

A COMPARISON RESULT FOR A LINEAR DIFFERENTIAL EQUATION WITH PIECEWISE CONSTANT DELAY

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ABSTRACT. We present a new comparison result for an equation of the form $u'(t) + mu(t) + nu([t]) = \sigma(t)$, $t \in [0, T]$ where $[\cdot]$ denotes the greatest integer function.

1. INTRODUCTION

The study and applications of functional differential equations and in particular of delay differential equations is of great interest. Many times the delay function is assumed to be constant or continuous. However, in some situations the delay is not continuous but piecewise continuous [1, 3, 4, 6, 7, 8, 9].

On the other hand, the theory of differential inequalities plays a prominent role in the study of differential equations.

In this paper we consider a functional differential inequality with piecewise constant delay of the form

$$u'(t) + mu(t) + nu([t]) \leq 0,$$

where $[\cdot]$ is the greatest integer function. We prove a new comparison result obtaining a better estimate than others already known.

2. PRELIMINARIES

Let $T > 0$, $m > 0$ and $I = [0, T]$. It is well-known that if $u \in C^1(I)$ is such that

$$u'(t) + mu(t) \leq 0, \quad t \in I,$$

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and

$$u(0) \leq u(T)$$

then $u(t) \leq 0$ for every $t \in I$.

If we consider the functional inequality

$$(1) \quad u'(t) + mu(t) + nu(h(t)) \leq 0, \quad t \in I,$$

where $n \geq 0$ and $h : I \rightarrow I$ with $h(t) \leq t$, $t \in I$ then at least a restriction on n is necessary as the following example shows.

EXAMPLE 2.1. Consider

$$u : I \rightarrow I, \quad u(t) = t - \frac{T}{2},$$

and

$$h : I \rightarrow I, \quad h(t) = 0, \quad t \in I.$$

Thus, for n sufficiently large we have for any $t \in I$ that

$$u'(t) + mu(t) + nu(h(t)) = 1 + m(t - \frac{T}{2}) - n\frac{T}{2} \leq 1 + (m - n)\frac{T}{2} \leq 0.$$

However, $u(T) = \frac{T}{2} > 0$, and $u(0) = -\frac{T}{2} < u(T)$.

It is possible to give an estimate for n , and we have the following result.

THEOREM 2.2. *Let h be continuous. If $u \in C^1(I)$ satisfies the inequality (1), $u(0) \leq u(T)$, and*

$$(2) \quad n \frac{e^{mT} - 1}{m} < 1,$$

then $u(t) \leq 0$, $t \in I$.

This result has been proved independently by Jiang and Wei (Th. 2.1, [2]) and Nieto (Th. 5 and Cor. 2, [5]). In fact in [5] it is considered a more general equation of the form

$$u'(t) + mu(t) + [p(u)](t) = \sigma(t), \quad t \in I$$

where $p : L^1(I) \rightarrow L^1(I)$ is a given functional (not necessarily continuous). Of course, it contains the case $[p(v)](t) = nu(h(t))$.

Note that the case $h(t) = [t]$ is not included in [2] since it is not continuous, but it is a particular case of the results of [5].

The purpose of this note is to consider the important case $h(t) = [t]$ to obtain a better estimate for n and m . Note that the results of [2] are not applicable to this type of delay since the greatest integer function is not continuous.

3. MAIN RESULT

Let $k \in \mathbf{N}$, such that $T \in (k, k + 1]$. Consider the space E of functions $u : I \rightarrow \mathbf{R}$ such that $u \in C(I)$, $u'(t)$ exists and is continuous on $[i - 1, i]$, $i = 1, 2, \dots, k$ and $[k, T)$.

THEOREM 3.1. *Suppose that $u \in E$ is such that*

$$(3) \quad u'(t) + mu(t) + nu([t]) \leq 0, \quad t \in I,$$

$$(4) \quad u(0) \leq u(T),$$

$$(5) \quad \frac{n(e^m - 1)}{m} < 1.$$

Then $u(t) \leq 0$ for every $t \in I$.

PROOF. Let $v(t) = e^{mt}u(t)$, $t \in I$. Thus, by (3), for every $t \in I$ we get

$$(6) \quad v'(t) = e^{mt}u'(t) + me^{mt}u(t) \leq -ne^{mt}e^{-m[t]}v([t]).$$

Now, for $i = 1, 2, \dots, k$, consider an interval of the form $[i - 1, i)$. For $t \in [i - 1, i]$ we can write

$$\begin{aligned} v(t) &= v(i - 1) + \int_{i-1}^t v'(s) ds \leq v(i - 1) - nv(i - 1) \int_{i-1}^t e^{m(s-[s])} ds \\ &= v(i - 1) - nv(i - 1) \frac{e^{m(t-i+1)} - 1}{m}. \end{aligned}$$

In particular, for $t = i$, $i = 1, 2, \dots, k$ we obtain

$$(7) \quad v(i) \leq v(i - 1) \left(1 - \frac{n}{m}(e^m - 1) \right).$$

Analogously, for $t \in [k, T]$ we have the inequality

$$v(t) \leq v(k) - nv(k) \frac{e^{m(t-k)} - 1}{m}.$$

and in particular

$$(8) \quad v(T) \leq v(k) \left(1 - \frac{n}{m}(e^{m(T-k)} - 1) \right).$$

We now show that

$$(9) \quad v(i) > 0 \text{ for every } i = 1, 2, \dots, k$$

is not possible. Indeed, if (9) holds, then using (6) we see that v is decreasing on each subinterval $[i - 1, i]$, $i = 1, 2, \dots, k$ and on $[k, T]$. Thus,

$$v(0) \geq v(1) \geq v(2) \geq \dots \geq v(k - 1) \geq v(k) \geq v(T),$$

and $u(0) = v(0) \geq v(T) = e^{mT}u(T)$ which contradicts (4).

Therefore, there exists $j \in \{0, 1, \dots, k\}$ such that $v(j) \leq 0$. Hence, (5) and (7) imply that

$$v(i) \leq 0, \quad i = j + 1, \dots, k,$$

and from (8) we deduce that $v(T) \leq 0$. Using (4) we get that

$$(10) \quad v(0) \leq 0.$$

This implies, in view of (5) and (7), that $v(1) \leq 0$ and $v(i) \leq 0, i = 1, 2, \dots, k$. Now, by (8) we also have $v(T) \leq 0$. On the other hand, (6) implies that v is decreasing on every subinterval $[i - 1, i], i = 1, 2, \dots, k$ and (8) that it is decreasing on $[k, T]$. Hence $v(t) \leq 0$ for every $t \in I$, and, in consequence, $u(t) \leq 0, t \in I$. \square

We point out that estimate (5) is better than estimate (2) since for $T > 1$ we have that $e^{mT} > e^m$. Of course, if $T \leq 1$ then $[t] = 0$ for every $t \in I$ and we have in (3) an ordinary differential inequality.

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