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CLASSIFICATION OF ROTATIONAL SURFACES IN PSEUDO-GALILEAN SPACE

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Abstract. In the present paper, we study rotational surfaces in the three dimensional pseudo-Galilean space G_3^1. Also, we characterize rotational surfaces in G_3^1 in terms of the position vector field, Gauss map and Laplacian operator of the second fundamental form on the surface.

1. Introduction

Let M be a connected n-dimensional submanifold of the m-dimensional Euclidean space $\mathbb{E}^m$, equipped with the induced metric. Denote by Δ the Laplacian of M acting on smooth functions on M. Takahashi ([17]) classified the submanifolds in $\mathbb{E}^m$ in terms of an isometric immersion $\mathbf{x}$ and the Laplacian of M. He proved that M satisfying $\Delta\mathbf{x} = \lambda\mathbf{x}$, that is, all coordinate functions are eigenfunctions of the Laplacian with the same eigenvalue $\lambda \in \mathbb{R}$ are either the minimal submanifolds of $\mathbb{E}^m$ or the minimal submanifolds of hypersphere $\mathbb{S}^{m-1}$ in $\mathbb{E}^m$. As a generalization of Takahashi's theorem for the case of hypersurfaces, Garay ([8]) considered the hypersurfaces in $\mathbb{E}^m$ whose coordinate functions are eigenfunctions of the Laplacian, that is, he studied the hypersurfaces satisfying the condition

$$\Delta\mathbf{x} = A\mathbf{x}, \tag{1.1}$$

where $A \in \mathrm{Mat}(m, \mathbb{R})$ is an $m \times m$- diagonal matrix.

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On the other hand, the study of an isometric immersion satisfying (1.1) can be extended to Gauss map on a hypersurface of Euclidean space. The Gauss map is a useful tool to examine the character of the hypersurfaces in Euclidean space. In [6], Dillen, Pas and Verstraelen studied the surfaces of revolution in Euclidean 3-space $\mathbb{E}^3$ such that its Gauss map G satisfies the condition

$$\Delta G = AG, \tag{1.2}$$

where $A \in \mathrm{Mat}(3, \mathbb{R})$ is a 3×3-real matrix. Several geometers have studied surfaces satisfying the conditions (1.1) and (1.2) in the ambient spaces, and many interesting results have been obtained in [1–4, 18, 19] etc.

If a surface M in the ambient spaces has a non-degenerate second fundamental form II or a non-degenerate third fundamental form III, then it is regarded as a new (pseudo-)Riemannian metric on M. So, considering the conditions (1.1) and (1.2), we may have a natural question as follows: *What are the surfaces in the ambient spaces satisfying the conditions*

$$\Delta^p \mathbf{x} = A\mathbf{x}, \tag{1.3}$$

$$\Delta^p G = AG, \tag{1.4}$$

where Δ^p *is the Laplacian with respect to* p *of* M, $p = II$ *or* III *and* $A \in Mat(3, \mathbb{R})$*?*

Several results for the above question were obtained, when the ambient spaces are the Euclidean space ([11]) and the Minkowski space ([5, 9, 10, 12, 14, 15].

The main purpose of this paper is to complete classification of rotational surfaces in the three dimensional pseudo-Galilean space G_3^1 satisfying (1.3) and (1.4) with $p = II$.

2. Preliminaries

The pseudo-Galilean geometry is one of the real Cayley-Klein geometries (of projective signature $(0, 0, +, -)$, explained in [13]). The absolute of the pseudo-Galilean geometry is an ordered triple $\{\omega, f, I\}$, where ω is the ideal (absolute) plane, f the line in ω and I the fixed hyperbolic involution of f.

In affine coordinates defined by $(x_0 : x_1 : x_2 : x_3) = (1 : x : y : z)$, the distance between the points $P_i = (x_i, y_i, z_i)$ $(i = 1, 2)$ is defined by (cf. [16])

$$d(P_1, P_2) = \begin{cases} |x_2 - x_1|, & \text{if } x_1 \neq x_2, \\ \sqrt{|(y_2 - y_1)^2 - (z_2 - z_1)^2|}, & \text{if } x_1 = x_2. \end{cases}$$

The group motions of G_3^1 is a six-parameter group given (in affine coordinates) by

$$\begin{aligned}\bar{x} &= a + x,\\ \bar{y} &= b + cx + y\cosh\varphi + z\sinh\varphi,\\ \bar{z} &= d + ex + y\sinh\varphi + z\cosh\varphi.\end{aligned}$$

Let $\mathbf{x} = (x_1, y_1, z_1)$ and $\mathbf{y} = (x_2, y_2, z_2)$ be vectors in G_3^1. A vector $\mathbf{x}$ is called isotropic if $x_1 = 0$, otherwise it is called nonisotropic. The pseudo-Galilean scalar product of $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\langle \mathbf{x}, \mathbf{y}\rangle = \begin{cases} x_1x_2, & \text{if } x_1 \neq 0 \quad\text{or}\quad x_2 \neq 0,\\ y_1y_2 - z_1z_2, & \text{if } x_1 = 0 \quad\text{and}\quad x_2 = 0.\end{cases}$$

From this, the pseudo-Galilean norm of a vector $\mathbf{x}$ in G_3^1 is given by $||\mathbf{x}|| = \sqrt{|\langle \mathbf{x}, \mathbf{x}\rangle|}$ and all unit nonisotropic vectors are the form $(1, y_1, z_1)$. There are four types of isotropic vectors: spacelike ($y_1^2 - z_1^2 > 0$), timelike ($y_1^2 - z_1^2 < 0$) and the two types of lightlike ($y_1 = \pm z_1$) vectors. A non-lightlike isotropic vector is a unit vector if $y_1^2 - z_1^2 = \pm 1$.

The pseudo-Galilean cross product of $\mathbf{x}$ and $\mathbf{y}$ on G_3^1 is defined by

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} 0 & -e_2 & e_3\\ x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\end{vmatrix},$$

where $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

Consider a C^r-surface M, $r \geq 1$, in G_3^1 parametrized by

$$\mathbf{x}(u_1, u_2) = (x(u_1, u_2), y(u_1, u_2), z(u_1, u_2)). \tag{2.1}$$

Let us denote $g_i = \frac{\partial x}{\partial u_i}$, $h_{ij} = \langle \frac{\partial \tilde{\mathbf{x}}}{\partial u_i}, \frac{\partial \tilde{\mathbf{x}}}{\partial u_j}\rangle (i, j = 1, 2)$, where $^\sim$ stands for the projection of a vector on the pseudo-Euclidean yz-plane. A surface M is called admissible if it does not have Euclidean tangent planes. Therefore a surface M is admissible if and only if $x_{,i} \neq 0$ for some $i = 1, 2$.

Let M be an admissible surface. Then the unit normal vector field U of a surface M is defined by

$$U = \frac{1}{W}(0, x_{,1}z_{,2} - x_{,2}z_{,1}, x_{,1}y_{,2} - x_{,2}y_{,1}), \tag{2.2}$$

where

$$W = \sqrt{|(x_{,1}y_{,2} - x_{,2}y_{,1})^2 - (x_{,1}z_{,2} - x_{,2}z_{,1})^2|}.$$

On the other hand, the matrix of the first fundamental form ds^2 of a surface M in G_3 is given by ([16])

$$ds^2 = \begin{pmatrix} ds_1^2 & 0\\ 0 & ds_2^2\end{pmatrix},$$

where $ds_1^2 = (g_1du_1 + g_2du_2)^2$ and $ds_2^2 = h_{11}du_1^2 + 2h_{12}du_1du_2 + h_{22}du_2^2$. Here $g_i = x_{,i}$ and $h_{ij} = \langle \tilde{\mathbf{x}}_{,i}, \tilde{\mathbf{x}}_{,j}\rangle$ $(i, j = 1, 2)$. In such case, we denote the coefficients of ds^2 by g^*_{ij}. A surface is spacelike or timelike if the determinant of the matrix (g^*_{ij}) is positive or negative, respectively.

The coefficients $L_{ij}, i,j = 1,2$ of the second fundamental form II, which are the normal components of $\mathbf{x}_{,i,j}, i,j = 1,2$, that is,

$$L_{ij} = \frac{1}{g_1}\langle g_1\tilde{\mathbf{x}}_{,i,j} - g_{i,j}\tilde{\mathbf{x}}_{,1}, U\rangle = \frac{1}{g_2}\langle g_2\tilde{\mathbf{x}}_{,i,j} - g_{i,j}\tilde{\mathbf{x}}_{,2}, U\rangle. \tag{2.3}$$

If a surface M in G^1_3 has a non-degenerate second fundamental form II, then it is regarded as a new (pseudo-)Riemannian metric on (M, II). Let $\{u_1, u_2\}$ be a local coordinate system of M and $L_{ij}(i,j = 1,2)$ be the coefficients of the second fundamental form II on M. We denote by (L^{ij}) (resp. $\mathfrak{L}$) the inverse matrix (resp. the determinant) of the matrix (L_{ij}). Then, the Laplacian Δ^{II} of the second fundamental form II on M is defined by

$$\Delta^{II} = -\frac{1}{\sqrt{|\mathfrak{L}|}}\sum_{i,j=1}^{2}\frac{\partial}{\partial u_i}(\sqrt{|\mathfrak{L}|}L^{ij}\frac{\partial}{\partial u_j}). \tag{2.6}$$

3. Rotational surfaces in G^1_3

In the pseudo-Galilean space G^1_3 we distinguish between two types of circles and between two types of rotational surfaces. The first type occurs as the result of a pseudo-Euclidean rotation and the second as the result of an isotropic rotation. It is well-known that the normal form of pseudo-Euclidean rotations is given by

$$\begin{aligned}\bar{x} &= x,\\ \bar{y} &= y\cosh t + z\sinh t,\\ \bar{z} &= y\sinh t + z\cosh t\end{aligned} \tag{3.1}$$

and the normal form of isotropic rotations is given by

$$\begin{aligned}\bar{x} &= x + bt,\\ \bar{y} &= y + xt + b\frac{t^2}{2},\\ \bar{z} &= z,\end{aligned} \tag{3.2}$$

where $t \in \mathbb{R}$ and $b = \text{constant} > 0$ (cf. [16]).

The trajectory of a single point under a pseudo-Euclidean rotation is a pseudo-Euclidean circle (i.e., a rectangular hyperbola)

$$x = \text{constant}, \quad y^2 - z^2 = r^2, \quad r \in \mathbb{R}.$$

The invariant r is the radius of the circle. Pseudo-Euclidean circles intersect the absolute line f in the fixed points of the hyperbolic involution (F_1, F_2). There are three kinds of pseudo-Euclidean circles: circles of real radius, of imaginary radius, and of radius zero. Circles of real radius are timelike curves (having timelike tangent vectors) and imaginary radius spacelike curves (having spacelike tangent vectors).

The trajectory of a point under an isotropic rotation is an isotropic circle whose normal form is

$$z = \text{constant}, \quad y = \frac{x^2}{2b}.$$

The invariant b is the radius of the circle. The fixed line of the isotropic rotation (3.2) is the absolute line f.

Let α be a plane curve and l a given line. It is convenient, but not necessary, to start with a plane curve α. A rotational surface is a surface obtained when α is displaced in a pseudo-Euclidean rotation or an isotropic rotation about l ([7]).

First of all, we consider a nonisotropic curve α parameterized by

$$\alpha(u) = (f(u), g(u), 0) \quad \text{or} \quad \alpha(u) = (f(u), 0, g(u))$$

around the x-axis by the pseudo-Euclidean rotation (3.1), where g is a positive function and f is a smooth function on an open interval I. Then the rotational surface M can be written as

$$\mathbf{x}(u, v) = (f(u), g(u)\cosh v, g(u)\sinh v), \tag{3.3}$$

or

$$\mathbf{x}(u, v) = (f(u), g(u)\sinh v, g(u)\cosh v), \tag{3.4}$$

for any $v \in \mathbb{R}$, which is called a *nonisotropic rotational surface.*

Next, we consider the isotropic rotations. By an isotropic curve $\alpha(u) = (0, f(u), g(u))$ about the z-axis by an isotropic rotation (3.2), we obtain a surface

$$\mathbf{x}(u, v) = \left(v, f(u) + \frac{v^2}{2b}, g(u)\right), \tag{3.5}$$

where f and g are smooth functions and $b \neq 0$. This surface is called an *isotropic rotational surface.*

4. Rotational surfaces satisfying $\Delta^{II}\mathbf{x} = A\mathbf{x}$

In this section, we classify rotational surfaces in G_3^1 satisfying the condition

$$\Delta^{II}\mathbf{x} = A\mathbf{x}, \tag{4.1}$$

where $A = (a_{ij}), i, j = 1, 2, 3.$

First of all, let M be a rotational surface in G_3^1 defined by (3.3). Assume that the rotated curve α is parameterized by arc-length, that is,

$$\alpha(u) = (u, g(u), 0).$$

In this case, the parametrization of M is given by

$$\mathbf{x}(u,v) = (u, g(u)\cosh v, g(u)\sinh v), \tag{4.2}$$

where g is a positive function.

From now on, we shall often not write the parameter u explicitly in our formulas. By (2.3), the coefficients of the second fundamental form II on M are obtained by

$$L_{11} = g'',\ L_{12} = 0,\ L_{22} = g. \tag{4.3}$$

Here the prime denotes the derivative with respect to u. Since the surface has a non-degenerate second fundamental form II, the functions g and g'' are non-vanishing everywhere. By a straightforward computation, the Laplacian Δ^{II} of the second fundamental form II on M can be expressible as

$$\Delta^{II} = -\frac{1}{2g''}(\frac{g'}{g} - \frac{g'''}{g''})\frac{\partial}{\partial u} - \frac{1}{g''}\frac{\partial^2}{\partial u^2} - \frac{1}{g}\frac{\partial^2}{\partial v^2}. \tag{4.4}$$

Suppose that M satisfies (4.1). Then, from (4.2) and (4.4), we have the following system of differential equations:

$$\begin{aligned} -\frac{1}{2g''}(\frac{g'}{g} - \frac{g'''}{g''}) &= a_{11}u + a_{12}g\cosh v + a_{13}g\sinh v,\\ \left(-\frac{g'}{2g''}(\frac{g'}{g} - \frac{g'''}{g''}) - 2\right)\cosh v &= a_{21}u + a_{22}g\cosh v + a_{23}g\sinh v,\\ \left(-\frac{g'}{2g''}(\frac{g'}{g} - \frac{g'''}{g''}) - 2\right)\sinh v &= a_{31}u + a_{32}g\cosh v + a_{33}g\sinh v. \end{aligned} \tag{4.5}$$

From (4.5) we easily deduce that $a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0$ and $a_{22} = a_{33}$, that is, the matrix A is diagonal. We put $a_{11} = \lambda$ and $a_{22} = a_{33} = \mu$. Then the system (4.5) reduces now to the following equations

$$\begin{aligned} g'g'' - gg''' &= -2\lambda ugg''^2,\\ g'(g'g'' - gg''') &= 2\mu g^2g''^2 + 4gg''^2. \end{aligned} \tag{4.6}$$

Combining two equations in (4.6) we get

$$\lambda ug' + \mu g + 2 = 0. \tag{4.7}$$

Thus its general solution is either

$$g(u) = -\frac{2}{\lambda}\ln u + c_1 \tag{4.8}$$

if $\lambda \neq 0$ and $\mu = 0$, or

$$g(u) = \frac{1}{\mu}(c_1u^{-\frac{\mu}{\lambda}} - 2) \tag{4.9}$$

if $\lambda \neq 0$ and $\mu \neq 0$, where c_1 is constant. The remained cases with respect to λ and μ are do not appear. Consequently, we have

THEOREM 4.1. *Let M be a nonisotropic rotational surface generated by a curve $\alpha(u) = (u, g(u), 0)$ in the three dimensional pseudo-Galilean space G_3^1. If M satisfies the condition (4.1), then M is a timelike surface and parameterized as*

$$\mathbf{x}(u,v) = (u, g(u)\cosh v, g(u)\sinh v),$$

where

(1) *either* $g(u) = -\frac{2}{\lambda}\ln u + c_1$,
(2) *or* $g(u) = \frac{1}{\mu}(c_1 u^{-\frac{\mu}{\lambda}} - 2)$ *with* $\lambda \neq 0, \mu \neq 0, c_1 \in \mathbb{R}$.

Remark. For the specific constants λ, μ, c_1 of (1) and (2) in Theorem 4.1, we have the graphs shown in Figure 1 and Figure 2, respectively.

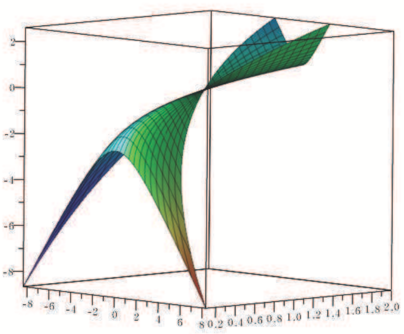

FIGURE 1. A nonoisotopic rotational surface with $g(u) = \ln u$.

Let M be a nonisotropic rotational surface given by (3.4) in the three dimensional pseudo-Galilean space G_3^1. Assume that a nonisotropic curve α is a unit speed curve, that is, $\alpha(u) = (u, 0, g(u))$. In the case, the surface M is parameterized by

$$\mathbf{x}(u,v) = (u, g(u)\sinh v, g(u)\cosh v). \tag{4.10}$$

Suppose that the surface M satisfies (4.1). Then, by using the similar method of Theorem 4.2, we can find the same equation (4.7). Consequently, we have

THEOREM 4.2. *Let M be a nonisotropic rotational surface given by (4.10) in the three dimensional pseudo-Galilean space G_3^1. If M satisfies the condition (4.1), then M is a spacelike surface and parameterized as*

$$\mathbf{x}(u,v) = (u, g(u)\sinh v, g(u)\cosh v),$$

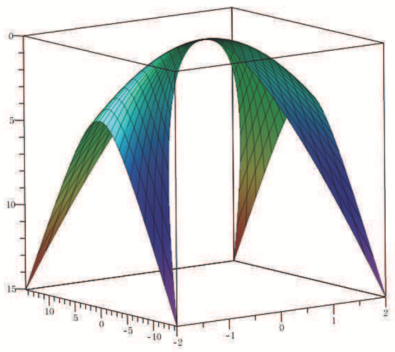

FIGURE 2. A nonoisotopic rotational surface with $g(u) = u^2 - 2$.

where

(1) *either* $g(u) = -\frac{2}{\lambda}\ln u + c_1$,
(2) *or* $g(u) = \frac{1}{\mu}(c_1 u^{-\frac{\mu}{\lambda}} - 2)$ *with* $\lambda \neq 0, \mu \neq 0, c_1 \in \mathbb{R}$.

Next, we consider rotational surfaces M in G_3^1 generated by an isotropic curve $\alpha(u) = (0, f(u), g(u))$. Assume that the parameter u is the arc-length parameter of α, that is,

$$f'(u)^2 - g'(u)^2 = -\epsilon(= \pm 1). \tag{4.11}$$

Then the parametrization of M is given by

$$\mathbf{x}(u, v) = \left(v, f(u) + \frac{v^2}{2b}, g(u)\right), \tag{4.12}$$

where f and g are smooth functions and $b \neq 0$.
On the other hand, the coefficients of the second fundamental form II on M are obtained by

$$L_{11} = \epsilon(f'g'' - f''g'),\ L_{12} = 0,\ L_{22} = \frac{\epsilon}{b}g'. \tag{4.13}$$

Since $\mathfrak{L} = -\frac{\epsilon}{b}f''$, the function f'' is non-vanishing everywhere. From this and (4.11), the function g' is also non-vanishing everywhere. By (4.11) we find $g'' = (f'/g')f''$, it follows that $L_{11} = -f''/g'$. Thus the Laplacian Δ^{II} of M is easily obtained by

$$\Delta^{II} = -\frac{1}{f''}\left(g'' - \frac{f'''g'}{2f''}\right)\frac{\partial}{\partial u} - \frac{g'}{f''}\frac{\partial^2}{\partial u^2} + \frac{\epsilon b}{g'}\frac{\partial^2}{\partial v^2}. \tag{4.14}$$

Equation (4.1) is equivalent to an expression

$$
\begin{aligned}
&0 = a_{11}v + a_{12}\left(f + \frac{v^2}{2b}\right) + a_{13}g,\\
&-\frac{f'}{f''}\left(g'' - \frac{f'''g'}{2f''}\right) - g' + \epsilon\frac{1}{g'} = a_{21}v + a_{22}\left(f + \frac{v^2}{2b}\right) + a_{23}g,\\
&-\frac{g'}{f''}\left(g'' - \frac{f'''g'}{2f''}\right) - \frac{g'g''}{f''} = a_{31}v + a_{32}\left(f + \frac{v^2}{2b}\right) + a_{33}g.
\end{aligned} \tag{4.15}
$$

We can easily find $a_{11} = a_{12} = a_{13} = a_{21} = a_{22} = a_{31} = a_{32} = 0$. In this case, (4.15) is rewritten as the following:

$$
\begin{aligned}
&-\frac{f'}{f''}\left(g'' - \frac{f'''g'}{2f''}\right) - g' + \epsilon\frac{1}{g'} = a_{23}g,\\
&-\frac{g'}{f''}\left(g'' - \frac{f'''g'}{2f''}\right) - \frac{g'g''}{f''} = a_{33}g.
\end{aligned} \tag{4.16}
$$

If we multiply the first equation of (4.16) by g' and the second equation of (4.16) by $-f'$, and add the resulting equations, we get $a_{33}f' = a_{23}g'$. Up to a rigid motion, there exists a real number $k \in \mathbb{R}$ such that $g(u) = kf(u)$. Consequently, the following theorem holds.

THEOREM 4.3. *Let M be an isotropic rotational surface generated by an isotropic curve in the three dimensional pseudo-Galilean space G_3^1. If M satisfies the condition (4.1), then, for any smooth function $f(u)$, M is parameterized as*

$$\mathbf{x}(u,v) = \left(v, f(u) + \frac{v^2}{2b}, kf(u)\right),$$

where k is constant.

5. ROTATIONAL SURFACES SATISFYING $\Delta^{II}G = AG$

In this section, we classify rotational surfaces in G_3^1 satisfying the condition

$$\Delta^{II}G = AG, \tag{5.1}$$

where $A = (a_{ij}), i, j = 1, 2, 3$.

First, we consider a nonisotropic rotational surface M in G_3^1 generated by a curve $\alpha(u) = (u, g(u), 0)$. Then the parametrization of M is given by

$$\mathbf{x}(u,v) = (u, g(u)\cosh v, g(u)\sinh v), \tag{5.2}$$

where g is a positive function. For the nondegeneracy of the second fundamental of M, we assume that g'' is nonvanishing everywhere. From (2.2) the Gauss map G of M is obtained by

$$G = \left(\frac{1}{||\mathbf{x}_u \times \mathbf{x}_v||}\right)\mathbf{x}_u \times \mathbf{x}_v = (0, \cosh v, \sinh v). \tag{5.3}$$

Then, the Laplacian $\Delta^{II}G$ of the Gauss map G together with (4.4) and (5.3) gives

$$\Delta^{II}G = -\frac{1}{g}G. \tag{5.4}$$

Theorem 5.1. *Let M be a nonisotropic rotational surface given by (5.2) in the three dimensional pseudo-Galilean space G_3^1. Then the Gauss map G of M satisfies (5.4).*

If a nonisotropic rotational surface M satisfies (5.1), then the function g is constant. It is a contradiction. Thus, we have

Theorem 5.2. *There is no nonisotropic rotational surfaces given by (5.2) satisfying (5.1) in the three dimensional pseudo-Galilean space G_3^1.*

On a nonisotropic rotational surface given by (4.10), we can also obtain the following result:

Theorem 5.3. *Let M be a nonisotropic rotational surface given by (4.10) in the three dimensional pseudo-Galilean space G_3^1. Then the following statements hold:*

(1) *The Gauss map G of M satisfies $\Delta^{II}G = \frac{1}{g}G$.*

(2) *There is no the surface M satisfying (5.1) in G_3^1.*

Next, let M be an isotropic rotational surface generated by a unit speed isotropic curve in G_3^1. Then M is parameterized by

$$\mathbf{x}(u,v) = \left(v, f(u) + \frac{v^2}{2b}, g(u)\right). \tag{5.5}$$

From this the Gauss map G of M is given by

$$G = (0, -g'(u), -f'(u)). \tag{5.6}$$

Suppose that M satisfies (5.1). Then from (4.14) and (5.6) we get the following system of differential equations:

$$a_{12}g'(u) + a_{13}f'(u) = 0, \tag{5.7}$$

$$-a_{22}g'(u) - a_{23}f'(u) = \frac{1}{f''}\left(\frac{1}{2}f'f''' + f''^2\right), \tag{5.8}$$

$$-a_{32}g'(u) - a_{33}f'(u) = \frac{1}{f''}\left(\frac{1}{2}f'''g' + f''g''\right). \tag{5.9}$$

If we multiply (5.8) by ${g'}^2$ and (5.9) by $-f'g'$, and add the resulting equations, we obtain

$$\epsilon f'' = (a_{32} - a_{23})f'({f'}^2 + \epsilon) + \left((a_{33} - a_{22}){f'}^2 - a_{22}\epsilon\right)g'. \tag{5.10}$$

Thus, (5.7) and (5.10) imply

$$\lambda f'' + \mu f'^3 + \nu f' = 0, \tag{5.11}$$

where

$$\begin{aligned} \lambda &= a_{12}, \\ \mu &= \epsilon\left(a_{13}(a_{33} - a_{22}) - a_{12}(a_{32} - a_{23})\right), \\ \nu &= a_{12}(a_{23} - a_{32}) - a_{13}a_{22}. \end{aligned}$$

Then a general solution of this equation is

$$f(u) = \pm \frac{\lambda}{\sqrt{|\mu\nu|}} \arctan\left(\frac{\sqrt{|e^{\frac{2\nu}{\lambda}(u+d_1)} - \mu|}}{\sqrt{|\mu|}} \right) + d_2. \tag{5.12}$$

From (5.7) and (5.12), the function $g(u)$ becomes

$$g(u) = \mp \frac{\delta}{\sqrt{|\mu\nu|}} \arctan\left(\frac{\sqrt{|e^{\frac{2\nu}{\lambda}(u+d_1)} - \mu|}}{\sqrt{|\mu|}} \right) + d_3, \tag{5.13}$$

where $\delta = a_{13}$ and d_1, d_2, d_3 are constants of integration. The surface generated by (5.12) and (5.13) is showed in Figure 3.

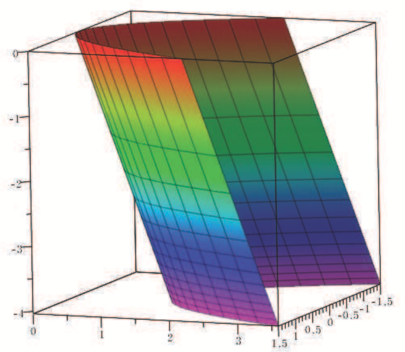

FIGURE 3. A isotropic rotational surface with $f(u) = \arctan(\sqrt{e^{2u} - 1})$ and $g(u) = -3f(u)$.

THEOREM 5.4. *An isotropic rotational surface in the three dimensional pseudo-Galilean space G_3^1 satisfies the condition $\Delta^{II} G = AG$ if and only if* *the surface is a sphere given by*

$$\mathbf{x}(u,v)=\left(v,f(u)+\frac{v^2}{2b},g(u)\right),$$

where $f(u)=\pm\frac{\lambda}{\sqrt{|\mu\nu|}}\arctan\left(\frac{\sqrt{|e^{\frac{2\nu}{\lambda}(u+d_1)}-\mu|}}{\sqrt{|\mu|}}\right)+d_2$ *and* $g(u)=-\frac{\delta}{\lambda}f(u)$.

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