

MORE ON INDUCED MAPS ON n -FOLD SYMMETRIC PRODUCT SUSPENSIONS

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ABSTRACT. We continue the work initiated by the first named author in *Induced maps on n -fold symmetric product suspensions*, *Topology Appl.* 158 (2011), 1192-1205. We consider classes of maps not included in the mentioned paper, namely: almost monotone, atriodic, freely decomposable, joining, monotonically refinable, refinable, semi-confluent, semi-open, simple and strongly freely decomposable maps.

1. INTRODUCTION

We continue the work initiated by the first named author in [2]. We consider classes of maps not included in the mentioned paper, namely: Almost monotone, atriodic, freely decomposable, joining, monotonically refinable, refinable, semi-confluent, semi-open, simple and strongly freely decomposable maps.

Let \mathcal{M} be one of the classes of maps between continua mentioned in the previous paragraph. We study the relationships between the following statements:

- (1) $f \in \mathcal{M}$,
- (2) $\mathcal{F}_n(f) \in \mathcal{M}$,
- (3) $\mathcal{SF}_n(f) \in \mathcal{M}$.

The paper contains eleven sections. After the first two sections, each section deals with one of the classes of maps mentioned above, except: Section 6,

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in which we work with both freely decomposable and strongly freely decomposable maps; and Section 8, in which we work with both, refinable and monotonically refinable maps.

2. DEFINITIONS AND NOTATIONS

If Z is a metric space, then given a subset A of Z , the interior of A is denoted by $\text{Int}_Z(A)$. If $\varepsilon > 0$, then $\mathcal{V}_\varepsilon(A)$ denotes the set of points of X whose distance to A is less than ε . Also, $\text{diam}(A)$ denotes the diameter of A .

Given a function $f: X \rightarrow Y$ between metric spaces, if A is a subset of X , then $f|_A$ denotes the restriction of f to A .

A *continuum* is a nonempty compact, connected and metric space. A *map* is a continuous function.

Let $f: X \rightarrow Y$ be a surjective map between continua. The function f is:

- *almost monotone* provided that for each subcontinuum with nonempty interior Q of Y , $f^{-1}(Q)$ is connected.
- *atriodic* if for every subcontinuum Q of Y , there exist two components C and D of $f^{-1}(Q)$ such that $f(C) \cup f(D) = Q$ and for each component E of $f^{-1}(Q)$, we have that either $f(E) = Q$, or $f(E) \subset f(C)$ or $f(E) \subset f(D)$.
- an ε -*map* provided that $\varepsilon > 0$ and $\text{diam}(f^{-1}(y)) < \varepsilon$ for each $y \in Y$.
- *freely decomposable* if whenever A and B are proper subcontinua of Y such that $Y = A \cup B$, then there exist two proper subcontinua A' and B' of X , such that $X = A' \cup B'$, $f(A') \subset A$ and $f(B') \subset B$.
- *joining* provided that for each subcontinuum Q of Y and for each two components C and D of $f^{-1}(Q)$, we have that $f(C) \cap f(D) \neq \emptyset$.
- *monotone* provided that $f^{-1}(Q)$ is connected, for each subcontinuum Q of Y .
- *monotonically refinable* provided that for each $\varepsilon > 0$, there exists a monotone ε -map $g: X \rightarrow Y$ such that $d(f(x), g(x)) < \varepsilon$ for all $x \in X$.
- *refinable* if for every $\varepsilon > 0$, there exists an ε -map $g: X \rightarrow Y$ such that $d(f(x), g(x)) < \varepsilon$ for all $x \in X$.
- *semi-confluent* provided that for each subcontinuum Q of Y and for each two components C and D of $f^{-1}(Q)$, either $f(C) \subset f(D)$ or $f(D) \subset f(C)$.
- *semi-open* if for each open subset U of X , $\text{Int}_Y(f(U)) \neq \emptyset$.
- *simple* provided that $f^{-1}(y)$ has at most two points for all $y \in Y$.
- *strongly freely decomposable* if whenever A and B are proper subcontinua of Y such that $Y = A \cup B$, we obtain that $f^{-1}(A)$ and $f^{-1}(B)$ are connected¹.

¹Strongly freely decomposable maps are also known in the literature as *feebly monotone* maps.

Given a continuum X , we define its hyperspaces as the following sets:

$$2^X = \{A \subset X \mid A \text{ is closed and nonempty}\};$$

$$\mathcal{C}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ components}\}, \quad n \in \mathbb{N};$$

$$\mathcal{F}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ points}\}, \quad n \in \mathbb{N}.$$

We topologize these sets with the Hausdorff metric, \mathcal{H} ([15, (0.1)]). $\mathcal{C}_n(X)$ is the n -fold hyperspace of X and $\mathcal{F}_n(X)$ is the n -fold symmetric product of X .

Let A_1, \dots, A_m be nonempty subsets of X . We define the set

$$\langle A_1, \dots, A_m \rangle = \{D \in 2^X \mid D \subset \cup_{i=1}^m A_i \text{ and } D \cap A_i \neq \emptyset \text{ for each } i\}.$$

Note that this set is nonempty (for each $j \in \{1, \dots, m\}$, let $a_j \in A_j$, then we have that $\{a_1, \dots, a_m\} \in \langle A_1, \dots, A_m \rangle$). It is known that the family $\{\langle U_1, \dots, U_l \rangle \mid U_1, \dots, U_l \text{ are open subsets of } X\}$ forms a base for a topology on 2^X called the Vietoris topology ([15, (0.11)]). It is well known that the Vietoris topology and the topology induced by the Hausdorff metric coincide ([15, (0.13)]).

NOTATION 2.1. Let X be a continuum, let U_1, \dots, U_m be a finite family of open subsets of X and let n be a positive integer. Then $\langle U_1, \dots, U_m \rangle_n$ denotes the set $\langle U_1, \dots, U_m \rangle \cap \mathcal{F}_n(X)$.

Let n be an integer greater than or equal to 2. Then the n -fold symmetric product suspension ([1]) of a continuum X , denoted by $\mathcal{SF}_n(X)$, is the quotient space $\mathcal{F}_n(X)/\mathcal{F}_1(X)$, with the quotient topology. The quotient map is denoted by $q_X^n: \mathcal{F}_n(X) \twoheadrightarrow \mathcal{SF}_n(X)$ and we denote $q_X^n(\mathcal{F}_1(X))$ by F_X^n .

Let n be an integer greater than or equal to 2, and let X and Y be continua. If $f: X \rightarrow Y$ is a map, then we define the function $\mathcal{F}_n(f): \mathcal{F}_n(X) \rightarrow \mathcal{F}_n(Y)$ by $\mathcal{F}_n(f)(A) = f(A)$ for all $A \in \mathcal{F}_n(X)$, and it is called the induced map of f on the n -fold symmetric products of X and Y . Note that $\mathcal{F}_n(f)$ is continuous ([13, 1.8.23]). We also define the function $\mathcal{SF}_n(f): \mathcal{SF}_n(X) \rightarrow \mathcal{SF}_n(Y)$ ([2]) by

$$\mathcal{SF}_n(f)(\chi) = \begin{cases} q_Y^n(\mathcal{F}_n(f)((q_X^n)^{-1}(\chi))), & \text{if } \chi \neq F_X^n; \\ F_Y^n, & \text{if } \chi = F_X^n. \end{cases}$$

Note that, by [6, 4.3, p. 126], $\mathcal{SF}_n(f)$ is continuous. In addition, the following diagram:

$$\begin{array}{ccc} \mathcal{F}_n(X) & \xrightarrow{\mathcal{F}_n(f)} & \mathcal{F}_n(Y) \\ q_X^n \downarrow & & \downarrow q_Y^n \\ \mathcal{SF}_n(X) & \xrightarrow{\mathcal{SF}_n(f)} & \mathcal{SF}_n(Y) \end{array} \quad (*)$$

is commutative.

3. PRELIMINARY RESULTS

In this section we present the results needed for the rest of the paper. In particular, we prove a couple of theorems that relate the classes of maps defined above and quotient spaces (Theorems 3.2 and 3.3).

Let X be a continuum and let K be a subcontinuum of X . We denote by X/K the quotient space obtained by shrinking K to a point. Then X/K is a continuum ([13, 1.7.3]). Let X and Y be continua, let K be a subcontinuum of X , and let $f: X \twoheadrightarrow Y$ be a surjective map. Consider the quotient spaces X/K and $Y/f(K)$. Let $q_X: X \twoheadrightarrow X/K$ and $q_Y: Y \twoheadrightarrow Y/f(K)$ be the quotient maps. We denote $q_X(K)$ and $q_Y(f(K))$ by K_X and K_Y , respectively. Note that f induces a function $f_\star: X/K \twoheadrightarrow Y/f(K)$ ([6, 7.7, p. 17]) given by

$$f_\star(\chi) = \begin{cases} q_Y(f((q_X)^{-1}(\chi))), & \text{if } \chi \neq K_X; \\ K_Y, & \text{if } \chi = K_X. \end{cases}$$

The continuity of f_\star follows from [6, 4.3, p. 126]. Observe that the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ q_X \downarrow & & \downarrow q_Y \\ X/K & \xrightarrow{f_\star} & Y/f(K) \end{array} \quad (**)$$

is commutative.

PROPOSITION 3.1. *Let X and Y be continua and let K be a subcontinuum of X . If $f: X \twoheadrightarrow Y$ is a surjective map, the following hold:*

- (a) *the maps q_X and q_Y are monotone;*
- (b) *the maps $q_X|_{X \setminus K}: X \setminus K \twoheadrightarrow X/K \setminus \{K_X\}$ and $q_Y|_{Y \setminus f(K)}: Y \setminus f(K) \twoheadrightarrow Y/f(K) \setminus \{K_Y\}$ are homeomorphisms;*
- (c) *if K and $f(K)$ are nowhere dense in X and Y , respectively, then q_X and q_Y are semi-open maps;*
- (d) *for each subcontinuum Γ of $Y/f(K)$, $f_\star^{-1}(\Gamma) = q_X(f^{-1}(q_Y^{-1}(\Gamma)))$;*
- (e) *if Δ and Γ are proper subcontinua of X/K such that $X/K = \Delta \cup \Gamma$, then $q_X^{-1}(\Delta)$ and $q_X^{-1}(\Gamma)$ are proper subcontinua of X such that $X = q_X^{-1}(\Delta) \cup q_X^{-1}(\Gamma)$;*
- (f) *if Γ is a subcontinuum of X/K such that $\text{Int}_{X/K}(\Gamma) \neq \emptyset$, then $q_X^{-1}(\Gamma)$ is a subcontinuum of X such that $\text{Int}_X(q_X^{-1}(\Gamma)) \neq \emptyset$;*
- (g) *if G is a subcontinuum of Y such that $G \cap f(K) = \emptyset$ and D is a component of $f^{-1}(G)$, then $q_X(D)$ is a component of $f_\star^{-1}(q_Y(G))$;*
- (h) *if G is a subcontinuum of Y such that $G \cap f(K) = \emptyset$ and Δ is a component of $f_\star^{-1}(q_Y(G))$, then $q_X^{-1}(\Delta)$ is a component of $f^{-1}(G)$;*
- (i) *if Γ is a subcontinuum of $Y/f(K)$ and D is a component of $f^{-1}(q_Y^{-1}(\Gamma))$, then $q_X(D)$ is a component of $f_\star^{-1}(\Gamma)$;*

(j) if Γ is a subcontinuum of $Y/f(K)$ and Δ is a component of $f_\star^{-1}(\Gamma)$, then there exists a component D of $f^{-1}(q_Y^{-1}(\Gamma))$ such that $q_X(D) = \Delta$.

PROOF. Note that (a) and (b) follow from the definitions. Also, the proofs of (c), (d), (e) and (f) are simple.

To see (g), let G be a subcontinuum of Y such that $G \cap f(K) = \emptyset$, and let D be a component of $f^{-1}(G)$. Since $D \subset f^{-1}(G)$, $q_Y(f(D)) \subset q_Y(G)$. Then, since $(**)$ is commutative, we have that $f_\star(q_X(D)) \subset q_Y(G)$. Let Δ be the component of $f_\star^{-1}(q_Y(G))$ containing $q_X(D)$. Hence, $D \subset q_X^{-1}(\Delta) \subset q_X^{-1}(f_\star^{-1}(q_Y(G)))$. Thus, since $(**)$ is commutative, $D \subset q_X^{-1}(\Delta) \subset f^{-1}(q_Y^{-1}(q_Y(G)))$. Since $G \cap f(K) = \emptyset$ and $q_Y|_{Y \setminus f(K)}$ is a homeomorphism (part (b)), we have that $D \subset q_X^{-1}(\Delta) \subset f^{-1}(G)$. Since q_X is monotone (part (a)) and D is a component of $f^{-1}(G)$, we obtain that $D = q_X^{-1}(\Delta)$. Hence, $q_X(D) = \Delta$. Therefore, $q_X(D)$ is a component of $f_\star^{-1}(q_Y(G))$.

To show (h), let G be a subcontinuum of Y such that $G \cap f(K) = \emptyset$ and let Δ be a component of $f_\star^{-1}(q_Y(G))$. Since $\Delta \subset f_\star^{-1}(q_Y(G))$, $q_X^{-1}(\Delta) \subset q_X^{-1}(f_\star^{-1}(q_Y(G)))$. Then, since $(**)$ is commutative, $q_X^{-1}(\Delta) \subset f^{-1}(q_Y^{-1}(q_Y(G)))$. Since $G \cap f(K) = \emptyset$ and $q_Y|_{Y \setminus f(K)}$ is a homeomorphism (part (b)), $q_X^{-1}(\Delta) \subset f^{-1}(G)$. Since q_X is monotone (part (a)), $q_X^{-1}(\Delta)$ is connected. Let D be the component of $f^{-1}(G)$ containing $q_X^{-1}(\Delta)$. Thus, $\Delta \subset q_X(D) \subset q_X(f^{-1}(G))$. This implies that $f_\star(q_X(D)) \subset q_Y(G)$. Hence, $q_X(D) \subset f_\star^{-1}(q_Y(G))$. Since $\Delta \subset q_X(D)$, we obtain that $\Delta = q_X(D)$. As a consequence of all this, we have that $D \subset q_X^{-1}(\Delta) \subset D$, and $D = q_X^{-1}(\Delta)$. Therefore, $q_X^{-1}(\Delta)$ is a component of $f^{-1}(G)$.

To prove (i), let Γ be a subcontinuum of $Y/f(K)$ and let D be a component of $f^{-1}(q_Y^{-1}(\Gamma))$. Since $D \subset f^{-1}(q_Y^{-1}(\Gamma))$ and $(**)$ commutes, $D \subset q_X^{-1}(f_\star^{-1}(\Gamma))$. Thus, $q_X(D) \subset f_\star^{-1}(\Gamma)$. Let Δ be the component of $f_\star^{-1}(\Gamma)$ containing $q_X(D)$. Then, since $(**)$ is commutative, we obtain that $D \subset q_X^{-1}(\Delta) \subset f^{-1}(q_Y^{-1}(\Gamma))$. Hence, since $q_X^{-1}(\Delta)$ is connected (q_X is monotone (part (a))), we have that $D = q_X^{-1}(\Delta)$. This implies that $q_X(D) = \Delta$. Therefore, $q_X(D)$ is a component of $f_\star^{-1}(\Gamma)$.

To see (j), let Γ be a subcontinuum of $Y/f(K)$ and let Δ be a component of $f_\star^{-1}(\Gamma)$. Since $\Delta \subset f_\star^{-1}(\Gamma)$ and $(**)$ is commutative, we obtain that $q_X^{-1}(\Delta) \subset f^{-1}(q_Y^{-1}(\Gamma))$. Note that $q_X^{-1}(\Delta)$ is connected (q_X is monotone (part (a))). Let D be the component of $f^{-1}(q_Y^{-1}(\Gamma))$ containing $q_X^{-1}(\Delta)$. Hence, by part (d), $\Delta \subset q_X(D) \subset q_X(f^{-1}(q_Y^{-1}(\Gamma))) = f_\star^{-1}(\Gamma)$. Since Δ is a component of $f_\star^{-1}(\Gamma)$, we obtain that $\Delta = q_X(D)$. □

THEOREM 3.2. *Let X and Y be continua, let K be a subcontinuum of X and let $f: X \rightarrow Y$ be a surjective map. Let \mathcal{M} be one of the following classes of maps: almost monotone, atriodic, freely decomposable, joining, semi-confluent and strongly freely decomposable. If $f \in \mathcal{M}$, then $f_\star \in \mathcal{M}$.*

PROOF. It is proved in [5, Proposition 7.5] that if f is either almost monotone or strongly freely decomposable, then f_* is almost monotone or strongly freely decomposable, respectively.

Suppose f is atriodic. Let Γ be a subcontinuum of $Y/f(K)$. Since q_Y is monotone (part (a) of Proposition 3.1), $q_Y^{-1}(\Gamma)$ is a subcontinuum of Y . Since f is atriodic, there exist two components D and L of $f^{-1}(q_Y^{-1}(\Gamma))$ such that $f(D) \cup f(L) = q_Y^{-1}(\Gamma)$ and for each component S of $f^{-1}(q_Y^{-1}(\Gamma))$, we have that either $f(S) = q_Y^{-1}(\Gamma)$ or $f(S) \subset f(D)$ or $f(S) \subset f(L)$. By part (i) of Proposition 3.1, $\Delta = q_X(D)$ and $\Lambda = q_X(L)$ are components of $f_*^{-1}(\Gamma)$. Since $f(D) \cup f(L) = q_Y^{-1}(\Gamma)$, we obtain that $q_Y(f(D)) \cup q_Y(f(L)) = q_Y(q_Y^{-1}(\Gamma))$. Hence, since $(**)$ is commutative and q_Y is surjective, $f_*(q_X(D)) \cup f_*(q_X(L)) = \Gamma$. This implies that $f_*(\Delta) \cup f_*(\Lambda) = \Gamma$. Now, let Σ be a component of $f_*^{-1}(\Gamma)$. By part (j) of Proposition 3.1, there exists a component S of $f^{-1}(q_Y^{-1}(\Gamma))$ such that $q_X(S) = \Sigma$. Since f is atriodic, either $f(S) = q_Y^{-1}(\Gamma)$ or $f(S) \subset f(D)$ or $f(S) \subset f(L)$. Then either $q_Y(f(S)) = q_Y(q_Y^{-1}(\Gamma))$ or $q_Y(f(S)) \subset q_Y(f(D))$ or $q_Y(f(S)) \subset q_Y(f(L))$. Thus, since $(**)$ is commutative and q_Y is surjective, we have that either $f_*(q_X(S)) = \Gamma$ or $f_*(q_X(S)) \subset f_*(q_X(D))$ or $f_*(q_X(S)) \subset f_*(q_X(L))$. This implies that either $f_*(\Sigma) = \Gamma$ or $f_*(\Sigma) \subset f_*(\Delta)$ or $f_*(\Sigma) \subset f_*(\Lambda)$. Therefore, f_* is atriodic.

Assume f is freely decomposable. Let Δ and Γ be proper subcontinua of $Y/f(K)$ such that $Y/f(K) = \Delta \cup \Gamma$. By part (e) of Proposition 3.1, $q_Y^{-1}(\Delta)$ and $q_Y^{-1}(\Gamma)$ are proper subcontinua of Y such that $Y = q_Y^{-1}(\Delta) \cup q_Y^{-1}(\Gamma)$. Since f is freely decomposable, there exist two proper subcontinua D and G of X such that $X = D \cup G$ and $f(D) \subset q_Y^{-1}(\Delta)$ and $f(G) \subset q_Y^{-1}(\Gamma)$. Let $\Delta' = q_X(D)$ and let $\Gamma' = q_X(G)$. Then Δ' and Γ' are proper subcontinua of X/K such that $X/K = \Delta' \cup \Gamma'$. Since $(**)$ is commutative, we obtain that $f_*(\Delta') \subset \Delta$ and $f_*(\Gamma') \subset \Gamma$. Therefore, f_* is freely decomposable.

Suppose f is joining. Let Γ be a subcontinuum of $Y/f(K)$ and let Δ and Λ be two components of $f_*^{-1}(\Gamma)$. By part (j) of Proposition 3.1, there exist two components D and L of $f^{-1}(q_Y^{-1}(\Gamma))$ such that $\Delta = q_X(D)$ and $\Lambda = q_X(L)$. Since f is joining, $f(D) \cap f(L) \neq \emptyset$. Since $q_Y(f(D) \cap f(L)) \subset q_Y(f(D)) \cap q_Y(f(L))$, we have that $q_Y(f(D)) \cap q_Y(f(L)) \neq \emptyset$. Then, since $(**)$ is commutative, $f_*(q_X(D)) \cap f_*(q_X(L)) \neq \emptyset$; i.e. $f_*(\Delta) \cap f_*(\Lambda) \neq \emptyset$. Therefore, f_* is joining.

Assume f is semi-confluent. Let Γ be a subcontinuum of $Y/f(K)$ and let Δ and Λ be two components of $f_*^{-1}(\Gamma)$. By part (j) of Proposition 3.1, there exist two components D and L of $f^{-1}(q_Y^{-1}(\Gamma))$ such that $\Delta = q_X(D)$ and $\Lambda = q_X(L)$. Since f is semi-confluent, either $f(D) \subset f(L)$ or $f(L) \subset f(D)$. Then either $q_Y(f(D)) \subset q_Y(f(L))$ or $q_Y(f(L)) \subset q_Y(f(D))$. Hence, since $(**)$ is commutative, either $f_*(q_X(D)) \subset f_*(q_X(L))$ or $f_*(q_X(L)) \subset f_*(q_X(D))$; i.e. either $f_*(\Delta) \subset f_*(\Lambda)$ or $f_*(\Lambda) \subset f_*(\Delta)$. Therefore, f_* is semi-confluent. \square

THEOREM 3.3. *Let X and Y be continua and let $f: X \twoheadrightarrow Y$ be a surjective map. Suppose K is a subcontinuum of X such that K and $f(K)$ are nowhere dense in X and Y , respectively. Then f is semi-open if and only if f_\star is semi-open.*

PROOF. Suppose f is semi-open. Let \mathcal{U} be a nonempty open subset of X/K . Since q_X is continuous, $q_X^{-1}(\mathcal{U})$ is a nonempty open subset of X . Since f is semi-open, $\text{Int}_Y(f(q_X^{-1}(\mathcal{U}))) \neq \emptyset$. Since $f(K)$ is nowhere dense in Y , by part (c) of Proposition 3.1, q_Y is semi-open. Then

$$\text{Int}_{Y/f(K)}(q_Y(\text{Int}_Y(f(q_X^{-1}(\mathcal{U})))) \neq \emptyset.$$

Since (**) is commutative, we have that $\text{Int}_{Y/f(K)}(q_Y(\text{Int}_Y(f(q_X^{-1}(\mathcal{U})))) \subset \text{Int}_{Y/f(K)}(f_\star(\mathcal{U}))$. Therefore, f_\star is semi-open.

Assume that f_\star is semi-open. Let U be a nonempty open subset of X . Since K is nowhere dense in X , there exists a nonempty open subset V of X such that $V \subset U \setminus K$. Hence, by part (b) of Proposition 3.1, $q_X(V)$ is a nonempty open subset of X/K . Since f_\star is semi-open, $\text{Int}_{Y/f(K)}(f_\star(q_X(V))) \neq \emptyset$. Thus, there exists a nonempty open subset \mathcal{W} of $Y/f(K)$ such that $\mathcal{W} \subset \text{Int}_{Y/f(K)}(f_\star(q_X(V))) \setminus \{K_Y\}$. Then, by part (b) of Proposition 3.1 and the commutativity of (**), $q_Y^{-1}(\mathcal{W}) \subset f(U)$. Hence, $\text{Int}_Y(f(U)) \neq \emptyset$. Therefore, f is semi-open. \square

The following result is [2, Theorem 3.3].

LEMMA 3.4. *Let X and Y be continua and let n be an integer greater than or equal to 2. If $f: X \rightarrow Y$ is a map, then the following are equivalent:*

- (1) $f: X \twoheadrightarrow Y$ is surjective;
- (2) $\mathcal{F}_n(f): \mathcal{F}_n(X) \twoheadrightarrow \mathcal{F}_n(Y)$ is surjective;
- (3) $\mathcal{SF}_n(f): \mathcal{SF}_n(X) \twoheadrightarrow \mathcal{SF}_n(Y)$ is surjective.

REMARK 3.5. As a consequence of Lemma 3.4 we obtain that all the maps we are working with are surjective.

LEMMA 3.6. *Let X and Y be continua, let n be an integer greater than or equal to 2, and let $f: X \twoheadrightarrow Y$ be a surjective map. If B_1, \dots, B_n are pairwise disjoint subcontinua of Y and E_j is a component of $f^{-1}(B_j)$, for each $j \in \{1, \dots, n\}$, then $\langle E_1, \dots, E_n \rangle_n$ is a component of $\mathcal{F}_n(f)^{-1}(\langle B_1, \dots, B_n \rangle_n)$.*

PROOF. Let B_1, \dots, B_n be pairwise disjoint subcontinua of Y and let E_j be a component of $f^{-1}(B_j)$, for each $j \in \{1, \dots, n\}$. Then $\langle E_1, \dots, E_n \rangle_n$ and $\langle B_1, \dots, B_n \rangle_n$ are subcontinua of $\mathcal{F}_n(X)$ and $\mathcal{F}_n(Y)$, respectively ([14, Lemma 1]). Clearly, $\mathcal{F}_n(f)(\langle E_1, \dots, E_n \rangle_n) \subset \langle B_1, \dots, B_n \rangle_n$. Let \mathcal{E} be the component of $\mathcal{F}_n(f)^{-1}(\langle B_1, \dots, B_n \rangle_n)$ containing $\langle E_1, \dots, E_n \rangle_n$. Then $\bigcup \mathcal{E}$ has at most n components ([13, 6.1.2]). Since $\bigcup_{j=1}^n E_j \subset \bigcup \mathcal{E}$ and $E_j \cap E_k = \emptyset$ if $j \neq k$, we obtain that $\bigcup \mathcal{E}$ has exactly n components, say E'_1, \dots, E'_n . Without loss of generality, we assume that $E_j \subset E'_j$, $j \in \{1, \dots, n\}$. Clearly,

$f(\bigcup \mathcal{E}) \subset \bigcup_{j=1}^n B_j$. Since E_j is a component of $f^{-1}(B_j)$, we have that $\bigcup \mathcal{E} = \bigcup_{j=1}^n E_j$, and $E'_j = E_j$ for all $j \in \{1, \dots, n\}$.

Let $A \in \mathcal{E}$. Then $A \cap E_j \neq \emptyset$ for every $j \in \{1, \dots, n\}$ ([8, Lemma 3.1]) and $A \subset \bigcup_{j=1}^n E_j$. This implies that $A \in \langle E_1, \dots, E_n \rangle_n$. Hence, $\langle E_1, \dots, E_n \rangle_n = \mathcal{E}$. Therefore, $\langle E_1, \dots, E_n \rangle_n$ is a component of $\mathcal{F}_n(f)^{-1}(\langle B_1, \dots, B_n \rangle_n)$. \square

LEMMA 3.7. *Let X and Y be continua, let n be an integer greater than or equal to 2, and let $f: X \rightarrow Y$ be a surjective map. If $\mathcal{SF}_n(f)^{-1}(F_Y^n) = \{F_X^n\}$, then f is a homeomorphism.*

PROOF. If f is not one-to-one, then there exist two distinct points x_1 and x_2 in X such that $f(x_1) = f(x_2)$. Then $q_X^n(\{x_1, x_2\})$ belongs to $\mathcal{SF}_n(f)^{-1}(F_Y^n)$ and $\mathcal{SF}_n(f)^{-1}(F_Y^n) \neq \{F_X^n\}$. \square

4. ALMOST MONOTONE MAPS

THEOREM 4.1. *Let X and Y be continua, let n be an integer greater than or equal to 2, and let $f: X \rightarrow Y$ be a surjective map. Consider the following statements:*

- (1) f is almost monotone;
- (2) $\mathcal{F}_n(f)$ is almost monotone;
- (3) $\mathcal{SF}_n(f)$ is almost monotone.

Then (2) implies (1) and (3).

PROOF. Suppose that $\mathcal{F}_n(f)$ is almost monotone, we see that f is almost monotone. To this end, let B be a subcontinuum of Y with nonempty interior. Then $\langle B \rangle_n$ is connected ([14, Lemma 1]) and it is easy to see that it is also closed. Hence, $\langle B \rangle_n$ is a subcontinuum of $\mathcal{F}_n(Y)$. Since $\langle \text{Int}_Y(B) \rangle_n \subset \langle B \rangle_n$, we have that $\text{Int}_{\mathcal{F}_n(Y)}(\langle B \rangle_n) \neq \emptyset$. Since $\mathcal{F}_n(f)$ is almost monotone, $\mathcal{F}_n(f)^{-1}(\langle B \rangle_n)$ is a connected subset of $\mathcal{F}_n(X)$. Hence, $\mathcal{F}_n(f)^{-1}(\langle B \rangle_n)$ is a subcontinuum of $\mathcal{F}_n(X)$. Thus, $\bigcup \mathcal{F}_n(f)^{-1}(\langle B \rangle_n)$ is a closed subset of X . Observe that $\bigcup \mathcal{F}_n(f)^{-1}(\langle B \rangle_n) = f^{-1}(B)$, and $\emptyset \neq \mathcal{F}_n(f)^{-1}(\langle B \rangle_n) \cap \mathcal{F}_1(X) \subset \mathcal{F}_n(f)^{-1}(\langle B \rangle_n) \cap \mathcal{C}_1(X)$. Thus, $f^{-1}(B)$ is connected ([15, (1.49)]). Therefore, f is almost monotone.

It follows from Theorem 3.2 that if $\mathcal{F}_n(f)$ is almost monotone, then $\mathcal{SF}_n(f)$ is almost monotone. \square

REMARK 4.2. In Theorem 4.1, when the range space Y is locally connected and n is greater than or equal to 3, we also have that (3) implies (1) and (2), for this see Corollary 6.3. In fact, we have the following:

THEOREM 4.3. *Let X and Y be continua, where Y is locally connected. Then a surjective map $f: X \rightarrow Y$ is almost monotone if and only if f is monotone.*

PROOF. Clearly, if f is monotone, then f is almost monotone. Suppose f is almost monotone. Let y be a point of Y . Since Y is locally connected, there exists a sequence $\{K_m\}_{m=1}^\infty$ of locally connected subcontinua of Y such that $\{y\} = \bigcap_{m=1}^\infty K_m$, and for all positive integer m , $y \in \text{Int}_Y(K_m)$ and $K_{m+1} \subset K_m$. Since f is almost monotone, $f^{-1}(K_m)$ is connected. Hence, $f^{-1}(y) = \bigcap_{m=1}^\infty f^{-1}(K_m)$ is connected ([13, 1.7.2]). Therefore, f is monotone ([13, 2.1.12]). \square

QUESTIONS 4.4. Let X and Y be continua, let n be an integer greater than or equal to 2, and let $f: X \twoheadrightarrow Y$ be a surjective map.

- (i) If f is almost monotone, then is $\mathcal{F}_n(f)$ almost monotone?
- (ii) If f is almost monotone, then is $\mathcal{SF}_n(f)$ almost monotone?
- (iii) If $\mathcal{SF}_n(f)$ is almost monotone and $\mathcal{SF}_n(Y)$ is not locally connected, then is f almost monotone?
- (iv) If $\mathcal{SF}_n(f)$ is almost monotone and $\mathcal{SF}_n(Y)$ is not locally connected, then is $\mathcal{F}_n(f)$ almost monotone?

REMARK 4.5. Note that by Theorem 4.3 and [2, Theorem 4.1], questions (i) and (ii) have positive answers when Y is locally connected.

5. ATRIODIC MAPS

THEOREM 5.1. *Let X and Y be continua, let n be an integer greater than or equal to 2, and let $f: X \twoheadrightarrow Y$ be a surjective map. Consider the following statements:*

- (1) f is atriodic;
- (2) $\mathcal{F}_n(f)$ is atriodic;
- (3) $\mathcal{SF}_n(f)$ is atriodic.

Then (2) implies (1) and (3); also (3) implies (1).

PROOF. Suppose $\mathcal{F}_n(f)$ is atriodic, we prove that f is atriodic. Let K be a subcontinuum of Y . If $K = Y$, then $f^{-1}(K)$ has only one component, namely X , and $f(X) = K$. Suppose $K \neq Y$ and let y_1, \dots, y_{n-1} be $n - 1$ distinct points in $Y \setminus K$. Let

$$\mathcal{K} = \{\{y_1, \dots, y_{n-1}\} \cup \{k\} \mid k \in K\}.$$

Observe that \mathcal{K} is a subcontinuum of $\mathcal{F}_n(Y)$ and $\mathcal{K} \cap \mathcal{F}_1(Y) = \emptyset$. Since $\mathcal{F}_n(f)$ is atriodic, there exist two components \mathcal{C} and \mathcal{D} of $\mathcal{F}_n(f)^{-1}(\mathcal{K})$ such that $\mathcal{F}_n(f)(\mathcal{C}) \cup \mathcal{F}_n(f)(\mathcal{D}) = \mathcal{K}$ and for each component \mathcal{E} of $\mathcal{F}_n(f)^{-1}(\mathcal{K})$ either $\mathcal{F}_n(f)(\mathcal{E}) = \mathcal{K}$ or $\mathcal{F}_n(f)(\mathcal{E}) \subset \mathcal{F}_n(f)(\mathcal{C})$ or $\mathcal{F}_n(f)(\mathcal{E}) \subset \mathcal{F}_n(f)(\mathcal{D})$. Note that $\bigcup \mathcal{C} \subset \left(\bigcup_{j=1}^{n-1} f^{-1}(y_j)\right) \cup f^{-1}(K)$, $\bigcup \mathcal{C} \cap f^{-1}(y_j) \neq \emptyset$ ($j \in \{1, \dots, n - 1\}$), and $\bigcup \mathcal{C} \cap f^{-1}(K) \neq \emptyset$. Since $\bigcup \mathcal{C}$ has at most n components ([13, 6.1.2]), we have that $\bigcup \mathcal{C}$ has exactly n components, say C_1, \dots, C_n . Without loss of generality, we assume that for each $j \in \{1, \dots, n - 1\}$, $C_j \subset f^{-1}(y_j)$ and

$C_n \subset f^{-1}(K)$. Let C be the component of $f^{-1}(K)$ containing C_n . Similarly, $\bigcup \mathcal{D}$ has exactly n components, say D_1, \dots, D_n . Without loss of generality, we assume that for each $j \in \{1, \dots, n-1\}$, $D_j \subset f^{-1}(y_j)$ and $D_n \subset f^{-1}(K)$. Let D be the component of $f^{-1}(K)$ containing D_n .

We show that $f(C) \cup f(D) = K$. Note that $f(C) \cup f(D) \subset K$. Let $k \in K$. Then $\{y_1, \dots, y_{n-1}\} \cup \{k\} \in \mathcal{K}$. Assume that $\{y_1, \dots, y_{n-1}\} \cup \{k\} \in \mathcal{F}_n(f)(\mathcal{C})$. Hence, there exists an element $A \in \mathcal{C}$ such that $\mathcal{F}_n(f)(A) = \{y_1, \dots, y_{n-1}\} \cup \{k\}$. Thus, $f(A) = \{y_1, \dots, y_{n-1}\} \cup \{k\}$. This implies that $k \in f(A) \subset f(\bigcup \mathcal{C}) = \bigcup_{j=1}^n f(C_j)$. Since $k \notin \{y_1, \dots, y_{n-1}\}$, $k \in f(C_n) \subset f(C)$. Similarly, if $\{y_1, \dots, y_{n-1}\} \cup \{k\} \in \mathcal{F}_n(f)(\mathcal{D})$, then $k \in f(C_n) \subset f(D)$. Hence, $K \subset f(C) \cup f(D)$, and $f(C) \cup f(D) = K$.

Let E be a component of $f^{-1}(K)$ and, for each $j \in \{1, \dots, n-1\}$, let E_j be a component of $f^{-1}(y_j)$. Then $\mathcal{E} = \langle E_1, \dots, E_{n-1}, E \rangle_n$ is a component of $\mathcal{F}_n(f)^{-1}(\mathcal{K})$, by Lemma 3.6. Since $\mathcal{F}_n(f)$ is atriodic, either $\mathcal{F}_n(f)(\mathcal{E}) = \mathcal{K}$ or $\mathcal{F}_n(f)(\mathcal{E}) \subset \mathcal{F}_n(f)(\mathcal{C})$ or $\mathcal{F}_n(f)(\mathcal{E}) \subset \mathcal{F}_n(f)(\mathcal{D})$.

Suppose $\mathcal{F}_n(f)(\mathcal{E}) = \mathcal{K}$, we prove that $f(E) = K$. Clearly, $f(E) \subset K$. Let $k \in K$. Then $\{y_1, \dots, y_{n-1}\} \cup \{k\} \in \mathcal{K}$. Hence, there exists $L \in \mathcal{E}$ such that $\mathcal{F}_n(f)(L) = \{y_1, \dots, y_{n-1}\} \cup \{k\}$; i.e. $f(L) = \{y_1, \dots, y_{n-1}\} \cup \{k\}$. Thus, $k \in f(L)$. Since $k \notin \{y_1, \dots, y_{n-1}\}$, $k \in f(E)$. Therefore, $f(E) = K$.

Assume $\mathcal{F}_n(f)(\mathcal{E}) \subset \mathcal{F}_n(f)(\mathcal{C})$, we show that $f(E) \subset f(C)$. Let $e \in E$. For each $j \in \{1, \dots, n-1\}$, let $e_j \in E_j$. Then $\{e_1, \dots, e_{n-1}, e\} \in \mathcal{E}$. Hence, $\mathcal{F}_n(f)(\{e_1, \dots, e_{n-1}, e\}) \in \mathcal{F}_n(f)(\mathcal{C})$. Then there exists $C' \in \mathcal{C}$ such that $\mathcal{F}_n(f)(C') = \mathcal{F}_n(f)(\{e_1, \dots, e_{n-1}, e\})$; i.e. $f(C') = f(\{e_1, \dots, e_{n-1}, e\})$. In particular, $f(e) \in f(C')$. Since $C' \subset \bigcup \mathcal{C} = \bigcup_{j=1}^n C_j$ and $e \notin \{e_1, \dots, e_{n-1}\}$, we obtain that $f(e) \in f(C_n) \subset f(C)$. Therefore, $f(E) \subset f(C)$.

Similarly, if $\mathcal{F}_n(f)(\mathcal{E}) \subset \mathcal{F}_n(f)(\mathcal{D})$, then $f(E) \subset f(D)$. Therefore, f is an atriodic map.

Suppose $\mathcal{SF}_n(f)$ is atriodic, we see that f is atriodic. The proof is similar to the previous one, we include the details for the convenience of the reader. Let K be a subcontinuum of Y . If $K = Y$, then $f^{-1}(K)$ has only one component, namely X , and $f(X) = K$. Suppose $K \neq Y$ and let y_1, \dots, y_{n-1} be $n-1$ distinct points in $Y \setminus K$. Let

$$\mathcal{K} = \{\{y_1, \dots, y_{n-1}\} \cup \{k\} \mid k \in K\}.$$

Observe that \mathcal{K} is a subcontinuum of $\mathcal{F}_n(Y)$ and $\mathcal{K} \cap \mathcal{F}_1(Y) = \emptyset$. Then $q_Y^n(\mathcal{K})$ is a subcontinuum of $\mathcal{SF}_n(Y) \setminus \{F_Y^n\}$. Since $\mathcal{SF}_n(f)$ is atriodic, there exist two components \mathfrak{C} and \mathfrak{D} of $\mathcal{SF}_n(f)^{-1}(q_Y^n(\mathcal{K}))$ such that $\mathcal{SF}_n(f)(\mathfrak{C}) \cup \mathcal{SF}_n(f)(\mathfrak{D}) = q_Y^n(\mathcal{K})$ and for each component \mathfrak{E} of $\mathcal{SF}_n(f)^{-1}(q_Y^n(\mathcal{K}))$, we have that either $\mathcal{SF}_n(f)(\mathfrak{E}) = q_Y^n(\mathcal{K})$ or $\mathcal{SF}_n(f)(\mathfrak{E}) \subset \mathcal{SF}_n(f)(\mathfrak{C})$ or $\mathcal{SF}_n(f)(\mathfrak{E}) \subset \mathcal{SF}_n(f)(\mathfrak{D})$. By part (h) of Proposition 3.1, we obtain that $(q_X^n)^{-1}(\mathfrak{C})$ and $(q_X^n)^{-1}(\mathfrak{D})$ are components of $\mathcal{F}_n(f)^{-1}(\mathcal{K})$. Also,

$$\mathcal{F}_n(f)((q_X^n)^{-1}(\mathfrak{C})) \cup \mathcal{F}_n(f)((q_X^n)^{-1}(\mathfrak{D})) = \mathcal{K}.$$

Note that $\bigcup (q_X^n)^{-1}(\mathfrak{C}) \subset \left(\bigcup_{j=1}^{n-1} f^{-1}(y_j)\right) \cup f^{-1}(K)$, $((q_X^n)^{-1}(\mathfrak{C})) \cap f^{-1}(y_j) \neq \emptyset$ ($j \in \{1, \dots, n-1\}$), and $((q_X^n)^{-1}(\mathfrak{C})) \cap f^{-1}(K) \neq \emptyset$. Since $(q_X^n)^{-1}(\mathfrak{C})$ has at most n components [13, 6.1.2], we have that $(q_X^n)^{-1}(\mathfrak{C})$ has exactly n components, say C_1, \dots, C_n . Without loss of generality, we assume that for each $j \in \{1, \dots, n-1\}$, $C_j \subset f^{-1}(y_j)$ and $C_n \subset f^{-1}(K)$. Let C be the component of $f^{-1}(K)$ containing C_n . Similarly, $(q_X^n)^{-1}(\mathfrak{D})$ has exactly n components, say D_1, \dots, D_n . Without loss of generality, we assume that for each $j \in \{1, \dots, n-1\}$, $D_j \subset f^{-1}(y_j)$ and $D_n \subset f^{-1}(K)$. Let D be the component of $f^{-1}(K)$ containing D_n .

We show that $f(C) \cup f(D) = K$. Note that $f(C) \cup f(D) \subset K$. Let $k \in K$. Then $\{y_1, \dots, y_{n-1}\} \cup \{k\} \in \mathcal{K}$. Assume that $\{y_1, \dots, y_{n-1}\} \cup \{k\} \in \mathcal{F}_n(f)((q_X^n)^{-1}(\mathfrak{C}))$. Hence, there exists an element $A \in (q_X^n)^{-1}(\mathfrak{C})$ such that $\mathcal{F}_n(f)(A) = \{y_1, \dots, y_{n-1}\} \cup \{k\}$. Thus, $f(A) = \{y_1, \dots, y_{n-1}\} \cup \{k\}$. This implies that $k \in f(A) \subset \bigcup \mathcal{F}_n(f)((q_X^n)^{-1}(\mathfrak{C})) = \bigcup_{j=1}^n f(C_j)$. Since $k \notin \{y_1, \dots, y_{n-1}\}$, $k \in f(C_n) \subset f(C)$. Similarly, if $\{y_1, \dots, y_{n-1}\} \cup \{k\} \in \mathcal{F}_n(f)((q_X^n)^{-1}(\mathfrak{D}))$, then $k \in f(D_n) \subset f(D)$. Hence, $K \subset f(C) \cup f(D)$, and $f(C) \cup f(D) = K$.

Let E be a component of $f^{-1}(K)$ and, for each $j \in \{1, \dots, n-1\}$, let E_j be a component of $f^{-1}(y_j)$. Then $\mathcal{E} = \langle E_1, \dots, E_{n-1}, E \rangle_n$ is a component of $\mathcal{F}_n(f)^{-1}(\mathcal{K})$, by Lemma 3.6. Thus, by part (g) of Proposition 3.1, $\mathfrak{E} = q_X^n(\mathcal{E})$ is a component of $\mathcal{SF}_n(f)^{-1}(q_Y^n(\mathcal{K}))$. Since $\mathcal{SF}_n(f)$ is atriodic, we have that either $\mathcal{SF}_n(f)(\mathfrak{E}) = q_Y^n(\mathcal{K})$ or $\mathcal{SF}_n(f)(\mathfrak{E}) \subset \mathcal{SF}_n(f)(\mathfrak{C})$ or $\mathcal{SF}_n(f)(\mathfrak{E}) \subset \mathcal{SF}_n(f)(\mathfrak{D})$. Hence, since $(*)$ commutes, we obtain that either $\mathcal{F}_n(f)(\mathcal{E}) = \mathcal{K}$ or $q_Y^n(\mathcal{F}_n(f)(\mathcal{E})) \subset \mathcal{SF}_n(f)(\mathfrak{C})$ or $q_Y^n(\mathcal{F}_n(f)(\mathcal{E})) \subset \mathcal{SF}_n(f)(\mathfrak{D})$.

Suppose that $\mathcal{F}_n(f)(\mathcal{E}) = \mathcal{K}$. We show that $f(E) = K$. Clearly, $f(E) \subset K$. Let $k \in K$. Then $\{y_1, \dots, y_{n-1}\} \cup \{k\} \in \mathcal{K}$. Hence, there exists $L \in \mathcal{E}$ such that $\mathcal{F}_n(f)(L) = \{y_1, \dots, y_{n-1}\} \cup \{k\}$; i.e. $f(L) = \{y_1, \dots, y_{n-1}\} \cup \{k\}$. Thus, $k \in f(L)$ and $k \in f(E)$. Therefore, $f(E) = K$.

Assume that $q_Y^n(\mathcal{F}_n(f)(\mathcal{E})) \subset \mathcal{SF}_n(f)(\mathfrak{C})$. We see that $f(E) \subset f(C)$. Let $e \in E$. For each $j \in \{1, \dots, n-1\}$, let $e_j \in E_j$. Then $\{e_1, \dots, e_{n-1}, e\} \in \mathcal{E}$. Hence, $q_Y^n(\mathcal{F}_n(f)(\{e_1, \dots, e_{n-1}, e\})) \in q_Y^n(\mathcal{F}_n(f)(\mathcal{E}))$. This implies that $q_Y^n(\mathcal{F}_n(f)(\{e_1, \dots, e_{n-1}, e\})) \in \mathcal{SF}_n(f)(\mathfrak{C})$. Thus, there exists $\mathcal{C} \in \mathfrak{C}$ such that $\mathcal{SF}_n(f)(\mathcal{C}) = q_Y^n(\mathcal{F}_n(f)(\{e_1, \dots, e_{n-1}, e\}))$. Hence, there exists $C' \in \mathcal{F}_n(X) \setminus \mathcal{F}_1(X)$ such that $q_X^n(C') = \mathcal{C}$. Since $(*)$ commutes, it follows that $q_Y^n(\mathcal{F}_n(f)(C')) = q_Y^n(\mathcal{F}_n(f)(\{e_1, \dots, e_{n-1}, e\}))$. Thus, by part (b) of Proposition 3.1, we obtain that $\mathcal{F}_n(f)(C') = \mathcal{F}_n(f)(\{e_1, \dots, e_{n-1}, e\})$; i.e. $f(C') = f(\{e_1, \dots, e_{n-1}, e\})$. In particular, $f(e) \in f(C')$. Since $C' \subset \bigcup (q_X^n)^{-1}(\mathfrak{C}) = \bigcup_{j=1}^n C_j$, we obtain that $f(e) \in f(C_n) \subset f(C)$. Therefore, $f(E) \subset f(C)$.

Similarly, if $q_Y^n(\mathcal{F}_n(f)(\mathcal{E})) \subset \mathcal{SF}_n(f)(\mathfrak{D})$, we have that $f(E) \subset f(D)$. Therefore, f is an atriodic map.

It follows from Theorem 3.2 that if $\mathcal{F}_n(f)$ is atriodic, then $\mathcal{SF}_n(f)$ is atriodic. \square

QUESTIONS 5.2. Let X and Y be continua, let n be an integer greater than or equal to 2, and let $f: X \twoheadrightarrow Y$ be a surjective map.

- (i) If f is atriodic, then is $\mathcal{F}_n(f)$ atriodic?
- (ii) If f is atriodic, then is $\mathcal{SF}_n(f)$ atriodic?
- (iii) If $\mathcal{SF}_n(f)$ is atriodic, then is $\mathcal{F}_n(f)$ atriodic?

6. FREELY DECOMPOSABLE AND STRONGLY FREELY DECOMPOSABLE MAPS

THEOREM 6.1. *Let X be a continuum and let Y be locally connected continuum. Let n be an integer greater than or equal to 2, and let $f: X \twoheadrightarrow Y$ be a surjective map. Then the following are equivalent:*

- (1) $\mathcal{F}_n(f)$ is freely decomposable;
- (2) $\mathcal{F}_n(f)$ is monotone;
- (3) $\mathcal{SF}_n(f)$ is freely decomposable;
- (4) $\mathcal{SF}_n(f)$ is monotone;
- (5) f is monotone.

PROOF. Suppose $\mathcal{F}_n(f)$ is freely decomposable. Since Y is locally connected, $\mathcal{F}_n(Y)$ is locally connected ([11, Lemma 2]). Also, by [11, Corollary 5], for each $B \in \mathcal{F}_n(Y)$, $\mathcal{F}_n(Y) \setminus \{B\}$ is connected. Hence, by [7, Theorem 7], we have that $\mathcal{F}_n(f)$ is monotone. It is clear that if $\mathcal{F}_n(f)$ is monotone, then $\mathcal{F}_n(f)$ is freely decomposable. By [2, Theorem 4.1], we have that f is monotone if and only if $\mathcal{F}_n(f)$ is monotone.

Assume $\mathcal{SF}_n(f)$ is freely decomposable. Since Y is locally connected, $\mathcal{SF}_n(Y)$ is locally connected ([1, Theorem 5.2]). Also, by [3, Theorem 3.5], for each $\chi \in \mathcal{SF}_n(Y)$, $\mathcal{SF}_n(Y) \setminus \{\chi\}$ is connected. Thus, by [7, Theorem 7], we have that $\mathcal{SF}_n(f)$ is monotone. It is clear that if $\mathcal{SF}_n(f)$ is monotone, then $\mathcal{SF}_n(f)$ is freely decomposable. By [2, Theorem 4.1], we have that f is monotone if and only if $\mathcal{SF}_n(f)$ is monotone. \square

THEOREM 6.2. *Let X and Y be continua, let n be an integer greater than or equal to 3, and let $f: X \twoheadrightarrow Y$ be a surjective map. Then*

- (1) $\mathcal{F}_n(f)$ is almost monotone if and only if $\mathcal{F}_n(f)$ is strongly freely decomposable;
- (2) $\mathcal{SF}_n(f)$ is almost monotone if and only if $\mathcal{SF}_n(f)$ is strongly freely decomposable.

PROOF. Clearly, every almost monotone map is strongly freely decomposable. Suppose $\mathcal{F}_n(f)$ ($\mathcal{SF}_n(f)$) is strongly freely decomposable. By [10, Theorem 8] ([1, Theorem 4.1]) $\mathcal{F}_n(X)$ ($\mathcal{SF}_n(X)$) is unicoherent. Therefore, by [4, Theorem 4.2], $\mathcal{F}_n(f)$ ($\mathcal{SF}_n(f)$) is almost monotone. \square

As a consequence of Theorems 4.1, 6.1 and 6.2, we obtain:

COROLLARY 6.3. *Let X be a continuum and let Y be locally connected continuum. Let n be an integer greater than or equal to 3, and let $f: X \twoheadrightarrow Y$ be a surjective map. Then the following are equivalent:*

- (1) $\mathcal{F}_n(f)$ is freely decomposable;
- (2) $\mathcal{F}_n(f)$ is strongly freely decomposable;
- (3) $\mathcal{F}_n(f)$ is almost monotone;
- (4) $\mathcal{F}_n(f)$ is monotone;
- (5) $\mathcal{SF}_n(f)$ is freely decomposable;
- (6) $\mathcal{SF}_n(f)$ is strongly freely decomposable;
- (7) $\mathcal{SF}_n(f)$ is almost monotone;
- (8) $\mathcal{SF}_n(f)$ is monotone;
- (9) f is monotone.

The following example shows a strongly freely decomposable map f defined between locally connected continua such that neither $\mathcal{F}_n(f)$ nor $\mathcal{SF}_n(f)$ is freely decomposable for any integer n greater than or equal to 3.

EXAMPLE 6.4. Let $\mathcal{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ and let $f: \mathcal{S}^1 \twoheadrightarrow [-1, 1]$ be given by $f((x, y)) = x$. Then f is strongly freely decomposable but it is neither monotone nor almost monotone. Hence, by Corollary 6.3, for an integer n greater than or equal to 3, $\mathcal{F}_n(f)$ and $\mathcal{SF}_n(f)$ are not freely decomposable.

THEOREM 6.5. *Let X and Y be continua, let n be an integer greater than or equal to 2, and let $f: X \twoheadrightarrow Y$ be a surjective map. Consider the following statements:*

- (1) f is freely decomposable;
- (2) $\mathcal{F}_n(f)$ is freely decomposable;
- (3) $\mathcal{SF}_n(f)$ is freely decomposable.

Then (2) implies (3). Also, if Y is locally connected and n is greater than or equal to 3, then (2) implies (1), (3) implies (2) and (3) implies (1). Moreover, if n is greater than or equal to 3, (1) does not imply (2) and (1) does not imply (3).

PROOF. It follows from Theorem 3.2 that (2) implies (3). If Y is locally connected and n is greater than or equal to 3, by Corollary 6.3, we have that (2) implies (1), (3) implies (2) and (3) implies (1). From Example 6.4, we obtain that if n is greater than or equal to 3, (1) does not imply (2) and (1) does not imply (3). □

THEOREM 6.6. *Let X and Y be continua, let n be an integer greater than or equal to 2, and let $f: X \twoheadrightarrow Y$ be a surjective map. Consider the following statements:*

- (1) f is strongly freely decomposable;
- (2) $\mathcal{F}_n(f)$ is strongly freely decomposable;
- (3) $\mathcal{SF}_n(f)$ is strongly freely decomposable.

Then (2) implies (3). Also, if n is greater than or equal to 3, (2) implies (1), (1) does not imply (2) and (1) does not imply (3). Moreover, if Y is locally connected and n is greater than or equal to 3, then (3) implies (1) and (3) implies (2).

PROOF. It follows from Theorem 3.2 that (2) implies (3). If $\mathcal{F}_n(f)$ is strongly freely decomposable and n is greater than or equal to 3, then, by Theorem 6.2, $\mathcal{F}_n(f)$ is almost monotone. Hence, by Theorem 4.1, f is almost monotone. Thus, f is strongly freely decomposable.

By Example 6.4, we have that, when n is greater than or equal to 3, neither (1) implies (2) nor (1) implies (3).

It follows from Corollary 6.3 that if Y is locally connected and n is greater than or equal to 3, then (3) implies (1) and (3) implies (2). \square

QUESTIONS 6.7. Let X and Y be continua, let n be an integer greater than or equal to 2, and let $f: X \rightarrow Y$ be a surjective map.

- (i) If f is freely decomposable, then is $\mathcal{F}_2(f)$ freely decomposable?
- (ii) If f is freely decomposable, then is $\mathcal{SF}_2(f)$ freely decomposable?
- (iii) If $\mathcal{F}_n(f)$ is freely decomposable, then is f freely decomposable?
- (iv) If $\mathcal{SF}_n(f)$ is freely decomposable, then is f freely decomposable?
- (v) If $\mathcal{SF}_n(f)$ is freely decomposable, then is $\mathcal{F}_n(f)$ freely decomposable?
- (vi) If f is strongly freely decomposable, then is $\mathcal{F}_2(f)$ strongly freely decomposable?
- (vii) If f is strongly freely decomposable, then is $\mathcal{SF}_2(f)$ strongly freely decomposable?
- (viii) If $\mathcal{F}_2(f)$ is strongly freely decomposable, then is f strongly freely decomposable?
- (ix) If $\mathcal{SF}_n(f)$ is strongly freely decomposable and Y is not locally connected, then is f strongly freely decomposable?
- (x) If $\mathcal{SF}_n(f)$ is strongly freely decomposable and Y is not locally connected, then is $\mathcal{F}_n(f)$ strongly freely decomposable?

7. JOINING MAPS

We begin with an example that shows that there exists a joining map f between continua such that the induced maps $\mathcal{F}_2(f)$ and $\mathcal{SF}_2(f)$ are not joining.

EXAMPLE 7.1. Let $X = Y = [0, 1]$ and let $f: X \twoheadrightarrow Y$ be the map given by:

$$f(x) = \begin{cases} 2x + \frac{1}{2}, & \text{if } x \in [0, \frac{1}{4}]; \\ -2x + \frac{3}{2}, & \text{if } x \in [\frac{1}{4}, \frac{3}{4}]; \\ 2x - \frac{3}{2}, & \text{if } x \in [\frac{3}{4}, 1]. \end{cases}$$

Then f is a joining map. We see that $\mathcal{F}_2(f)$ and $\mathcal{SF}_2(f)$ are not joining. To this end, let

$$\mathcal{B} = \left\{ \left\{ \frac{3}{4}, x \right\} \mid x \in \left[\frac{1}{4}, \frac{5}{8} \right] \right\} \cup \left\{ \left\{ \frac{1}{4}, x \right\} \mid x \in \left[\frac{3}{8}, \frac{3}{4} \right] \right\}.$$

Since $\{\{\frac{3}{4}, x\} \mid x \in [\frac{1}{4}, \frac{5}{8}]\}$ and $\{\{\frac{1}{4}, x\} \mid x \in [\frac{3}{8}, \frac{3}{4}]\}$ are subcontinua of $\mathcal{F}_2(Y)$ having the point $\{\frac{1}{4}, \frac{3}{4}\}$ in common, it follows that \mathcal{B} is a subcontinuum of $\mathcal{F}_2(Y)$. Observe that

$$\begin{aligned} \mathcal{F}_2(f)^{-1}(\mathcal{B}) &= \left\{ \left\{ \frac{1}{8}, x \right\} \mid x \in \left[0, \frac{1}{16} \right] \right\} \\ &\cup \left\{ \left\{ \frac{1}{8}, x \right\} \mid x \in \left[\frac{7}{16}, \frac{5}{8} \right] \right\} \cup \left\{ \left\{ \frac{1}{8}, x \right\} \mid x \in \left[\frac{7}{8}, 1 \right] \right\} \\ &\cup \left\{ \left\{ \frac{3}{8}, x \right\} \mid x \in \left[0, \frac{1}{16} \right] \right\} \cup \left\{ \left\{ \frac{3}{8}, x \right\} \mid x \in \left[\frac{7}{16}, \frac{5}{8} \right] \right\} \\ &\cup \left\{ \left\{ \frac{3}{8}, x \right\} \mid x \in \left[\frac{7}{8}, 1 \right] \right\} \cup \left\{ \left\{ \frac{5}{8}, x \right\} \mid x \in \left[0, \frac{1}{8} \right] \right\} \\ &\cup \left\{ \left\{ \frac{5}{8}, x \right\} \mid x \in \left[\frac{3}{8}, \frac{9}{16} \right] \right\} \cup \left\{ \left\{ \frac{3}{8}, x \right\} \mid x \in \left[\frac{15}{16}, 1 \right] \right\} \\ &\cup \left\{ \left\{ \frac{7}{8}, x \right\} \mid x \in \left[0, \frac{1}{8} \right] \right\} \cup \left\{ \left\{ \frac{7}{8}, x \right\} \mid x \in \left[\frac{3}{8}, \frac{9}{16} \right] \right\} \\ &\cup \left\{ \left\{ \frac{7}{8}, x \right\} \mid x \in \left[\frac{15}{16}, 1 \right] \right\}. \end{aligned}$$

Let $\mathcal{C} = \{\{\frac{1}{8}, x\} \mid x \in [0, \frac{1}{16}]\}$ and $\mathcal{D} = \{\{\frac{7}{8}, x\} \mid x \in [\frac{15}{16}, 1]\}$. Note that \mathcal{C} and \mathcal{D} are components of $\mathcal{F}_2(f)^{-1}(\mathcal{B})$. Also, $\mathcal{F}_2(f)(\mathcal{C}) = \{\{\frac{3}{4}, x\} \mid x \in [\frac{1}{2}, \frac{5}{8}]\}$ and $\mathcal{F}_2(f)(\mathcal{D}) = \{\{\frac{1}{4}, x\} \mid x \in [\frac{3}{8}, \frac{1}{2}]\}$. Hence, $\mathcal{F}_2(f)(\mathcal{C}) \cap \mathcal{F}_2(f)(\mathcal{D}) = \emptyset$. Thus, $\mathcal{F}_2(f)$ is not joining. To show that $\mathcal{SF}_2(f)$ is not joining, note that $q_Y^2(\mathcal{B})$ is a subcontinuum of $\mathcal{SF}_2(Y)$. Since $\mathcal{B} \cap \mathcal{F}_1(Y) = \emptyset$, by part (g) of Proposition 3.1, $q_X^2(\mathcal{C})$ and $q_X^2(\mathcal{D})$ are components of $\mathcal{SF}_2(f)^{-1}(q_Y^2(\mathcal{B}))$. Suppose $\mathcal{SF}_2(f)(q_X^2(\mathcal{C})) \cap \mathcal{SF}_2(f)(q_X^2(\mathcal{D})) \neq \emptyset$. Then, since $(*)$ commutes, $q_Y^2(\mathcal{F}_2(f)(\mathcal{C})) \cap q_Y^2(\mathcal{F}_2(f)(\mathcal{D})) \neq \emptyset$. Since $\mathcal{B} \cap \mathcal{F}_1(Y) = \emptyset$, by part (b) of Proposition 3.1, $\mathcal{F}_2(f)(\mathcal{C}) \cap \mathcal{F}_2(f)(\mathcal{D}) \neq \emptyset$, a contradiction. Hence, $\mathcal{SF}_2(f)(q_X^2(\mathcal{C})) \cap \mathcal{SF}_2(f)(q_X^2(\mathcal{D})) = \emptyset$. Thus, $\mathcal{SF}_2(f)$ is not joining.

THEOREM 7.2. *Let X and Y be continua, let n be an integer greater than or equal to 2, and let $f: X \rightarrow Y$ be a surjective map. Consider the following statements:*

- (1) f is joining;
- (2) $\mathcal{F}_n(f)$ is joining;
- (3) $\mathcal{SF}_n(f)$ is joining.

Then (2) implies (1) and (3), (3) implies (1), and (1) does not imply either (2) or (3) for $n = 2$.

PROOF. It follows from Theorem 3.2 that (2) implies (3). Suppose $\mathcal{SF}_n(f)$ is joining, we prove that f is joining. To this end, let B be a subcontinuum of Y . If $B = Y$, then $f^{-1}(B)$ only has one component, namely X , and $f(X) = B$. Suppose $B \neq Y$ and let C and D be two components of $f^{-1}(B)$. Since $B \neq Y$, there exist $n - 1$ distinct points y_1, \dots, y_{n-1} in $Y \setminus B$. Let

$$\mathcal{K} = \{\{y_1, \dots, y_{n-1}\} \cup \{b\} \mid b \in B\}.$$

Then \mathcal{K} is a subcontinuum of $\mathcal{F}_n(Y)$ and $\mathcal{K} \cap \mathcal{F}_1(Y) = \emptyset$. Hence, $q_Y^n(\mathcal{K})$ is a subcontinuum of $\mathcal{SF}_n(Y) \setminus \{F_Y^n\}$. For each $j \in \{1, \dots, n - 1\}$, let C_j be a component of $f^{-1}(y_j)$. Let $\mathcal{C} = \langle C_1, \dots, C_{n-1}, C \rangle_n$ and $\mathcal{D} = \langle C_1, \dots, C_{n-1}, D \rangle_n$. Then by Lemma 3.6, \mathcal{C} and \mathcal{D} are components of $\mathcal{F}_n(f)^{-1}(\mathcal{K})$. By part (g) of Proposition 3.1, $q_X^n(\mathcal{C})$ and $q_X^n(\mathcal{D})$ are components of $\mathcal{SF}_n(f)^{-1}(q_Y^n(\mathcal{K}))$. Since $\mathcal{SF}_n(f)$ is joining, we have that $\mathcal{SF}_n(f)(q_X^n(\mathcal{C})) \cap \mathcal{SF}_n(f)(q_X^n(\mathcal{D})) \neq \emptyset$. Hence, since $(*)$ is commutative, $q_X^n(\mathcal{F}_n(f)(\mathcal{C})) \cap q_X^n(\mathcal{F}_n(f)(\mathcal{D})) \neq \emptyset$. Thus, by part (b) of Proposition 3.1, $\mathcal{F}_n(f)(\mathcal{C}) \cap \mathcal{F}_n(f)(\mathcal{D}) \neq \emptyset$. Let $E \in \mathcal{F}_n(f)(\mathcal{C}) \cap \mathcal{F}_n(f)(\mathcal{D})$. Then there exist $A \in \mathcal{C}$ and $A' \in \mathcal{D}$ such that $\mathcal{F}_n(f)(A) = \mathcal{F}_n(f)(A') = E$. Without loss of generality, we may assume that $A = \{c_1, \dots, c_{n-1}, c\}$ and $A' = \{c'_1, \dots, c'_{n-1}, d\}$, where, for each $j \in \{1, \dots, n - 1\}$, c_j and c'_j belong to C_j , $c \in C$ and $d \in D$. Since $\mathcal{F}_n(f)(A) = \mathcal{F}_n(f)(A')$, we obtain that $\{f(c_1), \dots, f(c_{n-1}), f(c)\} = \{f(c'_1), \dots, f(c'_{n-1}), f(d)\}$. This implies that $f(c) = f(d)$ and $f(C) \cap f(D) \neq \emptyset$. Therefore, f is joining.

Since (2) implies (3) and (3) implies (1), we have that (2) implies (1). The fact that (1) does not imply either (2) or (3), when $n = 2$, follows from Example 7.1. □

QUESTIONS 7.3. Let X and Y be continua, let n be an integer greater than or equal to 2, and let $f: X \rightarrow Y$ be a surjective map.

- (i) If f is joining, then is $\mathcal{SF}_n(f)$ joining?
- (ii) If $\mathcal{SF}_n(f)$ is joining, then is $\mathcal{F}_n(f)$ joining?

REMARK 7.4. Note that, by Example 7.1, the answer to Question 7.3 (i) is negative when $n = 2$.

8. REFINABLE AND MONOTONICALLY REFINABLE MAPS

Let X be a continuum and let n be an integer greater than or equal to 2. We follow Sam B. Nadler, Jr. ([16], compare with [12, p. 149]) to define a metric on $\mathcal{SF}_n(X)$. Let

$$\mathfrak{S}_n(X) = \{\mathcal{F}_1(X) \cup \{A\} \mid A \in \mathcal{F}_n(X)\}.$$

Note that $\mathfrak{S}_n(X) \subset \mathcal{C}_2(\mathcal{F}_n(X))$. Let $G_n: \mathcal{SF}_n(X) \rightarrow \mathfrak{S}_n(X)$ be given by

$$G_n(\chi) = \mathcal{F}_1(X) \cup (q_X^n)^{-1}(\chi).$$

Then G_n is a homeomorphism. Next, define

$$\rho_X^n: \mathcal{SF}_n(X) \times \mathcal{SF}_n(X) \rightarrow [0, \infty)$$

by

$$\rho_X^n(\chi_1, \chi_2) = \mathcal{H}_X^2(G_n(\chi_1), G_n(\chi_2)),$$

where \mathcal{H}_X^2 is the Hausdorff metric on $\mathcal{C}_2(\mathcal{F}_n(X))$ induced by the Hausdorff metric \mathcal{H}_X on $\mathcal{F}_n(X)$. Then ρ_X^n is a metric. As a consequence of [16, (2.3)], with our terminology, we obtain:

LEMMA 8.1. *Let X and Y be continua, let n be an integer greater than or equal to 2, and let $\varepsilon > 0$. If $f: X \rightarrow Y$ is a surjective map such that $\mathcal{F}_n(f)$ is an ε -map, then $\mathcal{SF}_n(f)$ is an ε -map.*

LEMMA 8.2. *Let X and Y be continua, let n be an integer greater than or equal to 2, and let $\varepsilon > 0$. If $f: X \rightarrow Y$ is a surjective map, then the following are equivalent:*

- (1) f is an ε -map;
- (2) $\mathcal{F}_n(f)$ is an ε -map;
- (3) $\mathcal{SF}_n(f)$ is an ε -map.

PROOF. Suppose f is an ε -map, we show that $\mathcal{F}_n(f)$ is an ε -map. Since f is surjective, by Lemma 3.4, $\mathcal{F}_n(f)$ is surjective. Let $B \in \mathcal{F}_n(Y)$ and let A_1 and A_2 be two elements of $\mathcal{F}_n(f)^{-1}(B)$. Since $\mathcal{F}_n(f)(A_1) = \mathcal{F}_n(f)(A_2)$, for each $a_1 \in A_1$ there exists $a_2 \in A_2$ such that $f(a_1) = f(a_2)$. Since f is an ε -map, $d(a_1, a_2) < \varepsilon$. Hence, $A_1 \subset \mathcal{V}_\varepsilon(A_2)$. Similarly, $A_2 \subset \mathcal{V}_\varepsilon(A_1)$. Thus, $\mathcal{H}_X(A_1, A_2) < \varepsilon$. Therefore, $\mathcal{F}_n(f)$ is an ε -map.

Assume that $\mathcal{F}_n(f)$ is an ε -map, we prove that f is an ε -map. By Lemma 3.4, f is surjective. Let $y \in Y$ and let x_1 and x_2 be two points in $f^{-1}(y)$. Then $\mathcal{F}_n(f)(\{x_1\}) = \mathcal{F}_n(f)(\{x_2\})$. Since $\mathcal{F}_n(f)$ is an ε -map, $\mathcal{H}_X(\{x_1\}, \{x_2\}) < \varepsilon$. Thus, $d(x_1, x_2) < \varepsilon$. Therefore, f is an ε -map.

It follows from Lemma 8.1 that if $\mathcal{F}_n(f)$ is an ε -map, then $\mathcal{SF}_n(f)$ is an ε -map.

Suppose $\mathcal{SF}_n(f)$ is an ε -map, we show that $\mathcal{F}_n(f)$ is an ε -map. By Lemma 3.4, $\mathcal{F}_n(f)$ is surjective. Let $B \in \mathcal{F}_n(Y)$ and let A_1 and A_2 be two

elements of $\mathcal{F}_n(f)^{-1}(B)$. Since $\mathcal{F}_n(f)(A_1) = \mathcal{F}_n(f)(A_2)$ and $(*)$ is commutative, we have that $\mathcal{S}\mathcal{F}_n(f)(q_X^n(A_1)) = \mathcal{S}\mathcal{F}_n(f)(q_X^n(A_2))$. Since $\mathcal{S}\mathcal{F}_n(f)$ is an ε -map, $\rho_X^n(q_X^n(A_1), q_X^n(A_2)) < \varepsilon$. Hence, $\mathcal{H}_X^2(\mathcal{F}_1(X) \cup \{A_1\}, \mathcal{F}_1(X) \cup \{A_2\}) < \varepsilon$. Thus, $\mathcal{H}_X(A_1, A_2) < \varepsilon$. Therefore, $\mathcal{F}_n(f)$ is an ε -map. \square

LEMMA 8.3. *Let X and Y be continua, let n be an integer greater than or equal to 2, and let $\varepsilon > 0$. Let $f, g: X \rightarrow Y$ be surjective maps. Consider the following statements:*

- (1) *For each $x \in X$, $d(f(x), g(x)) < \varepsilon$;*
- (2) *For each $A \in \mathcal{F}_n(X)$, $\mathcal{H}_Y(\mathcal{F}_n(f)(A), \mathcal{F}_n(g)(A)) < \varepsilon$;*
- (3) *For each $\chi \in \mathcal{S}\mathcal{F}_n(X)$, $\rho_Y^n(\mathcal{S}\mathcal{F}_n(f)(\chi), \mathcal{S}\mathcal{F}_n(g)(\chi)) < \varepsilon$;*
- (4) *For each $A \in \mathcal{F}_n(X)$, $\mathcal{H}_Y(\mathcal{F}_n(f)(A), \mathcal{F}_n(g)(A)) \leq \varepsilon$.*

Then (1) and (2) are equivalent, (2) implies (3) and (3) implies (4).

PROOF. Suppose (1) holds, we prove (2) holds. Let $A \in \mathcal{F}_n(X)$. Then for each $a \in A$, $d(f(a), g(a)) < \varepsilon$. Hence, $\mathcal{F}_n(f)(A) \subset \mathcal{V}_\varepsilon(\mathcal{F}_n(g)(A))$ and $\mathcal{F}_n(g)(A) \subset \mathcal{V}_\varepsilon(\mathcal{F}_n(f)(A))$. Therefore, $\mathcal{H}_Y(\mathcal{F}_n(f)(A), \mathcal{F}_n(g)(A)) < \varepsilon$.

Assume (2) holds, we show that (1) holds. Let $x \in X$. Then $\mathcal{H}_Y(\mathcal{F}_n(f)(\{x\}), \mathcal{F}_n(g)(\{x\})) < \varepsilon$. Since $\mathcal{H}_Y(\mathcal{F}_n(f)(\{x\}), \mathcal{F}_n(g)(\{x\})) = d(f(x), g(x))$, we obtain that $d(f(x), g(x)) < \varepsilon$.

Suppose (2) holds, we prove (3) holds. Let $\chi \in \mathcal{S}\mathcal{F}_n(X)$. If $\chi = F_X^n$, then $\mathcal{S}\mathcal{F}_n(f)(\chi) = F_Y^n = \mathcal{S}\mathcal{F}_n(g)(\chi)$ and $\rho_Y^n(\mathcal{S}\mathcal{F}_n(f)(\chi), \mathcal{S}\mathcal{F}_n(g)(\chi)) = 0 < \varepsilon$. Assume $\chi \in \mathcal{S}\mathcal{F}_n(X) \setminus \{F_X^n\}$. Then, since $(*)$ is commutative, we obtain

$$\rho_Y^n(\mathcal{S}\mathcal{F}_n(f)(\chi), \mathcal{S}\mathcal{F}_n(g)(\chi)) =$$

$$\mathcal{H}_Y^2(\mathcal{F}_1(Y) \cup (q_Y^n)^{-1}(\mathcal{S}\mathcal{F}_n(f)(\chi)), \mathcal{F}_1(Y) \cup (q_Y^n)^{-1}(\mathcal{S}\mathcal{F}_n(g)(\chi))) =$$

$$\mathcal{H}_Y^2(\mathcal{F}_1(Y) \cup \{\mathcal{F}_n(f)((q_X^n)^{-1}(\chi))\}, \mathcal{F}_1(Y) \cup \{\mathcal{F}_n(g)((q_X^n)^{-1}(\chi))\}) < \varepsilon.$$

Suppose (3) holds, we show that (4) holds. Let $A \in \mathcal{F}_n(X)$. Assume first that $A \in \mathcal{F}_n(X) \setminus \mathcal{F}_1(X)$. Then $q_X^n(A) \in \mathcal{S}\mathcal{F}_n(X) \setminus \{F_X^n\}$. Hence, $\rho_Y^n(\mathcal{S}\mathcal{F}_n(f)(q_X^n(A)), \mathcal{S}\mathcal{F}_n(g)(q_X^n(A))) < \varepsilon$; i.e.

$$\mathcal{H}_Y^2(\mathcal{F}_1(Y) \cup (q_Y^n)^{-1}(\mathcal{S}\mathcal{F}_n(f)(q_X^n(A))), \mathcal{F}_1(Y) \cup (q_Y^n)^{-1}(\mathcal{S}\mathcal{F}_n(g)(q_X^n(A)))) =$$

$$\mathcal{H}_Y^2(\mathcal{F}_1(Y) \cup \{\mathcal{F}_n(f)((q_X^n)^{-1}(q_X^n(A)))\}, \mathcal{F}_1(Y) \cup \{\mathcal{F}_n(g)((q_X^n)^{-1}(q_X^n(A)))\}) =$$

$$\mathcal{H}_Y^2(\mathcal{F}_1(Y) \cup \{\mathcal{F}_n(f)(A)\}, \mathcal{F}_1(Y) \cup \{\mathcal{F}_n(g)(A)\}) =$$

$$\mathcal{H}_Y(\mathcal{F}_n(f)(A), \mathcal{F}_n(g)(A)) < \varepsilon.$$

Now suppose $A \in \mathcal{F}_1(X)$. Then $A = \{a\}$. Let $\{a_m\}_{m=1}^\infty$ be a sequence of points of X converging to a such that $a_m \neq a$ for any positive integer m . Then $\{a, a_m\} \in \mathcal{F}_n(X) \setminus \mathcal{F}_1(X)$. By the previous argument, $\mathcal{H}_Y(\mathcal{F}_n(f)(\{a, a_m\}), \mathcal{F}_n(g)(\{a, a_m\})) < \varepsilon$. Hence, since $\{\{a, a_m\}\}_{m=1}^\infty$ converges to $\{a\}$, by the continuity of $\mathcal{F}_n(f)$, $\mathcal{F}_n(g)$ and \mathcal{H}_Y , we have that $\mathcal{H}_Y(\mathcal{F}_n(f)(\{a\}), \mathcal{F}_n(g)(\{a\})) \leq \varepsilon$. \square

THEOREM 8.4. *Let X and Y be continua, let n be an integer greater than or equal to 2, and let $f: X \twoheadrightarrow Y$ be a surjective map. Consider the following statements:*

- (1) f is refinable;
- (2) $\mathcal{F}_n(f)$ is refinable;
- (3) $\mathcal{SF}_n(f)$ is refinable.

Then (1) implies (2) and (3).

PROOF. The fact that (1) implies (2) follows from [9, Theorem 3.49]. Suppose f is refinable, we show that $\mathcal{SF}_n(f)$ is refinable. Let $\varepsilon > 0$. Since f is refinable, there exists an ε -map $g: X \twoheadrightarrow Y$ such that for every $x \in X$, $d(f(x), g(x)) < \varepsilon$. Then, by Lemma 8.2, $\mathcal{SF}_n(g)$ is an ε -map and, by Lemma 8.3, for each $\chi \in \mathcal{SF}_n(X)$, $\rho_Y^n(\mathcal{SF}_n(f)(\chi), \mathcal{SF}_n(g)(\chi)) < \varepsilon$. Therefore, $\mathcal{SF}_n(f)$ is refinable. \square

THEOREM 8.5. *Let X and Y be continua, let n be an integer greater than or equal to 2, and let $f: X \twoheadrightarrow Y$ be a surjective map. Consider the following statements:*

- (1) f is monotonically refinable;
- (2) $\mathcal{F}_n(f)$ is monotonically refinable;
- (3) $\mathcal{SF}_n(f)$ is monotonically refinable.

Then (1) implies (2) and (3).

PROOF. Suppose f is monotonically refinable, we show that $\mathcal{F}_n(f)$ and $\mathcal{SF}_n(f)$ are monotonically refinable. Let $\varepsilon > 0$. Since f is monotonically refinable, there exists a monotone ε -map $g: X \twoheadrightarrow Y$ such that for every $x \in X$, $d(f(x), g(x)) < \varepsilon$. By Lemma 8.2, $\mathcal{F}_n(g)$ and $\mathcal{SF}_n(g)$ are ε -maps. By [2, Theorem 4.1], $\mathcal{F}_n(g)$ and $\mathcal{SF}_n(g)$ are monotone maps. Now, by Lemma 8.3, we have that for each $A \in \mathcal{F}_n(X)$, $\mathcal{H}_Y(\mathcal{F}_n(f)(A), \mathcal{F}_n(g)(A)) < \varepsilon$; and for each $\chi \in \mathcal{SF}_n(X)$, $\rho_Y^n(\mathcal{SF}_n(f)(\chi), \mathcal{SF}_n(g)(\chi)) < \varepsilon$. Therefore, $\mathcal{F}_n(f)$ and $\mathcal{SF}_n(f)$ are monotonically refinable maps. \square

QUESTIONS 8.6. Let X and Y be continua, let n be an integer greater than or equal to 2, and let $f: X \twoheadrightarrow Y$ be a surjective map.

- (i) If $\mathcal{F}_n(f)$ is (monotonically) refinable, then is f (monotonically) refinable?
- (ii) If $\mathcal{F}_n(f)$ is (monotonically) refinable, then is $\mathcal{SF}_n(f)$ (monotonically) refinable?
- (iii) If $\mathcal{SF}_n(f)$ is (monotonically) refinable, then is f (monotonically) refinable?
- (iv) If $\mathcal{SF}_n(f)$ is (monotonically) refinable, then is $\mathcal{F}_n(f)$ (monotonically) refinable?

9. SEMI-CONFLUENT MAPS

THEOREM 9.1. *Let X and Y be continua, let n be an integer greater than or equal to 2, and let $f: X \twoheadrightarrow Y$ be a surjective map. Consider the following statements:*

- (1) f is semi-confluent;
- (2) $\mathcal{F}_n(f)$ is semi-confluent;
- (3) $\mathcal{SF}_n(f)$ is semi-confluent.

Then (2) implies (1) and (3), (3) implies (1); and (1) does not imply either (2) or (3) for $n = 2$.

PROOF. Suppose $\mathcal{SF}_n(f)$ is semi-confluent, we show that f is semi-confluent. Let B be a subcontinuum of Y . If $B = Y$, then $f^{-1}(B)$ has only one component, namely X and $f(X) = B$. Suppose $B \neq Y$ and let C and D be two components of $f^{-1}(B)$. Let y_1, \dots, y_{n-1} be $n - 1$ distinct elements of $Y \setminus B$. Let

$$\mathcal{K} = \{\{y_1, \dots, y_{n-1}\} \cup \{b\} \mid b \in B\}.$$

Note that \mathcal{K} is a subcontinuum of $\mathcal{F}_n(Y) \setminus \mathcal{F}_1(Y)$. Hence, $q_Y^n(\mathcal{K})$ is a subcontinuum of $\mathcal{SF}_n(Y) \setminus \{F_Y^n\}$. For each $j \in \{1, \dots, n - 1\}$, let C_j be a component of $f^{-1}(y_j)$. Let $\mathcal{C} = \langle C_1, \dots, C_{n-1}, C \rangle_n$ and $\mathcal{D} = \langle C_1, \dots, C_{n-1}, D \rangle_n$. Then by Lemma 3.6, \mathcal{C} and \mathcal{D} are components of $\mathcal{F}_n(f)^{-1}(\mathcal{K})$. By part (g) of Proposition 3.1, $q_X^n(\mathcal{C})$ and $q_X^n(\mathcal{D})$ are components of $\mathcal{SF}_n(f)^{-1}(q_Y^n(\mathcal{K}))$. Since $\mathcal{SF}_n(f)$ is semi-confluent, we have that either $\mathcal{SF}_n(f)(q_X^n(\mathcal{C})) \subset \mathcal{SF}_n(f)(q_X^n(\mathcal{D}))$ or $\mathcal{SF}_n(f)(q_X^n(\mathcal{D})) \subset \mathcal{SF}_n(f)(q_X^n(\mathcal{C}))$. Hence, since $(*)$ is commutative, we have that either $q_Y^n(\mathcal{F}_n(f)(\mathcal{C})) \subset q_Y^n(\mathcal{F}_n(f)(\mathcal{D}))$ or $q_Y^n(\mathcal{F}_n(f)(\mathcal{D})) \subset q_Y^n(\mathcal{F}_n(f)(\mathcal{C}))$. Thus, by part (b) of Proposition 3.1, either $\mathcal{F}_n(f)(\mathcal{C}) \subset \mathcal{F}_n(f)(\mathcal{D})$ or $\mathcal{F}_n(f)(\mathcal{D}) \subset \mathcal{F}_n(f)(\mathcal{C})$. Assume $\mathcal{F}_n(f)(\mathcal{C}) \subset \mathcal{F}_n(f)(\mathcal{D})$, we see that $f(C) \subset f(D)$. Let $c \in C$ and, for each $j \in \{1, \dots, n - 1\}$, let $c_j \in C_j$. Then $\{c_1, \dots, c_{n-1}, c\} \in \mathcal{C}$. Since $\mathcal{F}_n(f)(\{c_1, \dots, c_{n-1}, c\}) \in \mathcal{F}_n(f)(\mathcal{D})$, there exist $d \in D$ and $c'_j \in C_j$, where $j \in \{1, \dots, n - 1\}$, such that

$$\mathcal{F}_n(f)(\{c_1, \dots, c_{n-1}, c\}) = \mathcal{F}_n(f)(\{c'_1, \dots, c'_{n-1}, d\}).$$

Hence, $f(\{c_1, \dots, c_{n-1}, c\}) = f(\{c'_1, \dots, c'_{n-1}, d\})$. Thus, $f(c) = f(d)$ and $f(C) \subset f(D)$. Similarly, if $\mathcal{F}_n(f)(\mathcal{D}) \subset \mathcal{F}_n(f)(\mathcal{C})$, then $f(D) \subset f(C)$. Therefore, f is semi-confluent.

It follows from Theorem 3.2 that (2) implies (3). Since (2) implies (3) and (3) implies (1), we have that (2) implies (1). The fact that (1) does not imply (2), when $n = 2$ follows from [9, Example 3.18]. By [2, Example 10.7], we have that (1) does not imply (3) when $n = 2$. □

QUESTION 9.1. Let X and Y be continua, let n be an integer greater than or equal to 2, and let $f: X \twoheadrightarrow Y$ be a surjective map. If $\mathcal{SF}_n(f)$ is semi-confluent, then is $\mathcal{F}_n(f)$ semi-confluent?

10. SEMI-OPEN MAPS

THEOREM 10.1. *Let X and Y be continua and let n be an integer greater than or equal to 2. If $f: X \rightarrow Y$ is a surjective map, then the following are equivalent:*

- (1) f is semi-open;
- (2) $\mathcal{F}_n(f)$ is semi-open;
- (3) $\mathcal{S}\mathcal{F}_n(f)$ is semi-open.

PROOF. Suppose f is semi-open, we show that $\mathcal{F}_n(f)$ is semi-open. Let \mathcal{U} be an open subset of $\mathcal{F}_n(X)$ and let $A_0 \in \mathcal{U}$. Then there exist open subsets U_1, \dots, U_k of X such that $A_0 \in \langle U_1, \dots, U_k \rangle_n \subset \mathcal{U}$. Since f is semi-open, for each $j \in \{1, \dots, k\}$, $\text{Int}_Y(f(U_j)) \neq \emptyset$. Thus, $\langle \text{Int}_Y(f(U_1)), \dots, \text{Int}_Y(f(U_k)) \rangle_n$ is an open subset of $\mathcal{F}_n(Y)$. We claim that $\langle \text{Int}_Y(f(U_1)), \dots, \text{Int}_Y(f(U_k)) \rangle_n \subset \mathcal{F}_n(f)(\mathcal{U})$. To see this, let $B \in \langle \text{Int}_Y(f(U_1)), \dots, \text{Int}_Y(f(U_k)) \rangle_n$. For every $j \in \{1, \dots, k\}$, let $B_j = B \cap \text{Int}_Y(f(U_j))$. Then there exists $A_j \subset U_j$ such that A_j and B_j both have the same cardinality and $f(A_j) = B_j$. Let $A = \bigcup_{j=1}^k A_j$. Then $A \in \langle U_1, \dots, U_k \rangle_n$ and $\mathcal{F}_n(f)(A) = B$. Hence, $B \in \mathcal{F}_n(f)(\langle U_1, \dots, U_k \rangle_n) \subset \mathcal{F}_n(f)(\mathcal{U})$. Thus, $\text{Int}_{\mathcal{F}_n(Y)}(\mathcal{F}_n(f)(\mathcal{U})) \neq \emptyset$. Therefore, $\mathcal{F}_n(f)$ is a semi-open map.

Assume $\mathcal{F}_n(f)$ is semi-open, we prove that f is semi-open. Let U be an open subset of X . Since $\mathcal{F}_n(f)$ is semi-open, $\text{Int}_{\mathcal{F}_n(Y)}(\mathcal{F}_n(f)(\langle U \rangle_n)) \neq \emptyset$. We consider two cases.

First, suppose that $\text{Int}_{\mathcal{F}_n(Y)}(\mathcal{F}_n(f)(\langle U \rangle_n)) \cap \mathcal{F}_1(Y) \neq \emptyset$. Let $\{y_0\} \in \text{Int}_{\mathcal{F}_n(Y)}(\mathcal{F}_n(f)(\langle U \rangle_n)) \cap \mathcal{F}_1(Y)$. Then there exist open subsets V_1, \dots, V_m of Y such that $\{y_0\} \in \langle V_1, \dots, V_m \rangle_n \subset \text{Int}_{\mathcal{F}_n(Y)}(\mathcal{F}_n(f)(\langle U \rangle_n))$. Note that $y_0 \in \bigcap_{j=1}^m V_j$. We claim that $\bigcap_{j=1}^m V_j \subset f(U)$. Let $y \in \bigcap_{j=1}^m V_j$. Then $\{y\} \in \langle V_1, \dots, V_m \rangle_n$, and there exists $A_y \in \langle U \rangle_n$ such that $\mathcal{F}_n(f)(A_y) = \{y\}$. This means that $A_y \subset U$ and $f(A_y) = \{y\}$. Thus, $y \in f(U)$ and $\bigcap_{j=1}^m V_j \subset f(U)$.

Now, assume that $\text{Int}_{\mathcal{F}_n(Y)}(\mathcal{F}_n(f)(\langle U \rangle_n)) \cap \mathcal{F}_1(Y) = \emptyset$. Let $\ell = \min\{j \in \{2, \dots, n\} \mid \text{Int}_{\mathcal{F}_n(Y)}(\mathcal{F}_n(f)(\langle U \rangle_n)) \cap \mathcal{F}_j(Y) \neq \emptyset\}$. Let

$$\{y_1, \dots, y_\ell\} \in \text{Int}_{\mathcal{F}_n(Y)}(\mathcal{F}_n(f)(\langle U \rangle_n)) \cap \mathcal{F}_\ell(Y).$$

Then there exist open subsets W_1, \dots, W_r of Y such that $\{y_1, \dots, y_\ell\} \in \langle W_1, \dots, W_r \rangle_n \subset \text{Int}_{\mathcal{F}_n(Y)}(\mathcal{F}_n(f)(\langle U \rangle_n))$. For each $j \in \{1, \dots, \ell\}$, let $Z_j = \bigcap \{W_k \in \{W_1, \dots, W_r\} \mid y_j \in W_k\}$. Note that each Z_j is an open subset of Y , and $\{y_1, \dots, y_\ell\} \in \langle Z_1, \dots, Z_\ell \rangle_n \subset \langle W_1, \dots, W_r \rangle_n$. Let $Z = \bigcup_{j=1}^\ell Z_j$. We claim that $Z \subset f(U)$. Let $z \in Z$. Without loss of generality, we assume that $z \in Z_1$. For every $j \in \{2, \dots, \ell\}$, let $z_j \in Z_j$. Then $\{z, z_2, \dots, z_\ell\} \in \langle Z_1, \dots, Z_\ell \rangle_n \subset \text{Int}_{\mathcal{F}_n(Y)}(\mathcal{F}_n(f)(\langle U \rangle_n))$. Hence, there exists $A_z \in \langle U \rangle_n$ such that $\mathcal{F}_n(f)(A_z) = \{z, z_2, \dots, z_\ell\}$. This implies that $A_z \subset U$ and $f(A_z) = \{z, z_2, \dots, z_\ell\}$. Thus, there exists $a \in A_z$ such that $f(a) = z$. Hence, $z \in f(U)$ and $Z \subset f(U)$. Therefore, f is a semi-open map.

The fact that (2) and (3) are equivalent follows from Theorem 3.3. \square

11. SIMPLE MAPS

THEOREM 11.1. *Let X and Y be continua, let n be an integer greater than or equal to 2, and let $f: X \rightarrow Y$ be a surjective map. If $\mathcal{F}_n(f)$ is a simple map, then f is a homeomorphism.*

PROOF. If f is not one-to-one, then there exist two distinct points x_1 and x_2 in X such that $f(x_1) = f(x_2)$. Let $f(x_1) = y$. Then $\{x_1\}$, $\{x_2\}$ and $\{x_1, x_2\}$ are 3 distinct points in $\mathcal{F}_n(f)^{-1}(\{y\})$. Therefore, $\mathcal{F}_n(f)$ is not simple. \square

The following example shows that there exists a simple map f between locally connected continua such that neither $\mathcal{F}_n(f)$ nor $\mathcal{SF}_n(f)$ is a simple map.

EXAMPLE 11.2. Let $f: \mathcal{S}^1 \rightarrow \mathcal{S}^1$ be given by $f(z) = z^2$. Then f is a simple map. Let z and z' be two distinct elements of \mathcal{S}^1 and let z_1, z_2, z'_1 and z'_2 be the elements of \mathcal{S}^1 such that $f(z_1) = f(z_2) = z$ and $f(z'_1) = f(z'_2) = z'$. Then $\{z_1\}$, $\{z_2\}$ and $\{z_1, z_2\}$ belong to $\mathcal{F}_n(f)^{-1}(\{z\})$. Thus, $\mathcal{F}_n(f)$ is not a simple map. Also, $q_{\mathcal{S}^1}^n(\{z_1, z'_1\})$, $q_{\mathcal{S}^1}^n(\{z_1, z'_2\})$, $q_{\mathcal{S}^1}^n(\{z_2, z'_1\})$ and $q_{\mathcal{S}^1}^n(\{z_2, z'_2\})$ all belong to $\mathcal{SF}_n(f)^{-1}(q_{\mathcal{S}^1}^n(\{z, z'\}))$. Hence, $\mathcal{SF}_n(f)$ is not a simple map.

THEOREM 11.3. *Let X and Y be continua, let n be an integer greater than or equal to 2, and let $f: X \rightarrow Y$ be a surjective map. If $\mathcal{SF}_n(f)$ is a simple map, then either f is a homeomorphism or f identifies two points. In particular, f is a simple map.*

PROOF. Assume $\mathcal{SF}_n(f)$ is a simple map. If $\mathcal{SF}_n(f)^{-1}(F_Y^n) = \{F_X^n\}$, then f is a homeomorphism by Lemma 3.7. Suppose there exists $\chi \in \mathcal{SF}_n(f)^{-1}(F_Y^n) \setminus \{F_X^n\}$. Then $(q_X^n)^{-1}(\chi) \in \mathcal{F}_2(X) \setminus \mathcal{F}_1(X)$. Otherwise, we could find k points x_1, \dots, x_k in X ($k \geq 3$) such that $q_X^n(\{x_1, \dots, x_k\}) = \chi$ and we would have that $q_X^n(\{x_1, x_2\})$, $q_X^n(\{x_1, x_3\})$ and $q_X^n(\{x_2, x_3\})$ would all belong to $\mathcal{SF}_n(f)^{-1}(F_Y^n)$, a contradiction. Assume that $(q_X^n)^{-1}(\chi) = \{x_1, x_2\}$. Observe that the previous argument implies that $f|_{X \setminus \{x_1, x_2\}}$ is one-to-one. Therefore, f identifies x_1 and x_2 . \square

The following example shows that there exists a simple map f between continua such that $\mathcal{F}_n(f)$ is not a simple map and $\mathcal{SF}_n(f)$ is a simple map.

EXAMPLE 11.4. Let $f: [0, 1] \rightarrow \mathcal{S}^1$ be given by $f(t) = (\sin(2\pi t), \cos(2\pi t))$. Note that f identifies two points. Then by Theorem 11.1, we have that $\mathcal{F}_n(f)$ is not simple. It is easy to see that $\mathcal{SF}_n(f)$ is simple.

THEOREM 11.5. *Let X and Y be continua, let n be an integer greater than or equal to 2, and let $f: X \rightarrow Y$ be a surjective map. If there exist two*

elements y_1 and y_2 of Y such that both $\mathcal{F}_n(f)^{-1}(\{y_1\})$ and $\mathcal{F}_n(f)^{-1}(\{y_2\})$ have two elements, then $\mathcal{SF}_n(f)$ is not simple.

PROOF. Let y_1 and y_2 be two elements of Y , let A_1 and B_1 be two distinct elements in $\mathcal{F}_n(f)^{-1}(\{y_1\})$, and let A_2 and B_2 be two distinct elements in $\mathcal{F}_n(f)^{-1}(\{y_2\})$. Without loss of generality, we assume that $A_1 \not\subset B_1$ and $A_2 \not\subset B_2$. Let $a_1 \in A_1 \setminus B_1$, $b_1 \in B_1$, $a_2 \in A_2 \setminus B_2$ and $b_2 \in B_2$. Then F_X^n , $q_X^n(\{a_1, b_1\})$ and $q_X^n(\{a_2, b_2\})$ belong to $\mathcal{SF}_n(f)^{-1}(F_Y^n)$. Therefore, $\mathcal{SF}_n(f)$ is not simple. \square

THEOREM 11.6. *Let X and Y be continua, let n be an integer greater than or equal to 2, and let $f: X \rightarrow Y$ be a surjective map. Consider the following statements:*

- (1) f is simple;
- (2) $\mathcal{F}_n(f)$ is simple;
- (3) $\mathcal{SF}_n(f)$ is simple.

Then (2) implies (1) and (3), (3) implies (1), (3) does not imply (2); and (1) does not imply either (2) or (3).

PROOF. If $\mathcal{F}_n(f)$ is simple, then f is a homeomorphism by Theorem 11.1. Thus, $\mathcal{SF}_n(f)$ is a homeomorphism ([2, Theorem 3.3]). Hence, f and $\mathcal{SF}_n(f)$ are both simple maps. It follows from Theorem 11.3 that if $\mathcal{SF}_n(f)$ is a simple map, then f is a simple map. The fact that (3) does not imply (2), follows from Example 11.4, and the fact that (1) does not imply either (2) or (3), follows from Example 11.2. \square

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