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LOCALIZED SVEP AND THE COMPONENTS OF QUASI-FREDHOLM RESOLVENT SET

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Abstract. In this paper, new characterizations of the single valued extension property are given, for a bounded linear operator T acting on a Banach space and its adjoint T^*, at $\lambda_0 \in \mathbb{C}$ in the case that $\lambda_0 I - T$ is quasi-Fredholm. With the help of a classical perturbation result concerning operators with eventual topological uniform descent, we show the constancy of certain subspace valued mappings on the components of quasi-Fredholm resolvent set. As a consequence, we obtain a classification of these components.

1. Introduction

Throughout this paper, $\mathcal{B}(X)$ will denote the set of all bounded linear operators on an infinite-dimensional complex Banach space X. For an operator $T \in \mathcal{B}(X)$, let T^* denote its adjoint, $N(T)$ its kernel and $R(T)$ its range. Two important subspaces of X are the *hyperrange* of T defined by $R(T^\infty) = \cap_{n=1}^{\infty} R(T^n)$, and the *hyperkernel* of T defined by $N(T^\infty) = \cup_{n=1}^{\infty} N(T^n)$, respectively. There are another two important subspaces of X, the *analytical core* $K(T)$ of T defined by

$$K(T) = \{x \in X : \text{there exist a sequence } \{x_n\}_{n=0}^{\infty} \subseteq X \text{ and a constant } \delta > 0 \text{ such that } x_0 = x, Tx_{n+1} = x_n \text{ and } \|x_n\| \leq \delta^n \|x\| \text{ for all } n \in \mathbb{N}\},$$

and the *quasi-nilpotent part* $H_0(T)$ of T defined by

$$H_0(T) = \{x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0\}.$$

It is well known that $K(T) \subseteq R(T^\infty)$ and $N(T^\infty) \subseteq H_0(T)$.

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Recall that $T \in \mathcal{B}(X)$ is called *bounded below* if T is injective and has closed range $R(T)$. An operator $T \in \mathcal{B}(X)$ is called *semi-regular* if $R(T)$ is closed and $N(T) \subseteq R(T^\infty)$ (or equivalently, $N(T^\infty) \subseteq R(T)$). The concept of semi-regular was originated from Kato's classical treatment [11] of perturbation theory, even if originally these operators were not named in this way. Trivial examples of semi-regular operators are surjective operators and bounded below operators.

The lattice of invariant subspaces of an operator $T \in \mathcal{B}(X)$ is denoted as $Lat(T)$. A pair of closed subspace (M, N) is said to reduce T (denoted as $(M, N) \in Red(T)$), if $X = M \oplus N$ and $M, N \in Lat(T)$. For $M \in Lat(T)$, $T|_M$ denotes the restriction of T to M. An operator $T \in \mathcal{B}(X)$ is said to be of *Kato type* if there exists $(M, N) \in Red(T)$ such that $T|_M$ is semi-regular and $T|_N$ is nilpotent. If we assume in the definition above that N is finite-dimensional, then T is said to be *essentially semi-regular*. Equivalently, essentially semi-regular operators can be characterized in such a way that $R(T)$ is closed and there exists a finite-dimensional subspace F of X for which $N(T) \subseteq R(T^\infty) + F$ (see [1, Theorem 1.48]]).

For each $n \in \mathbb{N}$, we set $c_n(T) = \dim R(T^n)/R(T^{n+1})$ and $c_n^{'}(T) = \dim N(T^{n+1})/N(T^n)$. It follows from [10, Lemmas 3.1 and 3.2] that, for every $n \in \mathbb{N}$,

$$c_n(T) = \dim X/(R(T) + N(T^n)), \quad c_n^{'}(T) = \dim N(T) \cap R(T^n).$$

Hence, it is easy to see that the sequences $\{c_n(T)\}_{n=0}^\infty$ and $\{c_n^{'}(T)\}_{n=0}^\infty$ are decreasing. Recall that the *descent* and the *ascent* of $T \in \mathcal{B}(X)$ are defined as $dsc(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$ and $asc(T) = \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$, respectively (the infimum of an empty set is defined to be ∞). That is,

$$dsc(T) = \inf\{n \in \mathbb{N} : c_n(T) = 0\}$$

and

$$asc(T) = \inf\{n \in \mathbb{N} : c_n^{'}(T) = 0\}.$$

Recall that an operator $T \in \mathcal{B}(X)$ is said to be *left Drazin invertible* if $p := asc(T) < \infty$ and $R(T^{p+1})$ is closed.

If $T \in \mathcal{B}(X)$, for each $n \in \mathbb{N}$, T induces a linear transformation from the vector space $R(T^n)/R(T^{n+1})$ to the space $R(T^{n+1})/R(T^{n+2})$. Let $k_n(T)$ be the dimension of the kernel of the induced map. From [9, Lemma 2.3] it follows that, for every $n \in \mathbb{N}$,

$$\begin{aligned} k_n(T) &= \dim(N(T) \cap R(T^n))/(N(T) \cap R(T^{n+1})) \\ &= \dim(R(T) + N(T^{n+1}))/(R(T) + N(T^n)). \end{aligned}$$

We remark that the sequence $\{k_n(T)\}_{n=0}^\infty$ is not always decreasing. For this, see the following simple example.

Example 1.1. An operator $T \in \mathcal{B}(l_2^{(1)} \oplus l_2^{(2)})$ is defined as follows:

$$T = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix} : l_2^{(1)} \oplus l_2^{(2)} \to l_2^{(1)} \oplus l_2^{(2)},$$

where $S : l_2^{(2)} \to l_2^{(1)}$ is an isomorphism. It is easy to know that $N(T) = l_2^{(1)} \oplus \{0\}, R(T) = l_2^{(1)} \oplus \{0\}$, and $R(T^n) = \{0\} \oplus \{0\}$ for all $n \geq 2$. Then we have that

$$k_0(T) = \dim \frac{N(T)}{N(T) \cap R(T)} = 0, \quad k_1(T) = \dim \frac{N(T) \cap R(T)}{N(T) \cap R(T^2)} = \infty,$$

$$k_n(T) = \dim \frac{N(T) \cap R(T^n)}{N(T) \cap R(T^{n+1})} = 0, \quad \text{for all } \ n \geq 2.$$

J. P. Labrousse in [13] introduced and studied quasi-Fredholm operators on Hilbert spaces. M. Mbekhta and V. Müller in [15] extended them to Banach spaces.

Definition 1.2. *Let $d \in \mathbb{N}$. An operator $T \in \mathcal{B}(X)$ is said to be quasi-Fredholm of degree d if $k_n(T) = 0$ for $n \geq d$, and the subspaces $N(T^d) + R(T)$ and $N(T) \cap R(T^d)$ are closed.*

An operator $T \in \mathcal{B}(X)$ is said to be quasi-Fredholm if it is quasi-Fredholm of some degree d.

Discussions of quasi-Fredholm operators may be found in [2, 4, 13, 15, 18]. The following lemma describes some equivalent conditions of the assumption that the subspaces $N(T^d) + R(T)$ and $N(T) \cap R(T^d)$ are closed.

Lemma 1.3 ([18, Proposition 3]). *Let $T \in \mathcal{B}(X)$, $d \in \mathbb{N}$ and let $k_n(T) = 0$ for all $n \geq d$. The following statements are equivalent:*

(1) *T is quasi-Fredholm, i.e. $N(T^d) + R(T)$ and $N(T) \cap R(T^d)$ are closed.*
(2) *$R(T^{d+1})$ is closed.*
(3) *$R(T^n)$ is closed for all $n \geq d$.*
(4) *$R(T^i) + N(T^j)$ is closed for all i, j with $i + j \geq d$.*

The next definition, which was introduced by S. Grabiner ([9]), is closely related to that of quasi-Fredholm operators.

Definition 1.4. *Let $d \in \mathbb{N}$. An operator $T \in \mathcal{B}(X)$ is said to be have topological uniform descent for $n \geq d$ if $k_n(T) = 0$ for $n \geq d$, and the subspace $N(T^d) + R(T)$ is closed.*

An operator $T \in \mathcal{B}(X)$ is said to be have eventual topological uniform descent if there exists $d \in \mathbb{N}$ such that it has topological uniform descent for $n \geq d$.

From Definition 1.4 we see easily that $T \in \mathcal{B}(X)$ is semi-regular if and only if T has topological uniform descent for $n \geq 0$. By Lemma 1.3, we know that quasi-Fredholm operators of degree d are precisely all operators $T \in \mathcal{B}(X)$ that have topological uniform descent for $n \geq d$ and closed range $R(T^{d+1})$.

The single valued extension property was introduced by N. Dunford in [6,7] and plays an important role in local spectral theory and Fredholm theory, see the recent monographs [1] by P. Aiena and [14] by K. B. Laursen and M. M. Neumann.

Definition 1.5. *An operator $T \in \mathcal{B}(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc U of λ_0, the only analytic function $f : U \longrightarrow X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ is the constant function $f \equiv 0$.*

An operator $T \in \mathcal{B}(X)$ is said to have the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$.

The notion of localized SVEP at a point dates back to J. Finch ([8]). Some characterizations of the SVEP were given by P. Aiena ([2]), for an operator $T \in \mathcal{B}(X)$ and its adjoint T^*, at $\lambda_0 \in \mathbb{C}$ in the case that $\lambda_0 I - T$ is quasi-Fredholm.

This paper is organized as follows. In section 2, as a continuation of [2], we give new characterizations of the SVEP, for T and T^*, at $\lambda_0 \in \mathbb{C}$ in the case that $\lambda_0 I - T$ is quasi-Fredholm. In section 3, with the help of a classical perturbation result concerning operators with eventual topological uniform descent, we show the constancy of certain subspace valued mappings on the components of quasi-Fredholm resolvent set. As a consequence, a classification of these components is obtained. This generalizes the corresponding results of P. Aiena and F. Villafañe ([3]).

2. New characterizations of the localized SVEP

V. Müller in [18] proved that if $T \in \mathcal{B}(X)$ is quasi-Fredholm of degree d then $T^* \in \mathcal{B}(X^*)$ is also quasi-Fredholm of the same degree d. The following result shows that the reverse is also true.

For a subspace M of X, let $M^\perp \subseteq X^*$ denote the annihilator of M. For a subspace N of X^*, let ${}^\perp N \subseteq X$ denote the pre-annihilator of N.

Theorem 2.1. *Let $d \in \mathbb{N}$. Then $T \in \mathcal{B}(X)$ is quasi-Fredholm of degree d if and only if $T^* \in \mathcal{B}(X^*)$ is quasi-Fredholm of degree d.*

Proof. For the "only if" part, see [18, Lemma 4].

For the "if" part, suppose that T^* is quasi-Fredholm of degree d. From Lemma 1.3, $R(T^{*j})$ is closed for all $j \geq d$. By the closed range theorem we know that $R(T^j)$ is closed for all $j \geq d$ and, we can get the following equation

$$R(T^{*j}) \cap N(T^*) = N(T^j)^\perp \cap R(T)^\perp = (N(T^j) + R(T))^\perp \tag{2.1}$$

for all $j \geq d$. Since $T^{(-j)}(R(T^{(j+1)})) = N(T^j) + R(T)$ for all $j \geq d$, $N(T^j) + R(T)$ is closed for all $j \geq d$. From the fact that $k_j(T^*) = 0$ for all $j \geq d$ and by equation (2.1), we can obtain that

$$N(T^j)+R(T) = {}^{\perp}((N(T^j)+R(T))^{\perp}) = {}^{\perp}((N(T^d)+R(T))^{\perp}) = N(T^d)+R(T)$$

for all $j \geq d$. Therefore $k_j(T) = 0$ for all $j \geq d$. By Lemma 1.3 again, it follows that T is quasi-Fredholm of degree d. ∎

P. Aiena in [2] gave some characterizations of the SVEP, for T, at $\lambda_0 \in \mathbb{C}$ in the case that $\lambda_0 I - T$ is quasi-Fredholm.

Proposition 2.2 ([2, Theorem 2.7]). *Let $T \in \mathcal{B}(X)$ be quasi-Fredholm of degree d. Then the following statements are equivalent:*

- (i) *T has SVEP at 0;*
- (ii) *$asc(T) < \infty$;*
- (iii) *$\sigma_{ap}(T)$ does not cluster at 0;*
- (iv) *there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and $T|_{R(T^n)}$ is bounded below;*
- (v) *T is left Drazin invertible;*
- (vi) *there exists $m \in \mathbb{N}$ such that $H_0(T) = N(T^m)$;*
- (vii) *$H_0(T)$ is closed;*
- $(viii)$ *$H_0(T) \cap K(T) = \{0\}$.*

Dually, P. Aiena ([2]) gave some characterizations of the SVEP, for T^*, at $\lambda_0 \in \mathbb{C}$ in the case that $\lambda_0 I - T$ is quasi-Fredholm.

Proposition 2.3 ([2, Theorem 2.11]). *Let $T \in \mathcal{B}(X)$ be quasi-Fredholm of degree d. Then the following statements are equivalent:*

- (i) *T^* has SVEP at 0;*
- (ii) *$dsc(T) < \infty$;*
- (iii) *$\sigma_{su}(T)$ does not cluster at 0;*
- (iv) *there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and $T|_{R(T^n)}$ is onto;*
- (v) *$X = H_0(T) + K(T)$;*
- (vi) *there exists $m \in \mathbb{N}$ such that $K(T) = R(T^m)$;*

We give new characterizations of the SVEP, for T, at $\lambda_0 \in \mathbb{C}$ in the case that $\lambda_0 I - T$ is quasi-Fredholm.

Theorem 2.4. *Let $T \in \mathcal{B}(X)$ be quasi-Fredholm of degree d. Then the conditions (i)-$(viii)$ of Proposition 2.2 are equivalent to the following assertions:*

1. *$N(T^\infty) \cap R(T^\infty) = \{0\}$;*
2. *$N(T^\infty)^\perp + R(T^\infty)^\perp = X^*$;*
3. *$N((T^*)^\infty) + R((T^*)^\infty)$ is weak*-dense in X^*;*
4. *$H_0(T^*) + K(T^*)$ is weak*-dense in X^*;*
5. *$H_0(T^*) + R(T^*)$ is weak*-dense in X^*.*

Proof. $(viii) \Rightarrow (1)$ Since T is quasi-Fredholm, by [2, Lemma 2.6], $R(T^\infty) = K(T)$. Therefore, $N(T^\infty) \cap R(T^\infty) \subseteq H_0(T) \cap R(T^\infty) = H_0(T) \cap K(T) = \{0\}$. Thus, $N(T^\infty) \cap R(T^\infty) = \{0\}$.

(1) $\Rightarrow$ (2) Since T is quasi-Fredholm of degree d, T has topological uniform descent for $n \geq d$. By part (a) of [9, Lemma 3.6] and part (e) of [9, Theorem 3.2], we conclude that $N(T^\infty)+R(T^\infty) = N(T^d)+R(T^\infty)$ is closed. Hence, by a classical theorem of T. Kato, $N(T^\infty)^\perp + R(T^\infty)^\perp = (N(T^\infty) \cap R(T^\infty))^\perp = X^*$ (see [12, Chapter Four, Theorem 4.8]).

(2) $\Rightarrow$ (3) Since T is quasi-Fredholm of degree d, by Theorem 2.1, T^* is quasi-Fredholm of degree d. Hence, by Lemma 1.3, $R((T^*)^n)$ is closed for all $n \geq d$. Therefore

$$N(T^\infty)^\perp \subseteq N(T^n)^\perp = R((T^*)^n) \text{ for all } n \geq d.$$

Thus

$$N(T^\infty)^\perp \subseteq \bigcap_{n=d}^{\infty} R((T^*)^n) = \bigcap_{n=1}^{\infty} R((T^*)^n) = R((T^*)^\infty).$$

Since T is quasi-Fredholm of degree d, by Lemma 1.3 again, $R(T^n)$ is closed for all $n \geq d$. Hence

$${}^\perp N((T^*)^\infty) \subseteq {}^\perp N((T^*)^n) = R(T^n) \text{ for all } n \geq d.$$

Thus

$${}^\perp N((T^*)^\infty) \subseteq \bigcap_{n=d}^{\infty} R(T^n) = \bigcap_{n=1}^{\infty} R(T^n) = R(T^\infty).$$

So

$$R(T^\infty)^\perp \subseteq ({}^\perp N((T^*)^\infty))^\perp = \overline{N((T^*)^\infty)}^{w^*}.$$

By the assumption of (2), we have $X^* = N(T^\infty)^\perp + R(T^\infty)^\perp \subseteq R((T^*)^\infty) + \overline{N((T^*)^\infty)}^{w^*} \subseteq \overline{N((T^*)^\infty) + R((T^*)^\infty)}^{w^*} \subseteq X^*$. Therefore, $N((T^*)^\infty) + R((T^*)^\infty)$ is weak*-dense in X^*.

(3) $\Rightarrow$ (4) Since T is quasi-Fredholm of degree d, by Theorem 2.1, T^* is quasi-Fredholm of degree d. Hence, by [2, Lemma2.6], $R((T^*)^\infty) = K(T^*)$ and the desired conclusion follows.

(4) $\Rightarrow$ (5) Since $K(T^*) \subseteq R(T^*)$, the desired conclusion follows.

(5) $\Rightarrow$ (i) See [1, Theorem 2.36]. ∎

The next result, which is dual to Theorem 2.4, give new characterizations of the SVEP, for T^*, at $\lambda_0 \in \mathbb{C}$ in the case that $\lambda_0 I - T$ is quasi-Fredholm.

Theorem 2.5. *Let $T \in \mathcal{B}(X)$ be quasi-Fredholm of degree d. Then the conditions (i)-$(viii)$ of Proposition 2.3 are equivalent to the following assertions:*

(1) $N(T^\infty) + R(T^\infty) = X$;
(2) $N(T^\infty)^\perp \cap R(T^\infty)^\perp = \{0\}$;
(3) $N((T^*)^\infty) \cap R((T^*)^\infty) = \{0\}$;

(4) $N(T^*) \cap R((T^*)^\infty) = \{0\}$.

Proof. $(ii) \Rightarrow (1)$ Let $dsc(T) = q < \infty$. Then $R(T^\infty) = R(T^q)$ and, by [1, Lemma 3.2], $N(T^\infty) + R(T^\infty) = N(T^\infty) + R(T^q) \supseteq N(T^q) + R(T^q) = X$. Therefore, $N(T^\infty) + R(T^\infty) = X$.

$(1) \Rightarrow (2)$ Since $N(T^\infty) + R(T^\infty) = X$, it follows that $N(T^\infty)^\perp \cap R(T^\infty)^\perp = (N(T^\infty) + R(T^\infty))^\perp = \{0\}$.

$(2) \Rightarrow (3)$ Since T is quasi-Fredholm of degree d, by Lemma 1.3, $R(T^n)$ is closed for all $n \geq d$. Hence

$$\begin{aligned} R((T^*)^\infty) &= \bigcap_{n=d}^{\infty} R((T^*)^n) = \bigcap_{n=d}^{\infty} N(T^n)^\perp \\ &= (\bigcup_{n=d}^{\infty} N(T^n))^\perp = N(T^\infty)^\perp. \end{aligned} \tag{2.2}$$

Since $N(T^\infty)^\perp \cap R(T^\infty)^\perp = \{0\}$, it follows that $(N(T^\infty) + R(T^\infty))^\perp = N(T^\infty)^\perp \cap R(T^\infty)^\perp = \{0\}$, hence $N(T^\infty) + R(T^\infty) = X$. Since T is quasi-Fredholm, by [2, Lemma 2.6], $R(T^\infty) = K(T)$. Therefore $X = N(T^\infty) + R(T^\infty) \subseteq H_0(T) + K(T) \subseteq X$, so $H_0(T) + K(T) = X$. By Proposition 2.3, $dsc(T) < \infty$. Hence $asc(T^*) \leq dsc(T) < \infty$. Let $dsc(T) = q < \infty$. It is easy to see that

$$N((T^*)^\infty) = N((T^*)^q) = R(T^q)^\perp = R(T^\infty)^\perp.$$

Thus $N((T^*)^\infty) \cap R((T^*)^\infty) = N(T^\infty)^\perp \cap R(T^\infty)^\perp = \{0\}$.

$(3) \Rightarrow (4)$ Since $N(T^*) \subseteq N((T^*)^\infty)$, the desired conclusion follows.

$(4) \Rightarrow (i)$ See [1, Theorem 2.22]. ∎

3. Components of quasi-Fredholm resolvent set

The following proposition, which was due to S. Grabiner, is a classical perturbation result concerning operators with eventual topological uniform descent.

Proposition 3.1 ([9, Theorem 4.7]). *Suppose that $T \in \mathcal{B}(X)$ has topological uniform descent for $n \geq d$, and that $S \in \mathcal{B}(X)$ commutes with T. If S is sufficiently small and invertible, then*

(a) *$T + S$ is semi-regular;*
(b) *$\overline{R((T+S)^\infty)} = \overline{N(T^\infty) + R(T^\infty)}$;*
(c) *$\overline{N((T+S)^\infty)} = \overline{N(T^\infty) \cap R(T^\infty)}$.*

For $T \in \mathcal{B}(X)$, the *Kato type spectrum* and the *quasi-Fredholm spectrum* are defined as $\sigma_{kt}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not of Kato type}\}$ and $\sigma_{qf}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not quasi-Fredholm}\}$, respectively. From [1, Theorem 1.42] it follows that $\sigma_{qf}(T) \subseteq \sigma_{kt}(T)$. It is known that $\sigma_{kt}(T)$ is closed, see [1, Corollary 1.45]. According to Proposition 3.1, it follows easily that $\sigma_{qf}(T)$ is also closed.

The *Kato type resolvent set* and the *quasi-Fredholm resolvent set* are defined as $\rho_{kt}(T) = \mathbb{C}\backslash\sigma_{kt}(T)$ and $\rho_{qf}(T) = \mathbb{C}\backslash\sigma_{qf}(T)$, respectively. The sets $\rho_{kt}(T)$ and $\rho_{qf}(T)$ are open subsets of $\mathbb{C}$, so they can be decomposed in connected disjoint open non-empty components.

M. Mbekhta and A. Ouahab ([16]) showed that the mappings

$$\lambda \longrightarrow H_0(\lambda I - T) + K(\lambda I - T), \quad \lambda \longrightarrow \overline{H_0(\lambda I - T)} \cap K(\lambda I - T) \tag{3.1}$$

are constant on the components of $\rho_{kt}(T)$. P. Aiena and F. Villafañe ([3]) proved that the mappings (3.1) and the mappings

$$\lambda \longrightarrow N((\lambda I - T)^\infty) + R((\lambda I - T)^\infty), \quad \lambda \longrightarrow \overline{N((\lambda I - T)^\infty)} \cap R((\lambda I - T)^\infty) \tag{3.2}$$

coincide, respectively, on the components of $\rho_{kt}(T)$.

We generalize these results to the components of $\rho_{qf}(T)$. We first show the constancy of the mappings (3.2) on the components of $\rho_{qf}(T)$.

LEMMA 3.2. *Let $T \in \mathcal{B}(X)$ be quasi-Fredholm of degree d. Then there exists an $\varepsilon > 0$ such that:*

(1) $N((\lambda I - T)^\infty) + R((\lambda I - T)^\infty) = N(T^\infty) + R(T^\infty)$ *for all* $0 < |\lambda| < \varepsilon$;
(2) $\overline{N((\lambda I - T)^\infty)} \cap R((\lambda I - T)^\infty) = \overline{N(T^\infty)} \cap R(T^\infty)$ *for all* $0 < |\lambda| < \varepsilon$.

PROOF. Since T is quasi-Fredholm of degree d, T has topological uniform descent for $n \geq d$. By Proposition 3.1, there exists an $\varepsilon > 0$ such that

$$\lambda I - T \text{ is semi-regular,}$$

$$R((\lambda I - T)^\infty) = N(T^\infty) + R(T^\infty)$$

and

$$\overline{N((\lambda I - T)^\infty)} = \overline{N(T^\infty) \cap R(T^\infty)}$$

for all $0 < |\lambda| < \varepsilon$. By [17, Theorem 1.2], $N((\lambda I - T)^\infty) \subseteq R((\lambda I - T)^\infty)$. Moreover, by [1, Theorem 1.24] $R((\lambda I - T)^\infty)$ is closed, consequently, $\overline{N((\lambda I - T)^\infty)} \subseteq R((\lambda I - T)^\infty)$. Hence

$$N((\lambda I - T)^\infty) + R((\lambda I - T)^\infty) = R((\lambda I - T)^\infty) = N(T^\infty) + R(T^\infty)$$

and

$$\begin{aligned}\overline{N((\lambda I - T)^\infty)} \cap R((\lambda I - T)^\infty) = \overline{N((\lambda I - T)^\infty)} &= \overline{N(T^\infty) \cap R(T^\infty)} \\ &\overset{[9,\text{ Lemma 3.6(d)}]}{=\!=\!=} \overline{\overline{N(T^\infty)} \cap R(T^\infty)} \\ &\overset{[2,\text{ Lemma 2.6}]}{=\!=\!=} \overline{N(T^\infty)} \cap R(T^\infty)\end{aligned}$$

for all $0 < |\lambda| < \varepsilon$. □

By using the classical Heine-Borel theorem, we obtain the following result.

COROLLARY 3.3. *Let $T \in \mathcal{B}(X)$. If Ω is a component of $\rho_{qf}(T)$ and $\lambda_0 \in \Omega$, then*

$$N((\lambda I - T)^\infty) + R((\lambda I - T)^\infty) = N((\lambda_0 I - T)^\infty) + R((\lambda_0 I - T)^\infty)$$

and

$$\overline{N((\lambda I - T)^\infty)} \cap R((\lambda I - T)^\infty) = \overline{N((\lambda_0 I - T)^\infty)} \cap R((\lambda_0 I - T)^\infty)$$

for all $\lambda \in \Omega$.

Therefore, the mappings

$$\lambda \longrightarrow N((\lambda I - T)^\infty) + R((\lambda I - T)^\infty)$$

and

$$\lambda \longrightarrow \overline{N((\lambda I - T)^\infty)} \cap R((\lambda I - T)^\infty)$$

are constant on the components of $\rho_{qf}(T)$.

The following theorem extends [3, Theorem 2.1].

THEOREM 3.4. *Let $\lambda I - T$ be quasi-Fredholm. Then*

(1) $N((\lambda I - T)^\infty) + R((\lambda I - T)^\infty) = H_0(\lambda I - T) + K(\lambda I - T)$.
(2) $\overline{N((\lambda I - T)^\infty)} \cap R((\lambda I - T)^\infty) = \overline{H_0(\lambda I - T)} \cap K(\lambda I - T)$.

PROOF. Without loss of generality, we may assume that $\lambda = 0$.

Since T is quasi-Fredholm of degree d, by Theorem 2.1, T^* is also quasi-Fredholm of degree d. Then by [2, Lemma 2.6], $R(T^\infty) = K(T)$ and $R((T^*)^\infty) = K(T^*)$. By [1, Theorem 1.70], $\overline{N(T^\infty)} \subseteq \overline{H_0(T)} \subseteq {}^\perp K(T^*)$. By equation (2.2), $\overline{N(T^\infty)}^\perp = R((T^*)^\infty) = K(T^*)$. So, $\overline{N(T^\infty)} = {}^\perp K(T^*)$. Hence, $\overline{N(T^\infty)} = \overline{H_0(T)}$. Consequently, $\overline{N(T^\infty)} \cap R(T^\infty) = \overline{H_0(T)} \cap K(T)$. This shows (2).

On one hand, $N(T^\infty) + R(T^\infty) \subseteq H_0(T) + R(T^\infty) = H_0(T) + K(T)$. On the other hand,

$$\begin{aligned} H_0(T) + K(T) \subseteq \overline{H_0(T)} + K(T) &= \overline{N(T^\infty)} + R(T^\infty) \\ &\overset{[9,\ \text{Lemma } 3.6(\text{a})]}{=\!=\!=\!=\!=} N(T^\infty) + R(T^\infty) \end{aligned}$$

Therefore, $N(T^\infty) + R(T^\infty) = H_0(T) + K(T)$. This shows (1). ∎

By Corollary 3.3 and Theorem 3.4, we obtain the next result which generalizes the corresponding result of M. Mbekhta and A. Ouahab ([16]).

COROLLARY 3.5. *The mappings*

$$\lambda \longrightarrow H_0(\lambda I - T) + K(\lambda I - T)$$

and

$$\lambda \longrightarrow \overline{H_0(\lambda I - T)} \cap K(\lambda I - T)$$

are constant on the components of $\rho_{qf}(T)$.

Combining Theorem 2.4 with Corollary 3.3, the following classification is obtained.

THEOREM 3.6. *Let $T \in \mathcal{B}(X)$ and Ω a component of $\rho_{qf}(T)$. Then the following alternative holds:*

(1) *T has the SVEP at every point of Ω. In this case, $asc(\lambda I - T) < \infty$ for all $\lambda \in \Omega$. Moreover, $\sigma_{ap}(T)$ does not have limit points in Ω; every point of Ω, except possibly for at most countably many isolated points, is not an eigenvalue of T.*
(2) *T has the SVEP at no point of Ω. In this case, $asc(\lambda I - T) = \infty$ for all $\lambda \in \Omega$. Every point of Ω is an eigenvalue of T.*

PROOF. (1) Suppose that T has the SVEP at $\lambda_0 \in \Omega$. Then by Proposition 2.2, $asc(\lambda_0 I - T) < \infty$, so $N((\lambda_0 I - T)^\infty)$ is closed. By Theorem 2.4, $\overline{N((\lambda_0 I - T)^\infty)} \cap R((\lambda_0 I - T)^\infty) = N((\lambda_0 I - T)^\infty) \cap R((\lambda_0 I - T)^\infty) = \{0\}$. By Corollary 3.3 the mapping

$$\lambda \longrightarrow \overline{N((\lambda I - T)^\infty)} \cap R((\lambda I - T)^\infty)$$

is constant on Ω, so $\overline{N((\lambda I - T)^\infty)} \cap R((\lambda I - T)^\infty) = \{0\}$ for all $\lambda \in \Omega$. Thus, $N((\lambda I - T)^\infty) \cap R((\lambda I - T)^\infty) = \{0\}$ for all $\lambda \in \Omega$. Therefore, again by Theorem 2.4, T has the SVEP at every $\lambda \in \Omega$. This is equivalent, by Proposition 2.2, to saying that $asc(\lambda I - T) < \infty$ for all $\lambda \in \Omega$. Moreover, from Proposition 2.2, $\sigma_{ap}(T)$ does not have limit points in Ω and, consequently, every point of Ω is not an eigenvalue of T, except possibly for at most countably many isolated points.

(2) Suppose that T has the SVEP at no point of Ω. Then by Proposition 2.2, $asc(\lambda I - T) = \infty$ for all $\lambda \in \Omega$ and, consequently, every point of Ω is an eigenvalue of T. ◻

Recall that $\lambda \in \mathbb{C}$ is said to be a *deficiency value* for if $\lambda I - T$ is not surjective. Combining Theorem 2.5 with Corollary 3.3, the following classification is obtained.

THEOREM 3.7. *Let $T \in \mathcal{B}(X)$ and Ω a component of $\rho_{qf}(T)$. Then the following alternative holds:*

(1) *T^* has the SVEP at every point of Ω. In this case, $dsc(\lambda I - T) < \infty$ for all $\lambda \in \Omega$. Moreover, $\sigma_{su}(T)$ does not have limit points in Ω; every point of Ω, except possibly for at most countably many isolated points, is not a deficiency value of T.*
(2) *T^* has the SVEP at no point of Ω. In this case, $dsc(\lambda I - T) = \infty$ for all $\lambda \in \Omega$. Every point of Ω is a deficiency value of T.*

PROOF. (1) Suppose that T^* has the SVEP at $\lambda_0 \in \Omega$. Then, by Theorem 2.5, $N((\lambda_0 I - T)^\infty) + R((\lambda_0 I - T)^\infty) = X$. By Corollary 3.3 the mapping

$$\lambda \longrightarrow R((\lambda I - T)^\infty) + N((\lambda I - T)^\infty)$$

is constant on Ω, so $R((\lambda I - T)^\infty) + N((\lambda I - T)^\infty) = X$ for all $\lambda \in \Omega$. Therefore, again by Theorem 2.5, T^* has the SVEP at every $\lambda \in \Omega$. This is equivalent, by Proposition 2.3, to saying that $dsc(\lambda I - T) < \infty$ for all $\lambda \in \Omega$. Moreover, from Proposition 2.3, $\sigma_{su}(T)$ does not have limit points in Ω and, consequently, every point of Ω is not a deficiency value of T, except possibly for at most countably many isolated points.

(2) Suppose that T^* has the SVEP at no point of Ω. Then by Proposition 2.3, $dsc(\lambda I - T) = \infty$ for all $\lambda \in \Omega$ and, consequently, every point of Ω is a deficiency value of T. □

At last, as an application, we give a characterization of finite-dimensional Banach spaces.

Corollary 3.8. *Let X be a Banach space. The following assertions are equivalent:*

(1) *X is finite-dimensional;*
(2) *$\sigma_{qf}(T) = \emptyset$ for every $T \in \mathcal{B}(X)$.*

Proof. (1) $\Longrightarrow$ (2) Clear.

(2) $\Longrightarrow$ (1) For every $T \in \mathcal{B}(X)$, since $\sigma_{qf}(T) = \emptyset$, $\rho_{qf}(T)$ has only one component $\Omega = \mathbb{C}$. Then by Theorem 3.7, $\sigma_{dsc}(T) := \{\lambda \in \mathbb{C} : dsc(T - \lambda) = \infty\} = \emptyset$. Consequently, by [5, Corollary 1.10], X is finite-dimensional. □

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References

[1] P. Aiena, Fredholm and local spectral theory, with application to multipliers, Kluwer Academic Publishers, Dordrecht, 2004.
[2] P. Aiena, *Quasi-Fredholm operators and localized SVEP,* Acta Sci. Math. (Szeged) **73** (2007), 251–263.
[3] P. Aiena and F. Villafañe, *Components of resolvent sets and local spectral theory,* Contemp. Math. **328** (2003), 1–14.
[4] M. Berkani, *Restriction of an operator to the range of its powers,* Studia Math. **140** (2000), 163–175.
[5] M. Burgos, A. Kaidi, M. Mbekhta and M. Oudghiri, *The descent spectrum and perturbations,* J. Operator Theory **56** (2006), 259–271.
[6] N. Dunford, *Spectral theory II. Resolutions of the identity,* Pacific J. Math. **2** (1952), 559–614.
[7] N. Dunford, *Spectral operators,* Pacific J. Math. **4** (1954), 321–354.
[8] J. Finch, *The single valued extension property on a banach space,* Pacific J. Math. **58** (1975), 61–69.
[9] S. Grabiner, *Uniform ascent and descent of bounded operators,* J. Math. Soc. Japan **34** (1982), 317–337.
[10] M. A. Kaashoek, *Ascent, descent, nullity and defect, a note on a paper by A. E. Taylor,* Math. Ann. **172** (1967), 105–115.

[11] T. Kato, *Perturbation theory for nullity, deficiency and other quantities of linear operators,* J. Analyse Math. **6** (1958), 261–322.

[12] T. Kato, Perturbation theory for linear operators, Spring-verlag, Berlin-Heidelberg-New York-Tokyo, 1995.

[13] J.-P. Labrousse, *Les opérateurs quasi Fredholm: une généralisation des opérateurs semi Fredholm,* Rend. Circ. Mat. Palermo (2) **29** (1980), 161-258.

[14] K. B. Laursen and M. M. Neumann, An introduction to local spectral theory, Oxford University Press, New York, 2000.

[15] M. Mbekhta and V. Müller, *On the axiomatic theory of spectrum. II,* Studia Math. **119** (1996), 129–147.

[16] M. Mbekhta and A. Ouahab, *Perturbation des opérateurs s-réguliers,* in Topics in operator theory, operator algebras and applications, Timisoara, 1994, Rom. Acad. Bucharest, 1995, 239–249.

[17] V. Müller, *On the regular spectrum,* J. Operator Theory **31** (1994), 363–380.

[18] V. Müller, *On the Kato-decomposition of quasi-Fredholm and B-Fredholm operators,* Vienna, Preprint ESI 1013, 2001.

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