

CLASSIFICATION OF FACTORABLE SURFACES IN THE PSEUDO-GALILEAN SPACE

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ABSTRACT. In this paper, we introduce the factorable surfaces in the pseudo-Galilean space \mathbb{G}_3^1 and completely classify such surfaces with null Gaussian and mean curvature. Also, in a special case, we investigate the factorable surfaces which fulfill the condition that the ratio of the Gaussian curvature and the mean curvature is constant in \mathbb{G}_3^1 .

1. INTRODUCTION

The geometry of Galilean Relativity acts like a “bridge” from Euclidean geometry to Special Relativity. The Galilean space which can be defined in three-dimensional projective space $P_3(\mathbb{R})$ is the space of Galilean Relativity. The geometries of Galilean space and pseudo-Galilean space have similarities, but, of course, are different. In the Galilean and the pseudo Galilean space, surfaces of revolution, ruled surfaces, translation surfaces and tubular surfaces have been studied in [3–5, 10, 11, 13–15, 17, 20]. For further study of surfaces in the pseudo Galilean space, we refer the reader to Šipuš and Divjak’s paper [15].

One of the main problems in classical differential geometry of surfaces is to classify ones with null curvature. In particular, a surface is said to be *developable* if it has null Gaussian curvature (NGC). In this case the surface can be flattened onto a plane without distortion. We remark that cylinders and cones are examples of developable surfaces.

There exist remarkable applications of the results obtained on the surfaces of null curvature in different fields, for example in microeconomics. When

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the graphs of production functions in microeconomics have NGC, one can realize a “good” analysis of isoquants by projections, without losing essential information about their geometry (see [1,19]). By motivating these, we target to study the factorable surfaces in the pseudo Galilean space \mathbb{G}_3^1 .

In this paper, we classify the factorable surfaces with NGC and minimal ones in \mathbb{G}_3^1 . Then, in a special case, we obtain a non-existence result for the factorable surfaces satisfying the condition that the ratio of the Gaussian curvature and the mean curvature is constant in \mathbb{G}_3^1 .

2. PRELIMINARIES

The pseudo-Galilean space \mathbb{G}_3^1 is one of the Cayley-Klein spaces with absolute figure that consists of the ordered triple $\{\omega, f, I\}$, where ω is the absolute plane in the three dimensional real projective space $P_3(\mathbb{R})$, f the absolute line in ω and I the fixed hyperbolic involution of points of f .

Homogenous coordinates of \mathbb{G}_3^1 can be given in the following way: The absolute plane ω is given by $x_0 = 0$, the absolute line f by $x_0 = x_1 = 0$ and the hyperbolic involution by $(0 : 0 : x_2 : x_3) \mapsto (0 : 0 : x_3 : x_2)$, which is equivalent to the requirement that the conic $x_2^2 - x_3^2 = 0$ is the absolute conic. The metric connections in \mathbb{G}_3^1 are introduced with respect to the absolute figure. In terms of the affine coordinates given by $(x_0 : x_1 : x_2 : x_3) = (1 : x : y : z)$, the *distance* between the points $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ is defined by (see [15,20])

$$d(X, Y) = \begin{cases} |y_1 - x_1|, & \text{if } x_1 \neq y_1, \\ \sqrt{|(y_2 - x_2)^2 - (y_3 - x_3)^2|}, & \text{if } x_1 = y_1. \end{cases}$$

The *pseudo-Galilean scalar product* of the vectors $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ is defined by

$$X \cdot Y = \begin{cases} x_1 y_1, & \text{if } x_1 \neq 0 \text{ or } y_1 \neq 0, \\ x_2 y_2 - x_3 y_3, & \text{if } x_1 = 0 \text{ and } y_1 = 0. \end{cases}$$

In this sense, the *pseudo-Galilean norm* of a vector X is $\|X\| = \sqrt{|X \cdot X|}$. A vector $X = (x_1, x_2, x_3)$ is called *isotropic (non-isotropic)* if $x_1 = 0$ ($x_1 \neq 0$). All unit non-isotropic vectors are of the form $(1, x_2, x_3)$. The isotropic vector $X = (0, x_2, x_3)$ is called *spacelike, timelike and lightlike* if $x_2^2 - x_3^2 > 0$, $x_2^2 - x_3^2 < 0$ and $x_2 = \pm x_3$, respectively. The *pseudo-Galilean cross product* of X and Y on \mathbb{G}_3^1 is given by

$$X \times Y = \begin{vmatrix} 0 & -e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix},$$

where $e_i = (\delta_{i1}, \delta_{i2}, \delta_{i3})$, $i = 2, 3$.

Let M^2 be a surface in the pseudo-Galilean space \mathbb{G}_3^1 parametrized by

$$r(u_1, u_2) = (r_1(u_1, u_2), r_2(u_1, u_2), r_3(u_1, u_2)).$$

Denote $(r_i)_{u_j} = \partial r_i / \partial u_j$ and $(r_i)_{u_j u_k} = \partial^2 r_i / \partial u_j \partial u_k$, $i = 1, 2, 3$ and $j, k = 1, 2$. Then such a surface is *admissible* if and only if $(r_1)_{u_j} = \partial r_1 / \partial u_j \neq 0$ for some $j = 1, 2$. The coefficients of the first fundamental form of M^2 are

$$g_i = (r_1)_{u_i} \text{ and } h_{ij} = (0, (r_2)_{u_i}, (r_3)_{u_i}) \cdot (0, (r_2)_{u_j}, (r_3)_{u_j}), \quad i, j = 1, 2,$$

and, in matrix form, it can be written as

$$ds^2 = \begin{pmatrix} ds_1^2 & 0 \\ 0 & ds_2^2 \end{pmatrix},$$

where $ds_1^2 = (g_1 du_1 + g_2 du_2)^2$ and $ds_2^2 = h_{11} du_1^2 + 2h_{12} du_1 du_2 + h_{22} du_2^2$.

Let us define the function W as

$$W = \sqrt{\left| \left((r_1)_{u_1} (r_2)_{u_2} - (r_1)_{u_2} (r_2)_{u_1} \right)^2 - \left((r_1)_{u_1} (r_3)_{u_2} - (r_1)_{u_2} (r_3)_{u_1} \right)^2 \right|}.$$

Then the normal vector field N of M^2 is given by

$$N = \frac{1}{W} (0, (r_1)_{u_1} (r_3)_{u_2} - (r_1)_{u_2} (r_3)_{u_1}, (r_1)_{u_1} (r_2)_{u_2} - (r_1)_{u_2} (r_2)_{u_1}),$$

where $N \cdot N = \epsilon = \pm 1$. Hence, two types of admissible surfaces can be distinguished: spacelike surfaces having timelike unit normals ($\epsilon = -1$) and timelike ones having spacelike unit normals ($\epsilon = 1$).

The coefficients of the second fundamental form of M^2 are

$$\begin{aligned} L_{ij} &= \epsilon \frac{1}{g_1} \left(g_1 (0, (r_2)_{u_i u_j}, (r_3)_{u_i u_j}) - (g_i)_{u_j} (0, (r_2)_{u_1}, (r_3)_{u_1}) \right) \cdot N \\ &= \epsilon \frac{1}{g_2} \left(g_2 (0, (r_2)_{u_i u_j}, (r_3)_{u_i u_j}) - (g_i)_{u_j} (0, (r_2)_{u_2}, (r_3)_{u_2}) \right) \cdot N. \end{aligned}$$

The Gaussian curvature and the mean curvature of M^2 are respectively defined by

$$K = -\epsilon \frac{L_{11} L_{22} - L_{12}^2}{W^2}$$

and

$$H = -\epsilon \frac{g_2^2 L_{11} - 2g_1 g_2 L_{12} + g_1^2 L_{22}}{2W^2}.$$

A surface in \mathbb{G}_3^1 is said to be *minimal* if its mean curvature vanishes.

3. FACTORABLE SURFACES WITH NGC IN \mathbb{G}_3^1

A surface M^2 in the pseudo-Galilean space \mathbb{G}_3^1 is called a *factorable surface* if it can be locally written as

$$(3.1) \quad r(u_1, u_2) := (u_1, u_2, f_1(u_1) f_2(u_2)),$$

or

$$(3.2) \quad r(u_1, u_3) := (u_1, f_1(u_1) f_3(u_3), u_3),$$

or

$$(3.3) \quad r(u_2, u_3) := (f_2(u_2) f_3(u_3), u_2, u_3),$$

where f_i are smooth functions of one variable, for all $i \in \{1, 2, 3\}$. We call the surfaces given by (3.1), (3.2) and (3.3) the *factorable surfaces of the first type, the second type* and *the third type*, respectively. The factorable surfaces in the Euclidean space, the pseudo-Euclidean space and the Heisenberg group have been studied in [2, 12, 18, 21].

The following result provides a complete classification of the factorable surfaces having NGC in \mathbb{G}_3^1 .

THEOREM 3.1. *Let M^2 be a factorable surface with NGC in \mathbb{G}_3^1 . If M^2 is a factorable surface of the first type (respectively the second type and the third type), then either*

- (a) *at least one of f_1, f_2 (respectively f_1, f_3 and f_2, f_3) is a constant function, or*
- (b) *$f_i(u_i) = \gamma_i e^{\beta_i u_i}$, where $\beta_i, \gamma_i \in \mathbb{R} \setminus \{0\}$, $i \in \{1, 2\}$, (respectively $i \in \{1, 3\}$ and $i \in \{2, 3\}$) or*
- (c) *$f_i(u_i) = [(1 - \alpha_i) \zeta_i u_i + \lambda_i]^{\frac{1}{1-\alpha_i}}$, where $\alpha_i \neq 0, 1$, $\alpha_i \in \mathbb{R}$ and $\alpha_i \alpha_j = 1$, $\zeta_i \in \mathbb{R} \setminus \{0\}$ and $\lambda_i \in \mathbb{R}$, $i \in \{1, 2\}$ (respectively $i \in \{1, 3\}$ and $i \in \{2, 3\}$).*

Conversely, the factorable surfaces satisfying the conditions in cases (a)-(c) have NGC.

PROOF. It is only proved for the factorable surfaces of the first type since one can be verified by using the following way for the factorable surfaces of the second and the third types.

Let M^2 be a factorable surface of the first type in \mathbb{G}_3^1 . Then the coefficients of the first and the second fundamental forms of M^2 are

$$(3.4) \quad \begin{aligned} g_1 &= 1, & g_2 &= 0, \\ h_{11} &= -(f_1' f_2)^2, & h_{12} &= -(f_1 f_2) f_1' f_2', & h_{22} &= 1 - (f_1 f_2')^2, \\ N &= \frac{(0, f_1 f_2', 1)}{W}, \\ L_{11} &= \frac{-\varepsilon}{W} (f_1'' f_2), & L_{12} &= \frac{-\varepsilon}{W} (f_1' f_2'), & L_{22} &= \frac{-\varepsilon}{W} (f_1 f_2''), \end{aligned}$$

where $f'_i = \frac{df_i}{du_i}$, $f''_i = \frac{d^2f_i}{du_i^2}$, $i \in \{1, 2\}$, and $W = \sqrt{|1 - (f_1 f'_2)^2|}$. It can be easily seen that the Gaussian curvature of M^2 is

$$(3.5) \quad K = \frac{-\varepsilon}{W^4} \left[(f''_1 f''_2) f_1 f_2 - (f'_1 f'_2)^2 \right].$$

Now, let us assume that M^2 has NGC. It immediately implies that f_1 or f_2 is a constant function which gives case (a). If f_1 and f_2 are non-constant functions, then by (3.5), we get

$$(3.6) \quad (f''_1 f''_2) f_1 f_2 - (f'_1 f'_2)^2 = 0.$$

Since f''_i , $i \in \{1, 2\}$, cannot be zero, the equation (3.6) can be written as

$$(3.7) \quad \frac{f''_1 f_1}{(f'_1)^2} - \frac{(f'_2)^2}{f'_2 f_2} = 0.$$

Thus equation (3.7) yields the following ordinary differential equation (ODE)

$$(3.8) \quad \frac{f''_i f_i}{(f'_i)^2} = \alpha_i, \quad i \in \{1, 2\}$$

for $\alpha_i \in \mathbb{R} \setminus \{0\}$ and $\alpha_i \alpha_j = 1$. Equation (3.8) can be reduced to the following Bernoulli ODE

$$(3.9) \quad f'_i - \beta_i f_i^{\alpha_i} = 0, \quad \beta_i \in \mathbb{R} \setminus \{0\}, \quad i \in \{1, 2\}.$$

For $\alpha_i = 1$, $i \in \{1, 2\}$, after solving the ODE (3.9), we get

$$f_i(u_i) = \gamma_i e^{\beta_i u_i}, \quad \gamma_i \in \mathbb{R} \setminus \{0\}, \quad i \in \{1, 2\},$$

which gives case (b). If $\alpha_i \neq 1$, $i \in \{1, 2\}$, then by solving the ODE (3.9), we obtain

$$f_i(u_i) = [(1 - \alpha_i) \zeta_i u_i + \lambda_i]^{\frac{1}{1-\alpha_i}},$$

where $\alpha_i \in \mathbb{R} \setminus \{0\}$ and $\alpha_i \alpha_j = 1$, $\zeta_i \in \mathbb{R} \setminus \{0\}$ and $\lambda_i \in \mathbb{R}$, for $i \in \{1, 2\}$, which proves case (c).

Conversely, it is direct to verify that all of the factorable surfaces given by in cases (a)-(c) have NGC. □

EXAMPLE 3.2. Let M^2 be a factorable surface of the first type having NGC in \mathbb{G}_3^1 . Taking the parametrizations $u_3 = 0.5(u_1)^{0.5}$, $u_3 = 1.5 \exp(u_1 + u_2) - 5$ and $u_3 = -0.25(u_1)^{-1}(u_2)^2$ defined as in three cases of Theorem 3.1, then we respectively draw those as in Figure 1, Figure 2, Figure 3.

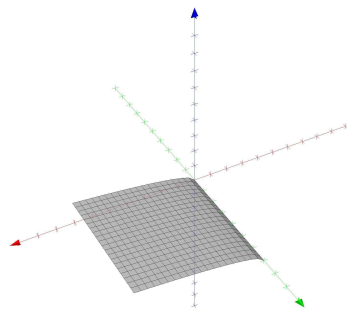


FIGURE 1. The factorable surface having NGC with $u_3 = 0.5(u_1)^{0.5}$.

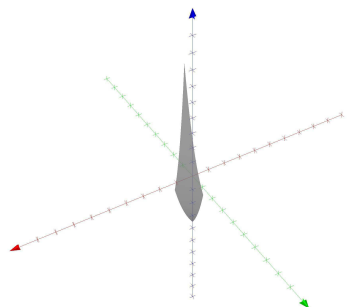


FIGURE 2. The factorable surface having NGC with $u_3 = 1.5 \exp(u_1 + u_2) - 5$.

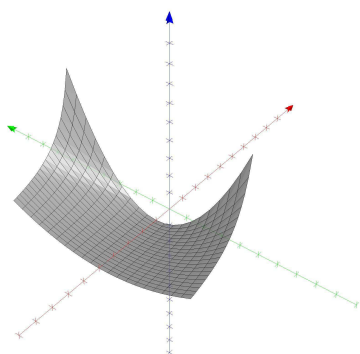


FIGURE 3. The factorable surface having NGC with $u_3 = -0.25(u_1)^{-1}(u_2)^2$.

4. MINIMAL FACTORABLE SURFACES IN \mathbb{G}_3^1

Next theorem provides a classification for minimal factorable surfaces in \mathbb{G}_3^1 .

THEOREM 4.1. *Let M^2 be a factorable surface in \mathbb{G}_3^1 . Then M^2 is minimal if and only if one of the following statements holds:*

- (a) *when M^2 is of the first type, f_2 is a linear function;*
- (b) *when M^2 is of the second type, f_3 is a linear function;*
- (c) *when M^2 is of the third type, either*
 - (c.i) *at least one of f_2 and f_3 is a constant function, or*
 - (c.ii) *$f_i(u_i) = \gamma_i e^{\beta_i u_i}$, where $\beta_i, \gamma_i \in \mathbb{R} \setminus \{0\}$, $i \in \{2, 3\}$, or*
 - (c.iii) *$f_i(u_i) = [(1 - \alpha_i)\zeta_i u_i + \lambda_i]^{\frac{1}{1-\alpha_i}}$, where $\alpha_i \neq 1$, $\alpha_i \in \mathbb{R}$ and $\alpha_2 + \alpha_3 = 2$, $\zeta_i \in \mathbb{R} \setminus \{0\}$ and $\lambda_i \in \mathbb{R}$, $i \in \{2, 3\}$.*

PROOF. Let M^2 be a factorable surface of the first type, i.e. it is parametrized by (3.1). It follows from (3.4) that the mean curvature of M^2 is

$$(4.1) \quad H = \frac{1}{2W^3} [f_1 f_2''].$$

Equation (4.1) implies that M^2 is minimal if and only if f_2 is a linear function. Similarly, it is not hard to prove that f_3 is a linear function when the factorable surface of the second type is minimal. This proves case (a) and case (b).

Now, let M^2 be a factorable surface of the third type. Then the coefficients of first and second fundamental forms of M^2 are

$$(4.2) \quad \begin{aligned} g_1 &= f_2' f_3, & g_2 &= f_2 f_3', & h_{11} &= 1, & h_{12} &= 0, & h_{22} &= -1, \\ N &= \frac{(0, f_2' f_3, -f_2 f_3')}{W}, \\ L_{11} &= \frac{-\varepsilon}{W} (f_2'' f_3), & L_{12} &= \frac{-\varepsilon}{W} (f_2' f_3'), & L_{22} &= \frac{-\varepsilon}{W} (f_2 f_3''). \end{aligned}$$

where $W = \sqrt{|(f_2' f_3)^2 - (f_2 f_3')^2|}$. From (4.2), the mean curvature of M^2 is

$$(4.3) \quad H = \frac{1}{2W^3} [(f_2 f_3')^2 f_2'' f_3 - 2 (f_2' f_3')^2 (f_2 f_3) + (f_2' f_3)^2 f_2 f_3''].$$

If M^2 is minimal, then (4.3) implies that

$$(f_2 f_3')^2 f_2'' f_3 - 2 (f_2' f_3')^2 (f_2 f_3) + (f_2' f_3)^2 f_2 f_3'' = 0.$$

If one of the f_2, f_3 is a constant function, then the previous formula holds. This gives the proof of case (c.i). Otherwise, we conclude

$$(4.4) \quad \frac{f_2'' f_2}{(f_2')^2} + \frac{f_3'' f_3}{(f_3')^2} = 2.$$

We divide the rest part of proof into two cases:

CASE (1): $f_i'' f_i / (f_i')^2 = 1$ for $i \in \{2, 3\}$. Then it is easily seen that $f_i(u_i) = \gamma_i e^{\beta_i u_i}$ for $\beta_i, \gamma_i \in \mathbb{R} \setminus \{0\}$, $i \in \{2, 3\}$, which gives case (c.ii).

CASE (2): $f_i'' f_i / (f_i')^2 \neq 1$ for $i \in \{2, 3\}$. From (4.4), we get

$$(4.5) \quad \frac{f_i'' f_i}{(f_i')^2} = \alpha_i, \quad \alpha_i \in \mathbb{R}, \quad \alpha_i \neq 1, i \in \{2, 3\},$$

such that $\alpha_2 + \alpha_3 = 2$. After solving the ODE (4.5), we derive

$$f_i(u_i) = [(1 - \alpha_i) \zeta_i u_i + \lambda_i]^{\frac{1}{1-\alpha_i}},$$

where $\zeta_i \in \mathbb{R} \setminus \{0\}$ and $\lambda_i \in \mathbb{R}$, $i \in \{2, 3\}$. This completes to prove the case (c.iii).

Conversely, it is easy to verify that all of the factorable surfaces given by in cases (a)-(c.iii) are minimal. □

EXAMPLE 4.2. Let M^2 be a minimal factorable surface in \mathbb{G}_3^1 . Choosing $u_3 = \cos(u_1)(u_2 + 1)$, $u_2 = \sin(u_1)(u_3 + 1)$ and $u_1 = (2u_2)^{0.5}(-2u_3)^{-0.5}$ given as in case (a), case (b) and case (c.iii) of Theorem 4.1, we respectively draw ones as in Figure 4, Figure 5, Figure 6.

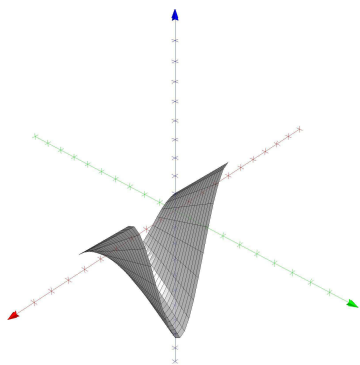


FIGURE 4. The minimal factorable surface of the first type with $u_3 = \cos(u_1)(u_2 + 1)$.

On the other hand, the authors in [8] introduced a new kind of curvature for the hypersurfaces of Euclidean $(n + 1)$ -spaces, called by *amalgamatic curvature* and explored its geometric meaning by proving an inequality related to the absolute mean curvature of the hypersurface. In three dimensional case, the amalgamatic curvature is indeed the harmonic ratio of the principal curvatures of any given surface, i.e., the ratio of the Gaussian curvature and the mean curvature.

Therefore, in the special case that f_1 is a linear function, we are able to analyze the factorable surfaces of first and second type having the ratio $K/H = const. \neq 0$. For this purpose, let M^2 be a factorable surfaces of the first type. Take $f_1(u_1) = \alpha_1 u_1 + \beta_1$. Because of the Theorem 4.1, f_2 cannot

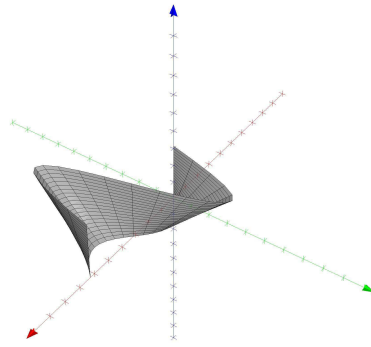


FIGURE 5. The minimal factorable surface of the second type with $u_2 = \sin(u_1)(u_3 + 1)$.

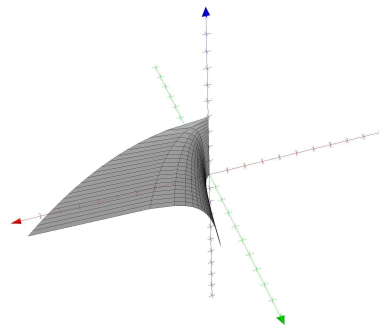


FIGURE 6. The minimal factorable surface of the third type with $u_1 = (2u_2)^{0.5}(-2u_3)^{-0.5}$.

be linear function. Then from (3.5) and (4.1), the ratio of the Gaussian curvature and the mean curvature of M^2 is

$$\frac{K}{H} = \frac{1}{2W} \frac{\alpha_1^2 (f_2')^2}{(\alpha_1 u_1 + \beta_1) f_2''}.$$

Suppose that $\frac{K}{H} = \lambda = \text{const}$. Then we get

$$(4.6) \quad \frac{1}{2W} \frac{\alpha_1^2 (f_2')^2}{(\alpha_1 u_1 + \beta_1) f_2''} = \lambda.$$

Taking the partial derivative of (4.6) with respect to u_1 , we derive

$$(4.7) \quad \left| 1 - (f_1 f_2')^2 \right| - (f_1 f_2')^2 = 0.$$

If M^2 is timelike, then $(f_1 f_2')^2 - 1 < 0$, which yields a contradiction. Otherwise, (4.7) implies that

$$(4.8) \quad (\alpha_1 u_1 + \beta_1)^2 (f_2')^2 = \frac{1}{2}.$$

Again by taking partial derivative of (4.8) with respect to u_1 , we obtain

$$\alpha_1 (\alpha_1 u_1 + \beta_1) (f_2')^2 = 0,$$

which is not possible. This implies that there does not exist a factorable surface of the first type fulfilling the ratio $K/H = \text{const.} \neq 0$ in \mathbb{G}_3^1 . Similarly, one can also be shown for the factorable surfaces of the second type.

Therefore, we have proved the following.

COROLLARY 4.3. *Let f_1 be a linear function. Then there does not exist a factorable surface of the first and the second type fulfilling the ratio $K/H = \text{const.} \neq 0$ in \mathbb{G}_3^1 .*

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