D : A(H) \to \mathcal{L}(H)$ satisfying the relation $D(A^n) = D(A)A^{*n-1} + AD(A^{n-2})A^* + A^{n-1}D(A)$ for all $A \in A(H)$ and some fixed integer $n > 1$. In this case there exists a unique $B \in \mathcal{L}(H)$ such that $D(A) = BA^* - AB$ holds for all $A \in A(H)$.

Throughout, R will represent an associative ring with center $Z(R)$. Given an integer $n \geq 2$, a ring R is said to be $n-$torsion free if for $x \in R$, $nx = 0$ implies $x = 0$. An additive mapping $x \mapsto x^*$ on a ring R is called an involution if $(xy)^* = y^*x^*$ and $x^{**} = x$ hold for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or *-ring. Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime in case $aRa = (0)$ implies $a = 0$. An additive mapping $D : R \to R$, where R is an arbitrary *-ring is called a *-derivation in case $D(xy) = D(x)y^* + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan *-derivation if $D(x^2) = D(x)x^* + xD(x)$ is fulfilled for all $x \in R$. It is easy to prove that there are no nonzero *-derivations on noncommutative prime *-rings (see [1] for the details). Note that the mapping $x \mapsto ax^* - xa$, where $a \in R$ is a fixed element, is a Jordan *-derivation; such Jordan *-derivations are said to be inner. By our knowledge the concept of Jordan *-derivations first appeared in [1]. The study of Jordan *-derivations has been motivated by the problem of the representability of quadratic forms by bilinear forms (for the results concerning this problem we refer to [6-10,13,14,16-19,22,23]). It turns

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out that the problem whether each quadratic form can be represented by some bilinear form is closely connected with the structure of Jordan *-derivations ([2, 15]). In [1] Brešar and the second named author of the present paper studied some algebraic properties of Jordan *-derivations. As a special case of Theorem 1 in [1] we have that every Jordan *-derivation of a complex algebra A with the identity element is inner. Let X be a real or complex Banach space, and let $\mathcal{L}(X)$ and $\mathcal{F}(X)$ denote the algebra of all bounded linear operators on X and the ideal of all finite rank operators in $\mathcal{L}(X)$, respectively. An algebra $A(X) \subseteq \mathcal{L}(X)$ is said to be standard in case $\mathcal{F}(X) \subset A(X)$. In case X is a Hilbert space we denote by A^* the adjoint operator of $A \in \mathcal{L}(X)$.

We start with the following result proved by Šemrl ([15]).

Theorem 1. *Let H be a real or complex Hilbert space and let $A(H)$ be a standard operator algebra. Suppose that $D : A(H) \to \mathcal{L}(H)$ is an additive mapping satisfying the relation*

$$D(A^2) = D(A)A^* + AD(A)$$

for all $A \in A(H)$. In this case there exists a unique $B \in \mathcal{L}(H)$ such that $D(A) = BA^* - AB$ holds for all $A \in A(H)$.

In case $D : R \to R$ is a Jordan *-derivation, where R is an arbitrary *-ring, one can easily prove that

$$D(xyx) = D(x)y^*x^* + xD(y)x^* + xyD(x), \tag{1}$$

holds for all pairs $x, y \in R$. The above relation has been considered in [5, 11, 23]. It seems natural to ask under what additional assumptions an additive mapping D satisfying the relation (1) is a Jordan *-derivation. The second named author of the present paper [20] has proved the following result.

Theorem 2. *Let R be a 6-torsion free semiprime *-ring, and let $D : R \to R$ be an additive mapping satisfying the relation* (1) *for all $x \in R$. In this case D is a Jordan *-derivation.*

Putting x^{n-2} for y in the relation (1), where $n > 1$ is some fixed integer, one obtains the relation bellow

$$D(x^n) = D(x)x^{*n-1} + xD(x^{n-2})x^* + x^{n-1}D(x), \qquad x \in R. \tag{2}$$

In case $n = 3$ the relation above reduces to the special case of the relation which has been considered in [21] . It is our aim in this paper to prove the result below, which is related to the equation (2).

Theorem 3. *Let H be a real or complex Hilbert space and let $A(H)$ be a standard operator algebra. Suppose we have an additive mapping $D : A(H) \to \mathcal{L}(H)$ satisfying the relation*

$$D(A^n) = D(A)A^{*n-1} + AD(A^{n-2})A^* + A^{n-1}D(A)$$

for all $A \in A(H)$ *and some fixed integer* $n > 1$. *In this case there exists a unique* $B \in \mathcal{L}(H)$ *such that* $D(A) = BA^* - AB$ *holds for all* $A \in A(H)$.

Let us point out that in the theorem above we obtain as a result the continuity of D under purely algebraic requirements concerning D, which means that the result above might be of some interest from the automatic continuity point of view. For results concerning automatic continuity we refer to [3, 4, 12].

Proof of Theorem 3. We have the relation

$$D(A^n) = D(A)A^{*n-1} + AD(A^{n-2})A^* + A^{n-1}D(A). \tag{3}$$

Let A be from $\mathcal{F}(H)$ and let $P \in \mathcal{F}(H)$ be a self-adjoint projection with $AP = PA = A$. Of course, we have also $A^*P = PA^* = A^*$. From the relation (3) one obtains

$$D(P) = D(P)P + PD(P)P + PD(P).$$

Right multiplication of the above relation by P gives

$$PD(P)P = 0. \tag{4}$$

Multiplying the relation (4) from the left side by A and from the right side by A^* we obtain

$$AD(P)A^* = 0. \tag{5}$$

Putting $A + P$ for A in the relation (3), we obtain

$$\begin{aligned} &\sum_{i=0}^{n} \binom{n}{i} D\left(A^{n-i}P^i\right) \\ &\quad = D(A+P)\left(\sum_{i=0}^{n-1} \binom{n-1}{i} A^{*n-1-i}P^i\right) \\ &\quad\quad + (A+P)\left[\sum_{i=0}^{n-2} \binom{n-2}{i} D\left(A^{*n-2-i}P^i\right)\right](A^*+P) \\ &\quad\quad + \left(\sum_{i=0}^{n-1} \binom{n-1}{i} A^{n-1-i}P^i\right) D(A+P). \end{aligned} \tag{6}$$

Using (3) and rearranging the equation (6) in sense of collecting together terms involving equal number of factors of P we obtain:

$$\sum_{i=1}^{n-1} f_i\,(A, P) = 0,$$

where $f_i\,(A, P)$ stands for the expression of terms involving i factors of P. Replacing A by $A + 2P$, $A + 3P$, ..., $A + (n-1)\,P$ in turn in the equation

(3), and expressing the resulting system of $n-1$ homogeneous equations of variables $f_i(A,P)$, $i=1,2,...,n-1$, we see that the coefficient matrix of the system is a van der Monde matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ n-1 & (n-1)^2 & \cdots & (n-1)^{n-1} \end{bmatrix}.$$

Since the determinant of the matrix is different from zero, it follows that the system has only a trivial solution. In particular,

$$\begin{aligned} f_{n-1}(A,P) = &\binom{n}{n-1} D(A) - (D(A)P + PD(A)) \\ &- \binom{n-1}{n-2}(AD(P) + D(P)A^*) - (AD(P)P + PD(P)A^*) \\ &- \binom{n-2}{n-3} PD(A)P = 0 \end{aligned}$$

and

$$\begin{aligned} f_{n-2}(A,P) = &\binom{n}{n-2} D\left(A^2\right) - \binom{n-1}{n-2}(D(A)A^* + AD(A)) \\ &- \binom{n-1}{n-3}\left(D(P)A^{*2} + A^2D(P)\right) \\ &- \binom{n-2}{n-3}(AD(A)P + PD(A)A^*) - AD(P)A^* \\ &- \binom{n-2}{n-4} PD\left(A^2\right)P = 0. \end{aligned}$$

The above equations reduce to

$$\begin{aligned} nD(A) = &D(A)P + PDA + (n-1)(AD(P) + D(P)A^*) \\ &+ AD(P)P + PD(P)A^* + (n-2)PD(A)P \end{aligned} \tag{7}$$

and

$$\begin{aligned} n(n-1)D\left(A^2\right) = &2(n-1)(AD(A) + D(A)A^*) \\ &+ (n-1)(n-2)\left(A^2D(P) + D(P)A^{*2}\right) \\ &+ 2(n-2)(AD(A)P + PD(A)A^*) \\ &+ (n-2)(n-3)PD\left(A^2\right)P + 2AD(P)A^*, \end{aligned} \tag{8}$$

respectively. Using (5) the relation (8) reduces to

$$(9)\quad \begin{aligned} n(n-1)D\left(A^2\right) &= 2(n-1)(AD(A)+D(A)A^*) \\ &+ (n-1)(n-2)\left(A^2D(P)+D(P)A^{*2}\right) \\ &+ 2(n-2)(AD(A)P+PD(A)A^*) \\ &+ (n-2)(n-3)PD\left(A^2\right)P. \end{aligned}$$

Applying the relation (4) and the fact that $AP = PA = A$ and $A^*P = PA^* = A^*$, we have $AD(P)P = A(PD(P)P) = 0$ and $PD(P)A^* = (PD(P)P)A^* = 0$. The relation (7) can now be written as

$$(10)\quad \begin{aligned} nD(A) = D(A)P + PD(A) + (n-1)(AD(P)+D(P)A^*) \\ + (n-2)PD(A)P. \end{aligned}$$

Left multiplication of the relation (10) by P gives

$$(11)\quad PD(A) = AD(P) + PD(A)P.$$

Similarly, one obtains

$$(12)\quad D(A)P = D(P)A^* + PD(A)P.$$

Subtracting the relation (11) from the relation (12) yields

$$(13)\quad D(A)P - PD(A) + AD(P) - D(P)A^* = 0.$$

Using relations (11) and (12) in (10) we obtain

$$(14)\quad D(A) = AD(P) + D(P)A^* + PD(A)P.$$

Using the relation (14) in the relation (10) gives

$$(15)\quad 2D(A) = D(A)P + PD(A) + D(P)A^* + AD(P).$$

Subtracting the relation (14) from the relation (15) gives

$$(16)\quad D(A) = D(A)P + PD(A) - PD(A)P.$$

Multiplying the relation (12) from the right side by A^* and relation (11) from the left side by A, we obtain

$$(17)\quad D(A)A^* = D(P)A^{*2} + PD(A)A^*$$

and

$$(18)\quad AD(A) = A^2D(P) + AD(A)P.$$

Combining relations (9), (17) and (18) we obtain

$$(19)\quad \begin{aligned} n(n-1)D\left(A^2\right) &= n(n-1)\left(A^2D(P)+D(P)A^{*2}\right) \\ &+ 2(2n-3)(AD(A)P+PD(A)A^*) \\ &+ (n-2)(n-3)PD\left(A^2\right)P. \end{aligned}$$

Combining relations (17) and (18) we obtain

$$D(A)A^* + AD(A) = D(P)A^{*2} + A^2D(P) + PD(A)A^* + AD(A)P. \tag{20}$$

By comparing (19) and (20) we obtain

$$\begin{aligned} n(n-1)D\left(A^2\right) = {} & n(n-1)\left(D(A)A^* + AD(A)\right) \\ & + (n-2)(n-3)\left(PD\left(A^2\right)P - AD(A)P - PD(A)A^*\right). \end{aligned} \tag{21}$$

Using the relations (17) and (18) in the above relation we arrive at

(22)

$$\begin{aligned} & n(n-1)D\left(A^2\right) = n(n-1)\left(D(A)A^* + AD(A)\right) \\ & + (n-2)(n-3)\left(PD\left(A^2\right)P + D(P)A^{*2} - D(A)A^* + A^2D(P) - AD(A)\right). \end{aligned}$$

Putting A^2 for A in the relation (14) gives

$$D\left(A^2\right) = A^2D(P) + D(P)A^{*2} + PD\left(A^2\right)P. \tag{23}$$

Using the relation (23) in the relation (22) one obtains after some calculation

$$D\left(A^2\right) = D(A)A^* + AD(A). \tag{24}$$

From the relation (16) one can conclude that D maps $\mathcal{F}(H)$ into itself. We have therefore an additive mapping D which maps $\mathcal{F}(H)$ into itself satisfying the relation (24) for all $A \in \mathcal{F}(H)$. In other words D is a Jordan *-derivation of $\mathcal{F}(H)$. Since all the assumptions of Theorem 1 are fulfilled, one can conclude that there exists a unique $B \in \mathcal{L}(H)$ such that

$$D(A) = BA^* - AB,$$

holds for all $A \in \mathcal{F}(H)$. It remains to prove that the above relation holds on $A(H)$ as well. Let us introduce $D_1 : A(H) \to \mathcal{L}(H)$ by $D_1(A) = BA^* - AB$ and consider $D_0 = D - D_1$.The mapping D_0 is, obviously, additive and satisfies the relation (3). Besides D_0 vanishes on $\mathcal{F}(H)$. Let $A \in A(H)$, let $P \in \mathcal{F}(H)$, be a self-adjoint projection and $S = A + PAP - (AP + PA)$. Since, obviously, $S - A \in \mathcal{F}(H)$, we have $D_0(S) = D_0(A)$. Besides $SP = PS = 0$ and $S^*P = PS^*$. We have therefore

$$D_0(A^n) = D_0(A)A^{*n-1} + AD_0(A^{n-2})A^* + A^{n-1}D_0(A)$$

for all $A \in A(H)$. Applying the above relation we obtain

$$\begin{aligned}
D_0(S)S^{*n-1} &+ SD_0(S^{n-2})S^* + S^{n-1}D_0(S) = D_0(S^n) \\
&= D_0(S^n + P) = D_0((S+P)^n) \\
&= D_0(S+P)\,(S+P)^{*n-1} + (S+P)\,D_0((S+P)^{n-2})\,(S+P)^* \\
&\quad + (S+P)^{n-1}\,D_0(S+P) \\
&= D_0(S)\left(S^{*n-1} + P\right) + (S+P)\,D_0(S^{n-2})\,(S^* + P) \\
&\quad + \left(S^{n-1} + P\right) D_0(S) \\
&= D_0(S)S^{*n-1} + D_0(S)P + SD_0(S^{n-2})S^* + SD_0(S^{n-2})P \\
&\quad + PD_0(S^{n-2})S^* + PD_0(S^{n-2})P + S^{n-1}D_0(S) + PD_0(S).
\end{aligned}$$

From the above relation it follows that

$$D_0(S)P + SD_0(S^{n-2})P + PD_0(S^{n-2})S^* + PD_0(S^{n-2})P + PD_0(S) = 0.$$

Since $D_0(S) = D_0(A)$, we obtain

$$\begin{aligned}
D_0(A)P &+ SD_0(A^{n-2})P + PD_0(A^{n-2})S^* \\
&+ PD_0(A^{n-2})P + PD_0(A) = 0.
\end{aligned} \tag{25}$$

Two-sided multiplication of the above relation by P gives

$$2PD_0(A)P + PD_0(A^{n-2})P = 0. \tag{26}$$

Putting $2A$ for A in the above relation, we obtain

$$2PD_0(A)P + 2^{n-3}PD_0(A^{n-2})P = 0. \tag{27}$$

Subtracting the relation (26) from the relation (27) gives

$$PD_0(A^{n-2})P = 0, \tag{28}$$

which means that

$$PD_0(A)P = 0 \tag{29}$$

as well. Right multiplication of the relation (25) by P and using the relations (28) and (29) give

$$D_0(A)P + SD_0(A^{n-2})P = 0.$$

Putting $2A$ for A in the above relation, we obtain (see how the relation (28) was obtained from the relation (26))

$$D_0(A)P = 0.$$

Since P is an arbitrary one-dimensional projection, it follows from the above relation that $D_0(A) = 0$ for any $A \in A(H)$. The proof of the theorem is therefore complete. □

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I. Kosi-Ulbl
Faculty of Mechanical Engineering
University of Maribor
Smetanova 17, 2000 Maribor
Slovenia
E-mail: irena.kosi@um.si

J. Vukman
Koroška cesta 57, 2000 Maribor
Slovenia
E-mail: joso.vukman@guest.um.si

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