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DERIVATION OF THE REYNOLDS TYPE EQUATION WITH MEMORY EFFECTS, GOVERNING TRANSIENT FLOW OF LUBRICANT

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Abstract. In this paper we derive the Reynolds type lubrication equation with memory effects for lubrication of a rotating shaft. Starting from the nonstationary Stokes equations and using the asymptotic expansion we obtain a law that is nonlocal in time and gives a linear connection between velocity and pressure gradient. Our law tends to the classical Reynolds equation when time tends to infinity.

We prove a convergence theorem for velocity and pressure in appropriate functional space.

1. Introduction

We consider the lubrication process of journal bearings. A circular shaft of radius R and length L rotates on a lubricated support with angular velocity ω. Between the shaft and the support there is a thin domain completely filled with a viscous incompressible lubricant injected by some prescribed velocity. The flow regime is assumed to be laminar, and heating effects are neglected.

Let the length scale in the "plane" of the film be denoted by $L_{\varphi z}$ (as $L_{\varphi z}$ we can choose L or $2R\pi$), and let l be the film thickness; for conventional bearing geometries $l/L_{\varphi z} = O(10^{-3})$. We use these length scales to normalize the equations of motion. Nondimensional coordinates, denoted by overbar, we define as follows

$$\{\overline{r}, \overline{\varphi}, \overline{z}\} = \frac{1}{L_{\varphi z}}\{\frac{L_{\varphi z}}{l}r, \varphi, z\}.$$

Let U_* represent the characteristic velocity in the "plane" of the film. The equation of continuity then requires $U_* \frac{l}{L_{\varphi z}}$ to be the velocity scale across

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the film, and we arrive the following definition for normalized velocity:

$$\overline{\mathbf{u}} = \{\overline{u}_r, \overline{u}_\varphi, \overline{u}_z\} = \frac{1}{U_*}\{\frac{L_{\varphi z}}{l}u_r, u_\varphi, u_z\},$$

where the overbar again denotes normalized, i.e. O(1), nondimensional quantity. The nondimensional pressure and time are chosen to be

$$\overline{p} = \frac{p}{\rho U_*^2}\{\frac{l}{L_{\varphi z}}\}Re, \qquad \overline{t} = \Omega t.$$

Here Ω is the characteristic frequency of the flow, and the Reynolds number has the definition $Re = lU_*/\nu$.

Substituting into the Navier-Stokes equations and neglecting terms of order $(\frac{l}{L_{\varphi z}})^2$, we obtain

$$\begin{aligned}
\Omega^* \frac{\partial \overline{u}_\varphi}{\partial t} + Re^* \overline{\mathbf{u}} \cdot \nabla \overline{u}_\varphi &= -\frac{1}{R}\frac{\partial \overline{p}}{\partial \varphi} + \frac{\partial^2 \overline{u}_\varphi}{\partial \overline{r}^2}, \\
\Omega^* \frac{\partial \overline{u}_z}{\partial t} + Re^* \overline{\mathbf{u}} \cdot \nabla \overline{u}_z &= -\frac{\partial \overline{p}}{\partial \overline{z}} + \frac{\partial^2 \overline{u}_z}{\partial \overline{r}^2},
\end{aligned}$$

Here $\Omega^* = \frac{l^2\Omega}{\nu}$ and $Re^* = Re\frac{l}{L}$ are the reduced frequency and the reduced Reynolds number, respectively, and $p = p(\varphi, z)$.

Contrary to [4] where, starting from the full Navier-Stokes equations, we derived rigorously the classical Reynolds equation for lubrication of a rotating shaft, in this paper we consider temporal inertia limit of the Navier-Stokes equations, which is characterized by $Re^* \to 0, \Omega^* > 1$, and derive a law which is nonlocal in time and properly describes the transient effects.

According to Szeri [10], when one of the bearing surfaces undergoes rapid small-amplitude oscillation, the condition $\Omega^* >> Re^*$ is approximately satisfied. In this case, we retain the temporal inertia terms but drop the terms representing convective inertia. Expressed in primitive variables, the equation of motion have now the reduced form

$$\begin{aligned}
\frac{\partial u_\varphi}{\partial t} &= -\frac{1}{R}\frac{\partial p}{\partial \varphi} + \mu \frac{\partial^2 u_\varphi}{\partial r^2}, \\
\frac{\partial u_z}{\partial t} &= -\frac{\partial p}{\partial z} + \mu \frac{\partial^2 u_z}{\partial r^2}.
\end{aligned}$$

From the asymptotic analysis point of view, these are equations for the lowest approximation of the velocity u. For the above-described situation the fluid flow in a thin film is described by the nonstationary Stokes system where the term with Δu is equal $\mu\varepsilon^2$.

Starting from nonstationary Stokes system and performing asymptotic analysis of this perturbed problem we study the behavior of the flow as $\varepsilon \to 0$.

At the limit, we find the Reynolds equation with memory effects governing the 2-dimensional macroscopic flow, as an approximation of the Stokes system in thin 3-dimensional domain. The law is nonlocal in time and gives a linear connection between velocity and the pressure gradient. According to the described physical situation, memory effects appear only for short period of time. We prove that our law tends to the ordinary Reynolds equation when time tends to infinity.

To achieve these results we use some properties of the Laplace transform in a similar way it has been done in [8], where the Darcy type law with memory effects was derived governing transient flow through porous media.

The mathematical justification of the Reynolds equation from the fundamental hydrodynamic equations has been subject of a lot papers.

The derivation of the classical Reynolds equations for a flow between two plain surfaces was given in [1] und [9]. A precise study of asymptotic behavior of the Newtonian flow in a thin domain between a rotating shaft and lubricated support was given in [4] and the difference between the solution of the Navier-Stokes system in thin domain and the solution of the Reynolds equation in terms of the domain thickness was estimated.

In [2] we considered lubrication of a rotating shaft with non-Newtonian fluid and starting from Navier-Stokes (Stokes) equations with nonlinear viscosity, being a power of the shear rate (power law), we derived non-linear Reynolds lubrication equation.

2. Description of the ε-problem

We consider the nonstationary Newtonian flow through a thin cylinder

$$\mathcal{C}_\varepsilon = \{\Xi^{-1}(r,\varphi,z) \in \mathbf{R}^3 \ ; \ \varphi \in]0,2\pi[, z \in]0,L[, R < r < R + \varepsilon h(\varphi,z)\}.$$

Function $h :]0,2\pi[\times]0,L[\to \mathbf{R}^+$ is of class C^1, 2π-periodic in variable φ, and $0 < \beta_1 \le h(\varphi,z) \le \beta_2$, $\varphi \in]0,2\pi[$, $z \in]0,L[$.

The flow in cylinder $\mathcal{C}_\varepsilon$ is described by the nonstationary Stokes system

$$\left\{ \begin{array}{l} \dfrac{\partial u^\varepsilon}{\partial t} - \mu\varepsilon^2 \Delta u^\varepsilon + \nabla p^\varepsilon = 0 \ \text{ in } \mathcal{C}_\varepsilon \times (0,T), \\ \text{div } u^\varepsilon = 0 \ \text{ in } \mathcal{C}_\varepsilon \times (0,T), \\ u^\varepsilon(x,0) = w^0(\varphi,z) \ \text{ in } \mathcal{C}_\varepsilon, \\ \quad \left. \begin{array}{l} u^\varepsilon = 0, \ \text{ for } r = R + \varepsilon h(\varphi,z), \\ u^\varepsilon = \omega(t)\vec{e}_\varphi, \ \text{ for } r = R, \\ u^\varepsilon = g_0(\dfrac{r-R}{\varepsilon},\varphi,t), \ \text{ for } z = 0, \\ u^\varepsilon = g_L(\dfrac{r-R}{\varepsilon},\varphi,t), \ \text{ for } z = L, \end{array} \right. \quad t \in (0,T), \end{array} \right. \tag{2.1}$$

where p^ε and u^ε are the pressure and the velocity, respectively. In order to have a well-posed problem we suppose that the functions $g_\alpha \in W^{1,\infty}_{loc}(0,\infty; H^1(\mathcal{S}_\alpha))$, $\alpha = 0, L$, $\mathcal{S}_\alpha = \{(\rho,\varphi); \rho \in]0,h(\varphi,\alpha)[, \varphi \in]0,2\pi[\}$, are continuous in variables ρ and φ. They are 2π-periodic in variable φ and for all $t \geq 0$ satisfy the hypothesis

(H1) $\displaystyle\int_0^{2\pi}\int_0^{h(\varphi,0)} \rho\vec{e}_z \cdot g_0(\rho,\varphi,t)d\rho d\varphi = \int_0^{2\pi}\int_0^{h(\varphi,L)} \rho\vec{e}_z \cdot g_L(\rho,\varphi,t)d\rho d\varphi,$

(H2) $\displaystyle\int_0^{2\pi}\int_0^{h(\varphi,0)} \vec{e}_z \cdot g_0(\rho,\varphi,t)d\rho d\varphi = \int_0^{2\pi}\int_0^{h(\varphi,L)} \vec{e}_z \cdot g_L(\rho,\varphi,t)d\rho d\varphi,$

(H3) $g_\alpha(h(\varphi),\varphi,t) = 0, \quad g_\alpha(0,\varphi,t) = \omega(t)\vec{e}_\varphi\ , \ \alpha = 0, L.$

We suppose that $\omega \in W^{1,\infty}_{loc}(0,\infty)$, $w^0 \in L^2(\Omega)$ is 2π-periodic in variable φ and $\mathrm{div}_{\varphi,z} w^0 = \frac{\partial w^0}{\partial \varphi} + R\frac{\partial w^0}{\partial z} = 0$, where $\Omega =]0,2\pi[\times]0,L[$.

Under the assumptions

$$
\begin{aligned}
&|g_\alpha(t) - g_\alpha^\infty|_{H^1(S_\alpha)} \to 0, \quad \text{for } \alpha = 0, L,\\
&|\omega(t) - \omega^\infty| \to 0,
\end{aligned}
$$

when $t \to \infty$, we want to investigate the behavior of the limit problem when time tends to infinity.

Lemma 6.1 gives the existence of a solenoidal vector valued function $\phi^\varepsilon \in W^{1,\infty}_{loc}(0,\infty; H^1(\mathcal{C}_\varepsilon))$ such that $\phi^\varepsilon_{/\partial\mathcal{C}_\varepsilon} = u^\varepsilon_{/\partial\mathcal{C}_\varepsilon}$.

Classical theory (see Temam [11]) gives the existence of a unique weak solution $u^\varepsilon \in L^2(0,T;H^1(\mathcal{C}_\varepsilon))$, $p^\varepsilon \in H^{-1}(0,T;L^2_0(\mathcal{C}_\varepsilon))$ such that $u^\varepsilon - \phi^\varepsilon \in L^2(0,T;V^\varepsilon)$, $u^\varepsilon \in L^\infty(0,T;L^2(\mathcal{C}_\varepsilon))$ and $\frac{\partial}{\partial t}(u^\varepsilon - \phi^\varepsilon) \in L^{4/3}(0,T;V^\varepsilon)$, where V_ε is the functional space given by

$$V_\varepsilon = \{u \in H^1_0(\mathcal{C}_\varepsilon),\ \ \mathrm{div}\ u = 0 \ \text{ u } \mathcal{C}_\varepsilon\}$$

and $T > 0$.

3. Statement of the main results

Let us now state the main results of this paper whose proofs will be found in the following sections. The first theorem shows that the Reynolds type equation with memory effects is an approximation of the Stokes system. We recall that $\Omega =]0,2\pi[\times]0,L[$.

Theorem 3.1. *Let $(u^\varepsilon, p^\varepsilon)$ be a solution of the nonstationary Stokes problem (2.1). Let p be a solution of the nonstationary Reynolds problem*

$$
(3.1)\quad \begin{cases}
\dfrac{1}{R}\dfrac{\partial}{\partial\varphi}(-\xi * \dfrac{\partial p}{\partial \varphi} + \xi w^0_\varphi) + R\dfrac{\partial}{\partial z}(-\xi * \dfrac{\partial p}{\partial z} + \xi w^0_z) = -\eta * \omega \ \textit{in}\ \Omega\times]0,T[,\\[2mm]
\left[-\xi * \dfrac{\partial p}{\partial z} + \xi w^0_z\right]_{z=0} = \dfrac{1}{R}\displaystyle\int_0^{h(\varphi,0)} \vec{e}_z \cdot g_0(\rho,\varphi,t)d\rho,\\[2mm]
\left[-\xi * \dfrac{\partial p}{\partial z} + \xi w^0_z\right]_{z=L} = \dfrac{1}{R}\displaystyle\int_0^{h(\varphi,L)} \vec{e}_z \cdot g_L(\rho,\varphi,t)d\rho,\\[2mm]
p \textit{ is } 2\pi\textit{-periodic in variable } \varphi,
\end{cases}
$$

and let u be given by

$$(3.2)\qquad \begin{aligned} u_\varphi &= G*[-\frac{1}{R}\frac{\partial p}{\partial\varphi}] + Gw^0_\varphi + g*\omega,\\ u_z &= G*\frac{\partial p}{\partial z} + Gw^0_z. \end{aligned}$$

Then

$$\begin{aligned} &\frac{1}{\varepsilon}\int_R^{R+\varepsilon h} u^\varepsilon \rightharpoonup \int_0^h u \text{ weakly in } L^2(\Omega\times(0,T)),\\ &\frac{1}{\varepsilon}\int_R^{R+\varepsilon h} p^\varepsilon \rightharpoonup p \text{ weakly in } H^{-1}(0,T;L^2_0(\Omega)). \end{aligned}$$

This theorem is a direct consequence of the Propositions 7.1 and 7.2 whose proof will be found in Section 7.

Remark 3.2. The existence and uniqueness of the problem (3.1) is given in Section 5, while the functions ξ, η, g and G are defined in Section 4.

The following theorems establish the connection between the nonlocal Reynolds-type problem (3.1) and the ordinary Reynolds problem from [4].

Theorem 3.3. *Let (u,p) be a solution of the problem (3.1)–(3.2). Let $p^\infty \in H^1(\Omega)\cap L^2_0(\Omega)$ be a solution of the problem*

$$(3.3)\qquad \begin{cases} \frac{1}{R}\frac{\partial}{\partial\varphi}(h^3\frac{\partial p^\infty}{\partial\varphi}) + R\frac{\partial}{\partial z}(h^3\frac{\partial p^\infty}{\partial z}) = 6\frac{\partial h}{\partial\varphi}\mu\omega^\infty \quad \text{in } \Omega,\\ \frac{\partial p^\infty}{\partial z} = -\frac{12\mu}{h^3(\varphi,0)}\int_0^{h(\varphi,0)} \vec{e}_z\cdot g_0^\infty(\rho,\varphi)d\rho, \quad \text{for } z=0,\\ \frac{\partial p^\infty}{\partial z} = -\frac{12\mu}{h^3(\varphi,L)}\int_0^{h(\varphi,L)} \vec{e}_z\cdot g_L^\infty(\rho,\varphi)d\rho, \quad \text{for } z=L,\\ p^\infty \text{ is } 2\pi\text{-periodic in } \varphi, \end{cases}$$

and let u^∞ be defined by

$$(3.4)\qquad \begin{cases} u^\infty_\varphi = \frac{1}{2\mu R}(\rho-h)\rho\frac{\partial p^\infty}{\partial\varphi} + \omega(1-\frac{\rho}{h})\\ u^\infty_z = \frac{1}{2\mu}(\rho-h)\rho\frac{\partial p^\infty}{\partial z}. \end{cases}$$

Then

$$\begin{cases} p(t)\to p^\infty \quad \text{in } H^1(\Omega)\cap L^2_0(\Omega),\\ u(t)\to u^\infty \quad \text{in } L^2(\mathcal{C}), \end{cases}$$

when $t\to\infty$.

THEOREM 3.4. *Let p^∞, u^∞ be a solution of the problem (3.3)-(3.4). Under assumptions*

$$|g_\alpha(t) - g_\alpha^\infty|_{H^1(S_\alpha)} \le C\exp(-\lambda_1 t), \quad \text{for } \alpha = 0, L, \tag{3.5}$$

$$|\omega(t) - \omega^\infty| \le C\exp(-\lambda_1 t), \tag{3.6}$$

for some $\lambda_1 > 0$, we have

$$|p(t) - p^\infty|_{H^1(\Omega)\cap L_0^2(\Omega)} \le C\exp(-\lambda t),$$

$$|\int_0^h u(t)d\rho - \int_0^h u^\infty d\rho|_{L^2(\Omega)} \le C\exp(-\lambda t),$$

for $t \ge t_0$ and $\lambda < \min\{\lambda_1, \frac{\pi^2}{\beta_2^2}\}$.

Therefore, when time tends to infinity the nonlocal Reynolds equation (3.1) tends to the ordinary Reynolds equation that is valid for stabilized flow. That stabilization occurs in a very short time when data satisfy (3.5)–(3.6).

4. Asymptotic expansion

Due to the geometry of the domain, we work in the cylindrical coordinate system. We read the Stokes equations in cylindrical coordinate system and try to find an ansatz that fits the system and the boundary conditions on $r = R, R + \varepsilon h$. We seek expansions for u^ε and p^ε in the form

$$u^\varepsilon \sim u^0(\rho, \varphi, z, t) + \varepsilon u^1(\rho, \varphi, z, t) + ..., \tag{4.1}$$

$$p^\varepsilon \sim p^0(\rho, \varphi, z, t) + \varepsilon p^1(\rho, \varphi, z, t) + ..., \tag{4.2}$$

where $\rho = \frac{r-R}{\varepsilon}$. Substituting these expansions into the Stokes equations and collecting equal powers of ε leads to the lowest terms in the form

$$\frac{1}{\varepsilon}: \quad \frac{\partial p^0}{\partial \rho} = 0,$$

$$\varepsilon^0: \quad \begin{cases} \dfrac{\partial}{\partial t}u_r^0 - \mu\dfrac{\partial^2 u_r^0}{\partial \rho^2} + \dfrac{\partial p^1}{\partial \rho} = 0, \\ \dfrac{\partial}{\partial t}u_\varphi^0 - \mu\dfrac{\partial^2 u_\varphi^0}{\partial \rho^2} + \dfrac{1}{R}\dfrac{\partial p^0}{\partial \varphi} = 0, \\ \dfrac{\partial}{\partial t}u_z^0 - \mu\dfrac{\partial^2 u_z^0}{\partial \rho^2} + \dfrac{\partial p^0}{\partial z} = 0. \end{cases}$$

From the incompressibility equation we get

$$\frac{1}{\varepsilon}: \quad \frac{\partial u_r^0}{\partial \rho} = 0,$$

$$\varepsilon^0: \quad \frac{\partial u_\varphi^0}{\partial \varphi} + R\frac{\partial u_z^0}{\partial z} + u_r^0 = 0.$$

First we conclude that $u_r^0 = 0$, $p^0 = p^0(\varphi, z)$ and $p^1 = p^1(\varphi, z)$. We compute only the first term in the asymptotic expansion and we further denote that term by (u, p) instead of (u^0, p^0). So, the first term (u, p) in the asymptotic expansion is a solution of the following linear problem

$$\begin{cases} \dfrac{\partial u_\varphi}{\partial t} - \mu \dfrac{\partial^2 u_\varphi}{\partial \rho^2} + \dfrac{1}{R}\dfrac{\partial p}{\partial \varphi} = 0, \\ \dfrac{\partial u_z}{\partial t} - \mu \dfrac{\partial^2 u_z}{\partial \rho^2} + \dfrac{\partial p}{\partial z} = 0, \\ \dfrac{\partial u_\varphi}{\partial \varphi} + R\dfrac{\partial u_z}{\partial z} = 0, \\ u_\varphi = 0, \ \text{for } \rho = 0, \quad u_\varphi = \omega(t), \ \text{for } \rho = h, \\ u_z = 0, \ \text{for } \rho = 0, \ h, \\ u(\cdot, 0) = w^0 \end{cases} \tag{4.3}$$

To solve this problem we use the Laplace transform. Here $\hat{u}$ denotes the Laplace transform of function u in variable t.

Extending functions u and p by zero for $t < 0$ and using the Laplace transform in variable t we get the system

$$\begin{cases} \tau \hat{u}_\varphi - w_\varphi^0 - \mu \dfrac{\partial^2 \hat{u}_\varphi}{\partial \rho^2} + \dfrac{1}{R}\dfrac{\partial \hat{p}}{\partial \varphi} = 0, \\ \tau \hat{u}_z - w_z^0 - \mu \dfrac{\partial^2 \hat{u}_z}{\partial \rho^2} + \dfrac{\partial \hat{p}}{\partial z} = 0, \\ \dfrac{\partial \hat{u}_\varphi}{\partial \varphi} + R\dfrac{\partial \hat{u}_z}{\partial z} = 0, \\ \hat{u}_\varphi = 0, \ \text{for } \rho = 0, \quad \hat{u}_\varphi = \hat{\omega}(\tau), \ \text{for } \rho = h, \\ \hat{u}_z = 0, \ \text{for } \rho = 0, \ h. \end{cases} \tag{4.4}$$

We can compute $\hat{u}_\varphi$ and $\hat{u}_z$ as

$$\begin{aligned} \hat{u}_\varphi &= \hat{G}(\rho, \varphi, z, \tau)\left[-\frac{1}{R}\frac{\partial \hat{p}}{\partial \varphi} + w_\varphi^0\right] + \hat{g}(\rho, \varphi, z, \tau)\hat{\omega}, \\ \hat{u}_z &= \hat{G}(\rho, \varphi, z, \tau)\left[-\frac{\partial \hat{p}}{\partial z} + w_z^0\right], \end{aligned}$$

where $\hat{G}$ and $\hat{g}$ are complex functions of three real variables (ρ, φ, z) and a complex variable τ:

$$\begin{aligned} \hat{G}(\rho, \varphi, z, \tau) &= \frac{1 + e^{h\sqrt{\tau/\mu}} - e^{(h-\rho)\sqrt{\tau/\mu}} - e^{\rho\sqrt{\tau/\mu}}}{(1 + e^{h\sqrt{\tau/\mu}})\tau/\mu}, \\ \hat{g}(\rho, \varphi, z, \tau) &= \frac{e^{(h-\rho)\sqrt{\tau/\mu}}(-1 + e^{2\rho\sqrt{\tau/\mu}})}{-1 + e^{2h\sqrt{\tau/\mu}}}. \end{aligned} \tag{4.5}$$

We define the functions $\hat{\xi}$ and $\hat{\eta}$ as follows:

$$\hat{\xi}(\varphi, z, \tau) = \int_0^{h(\varphi,z)} \hat{G}(\rho, \varphi, z, \tau) d\rho = 2\frac{1 - e^{h\sqrt{\tau/\mu}}}{(1 + e^{h\sqrt{\tau/\mu}})(\tau/\mu)^{3/2}} + \frac{h}{\tau/\mu}$$

and

$$\hat{\eta}(\varphi, z, \tau) = \frac{2e^{h\sqrt{\tau/\mu}}}{(1 + e^{h\sqrt{\tau/\mu}})^2}\frac{\partial h}{\partial \varphi}.$$

Let us note that $\hat{\xi}$ and $\hat{\eta}$ are analytic functions in the half plane $Re\,\tau > -\frac{\pi^2}{\beta_2^2}\mu$.

Integrating the equation

$$\frac{\partial \hat{u}_\varphi}{\partial \varphi} + R\frac{\partial \hat{u}_z}{\partial z} = 0$$

with respect to ρ over $]0, h(\varphi, z)[$, we get the equation

$$\frac{1}{R}\frac{\partial}{\partial \varphi}(-\hat{\xi}(\varphi, z, \tau)\frac{\partial \hat{p}}{\partial \varphi} + \hat{\xi} w_\varphi^0) + R\frac{\partial}{\partial z}(-\hat{\xi}(\varphi, z, \tau)\frac{\partial \hat{p}}{\partial z} + \hat{\xi} w_z^0) = -\hat{\eta}(\varphi, z, \tau)\hat{\omega}(\tau)$$

in $\Omega =]0, 2\pi[\, \times\,]0, L[$.

Using the inverse Laplace transform we come back to the domain $t \geq 0$ and find function u in the form

$$u_\varphi = G * [-\frac{1}{R}\frac{\partial p}{\partial \varphi}] + Gw_\varphi^0 + g * \omega,$$
$$u_z = G * \frac{\partial p}{\partial z} + Gw_z^0,$$

where $G = G(\rho, \varphi, z, t)$ is the inverse Laplace transform of $\hat{G}$ and $g = g(\rho, \varphi, z, t)$ is the inverse Laplace transform of $\hat{g}$. The pressure p satisfies Reynolds equation

$$\frac{1}{R}\frac{\partial}{\partial \varphi}(-\xi * \frac{\partial p}{\partial \varphi} + \xi w_\varphi^0) + R\frac{\partial}{\partial z}(-\xi * \frac{\partial p}{\partial z} + \xi w_z^0) = -\eta * \omega \quad \text{(a.e.) in } \Omega \times]0, T[$$

where $\xi = \xi(\varphi, z, t)$ is the inverse Laplace transform of $\hat{\xi}$ and $\eta = \eta(\varphi, z, t)$ is the inverse Laplace transform of $\hat{\eta}$. One boundary condition for p is 2π-periodicity with respect to φ. The other boundary condition follows from

$$\vec{e}_z \cdot \int_0^{h(\varphi,\alpha)} u(\rho, \varphi, \alpha, t) d\rho = \int_0^{h(\varphi,\alpha)} \vec{e}_z \cdot g_\alpha(\rho, \varphi, t) d\rho, \quad \text{for } \alpha = 0, L,$$

and should be of the form

$$R\left[-\xi * \frac{\partial p}{\partial z} + \xi w_z^0\right]_{z=\alpha} = \int_0^{h(\varphi,\alpha)} \vec{e}_z \cdot g_\alpha(\rho, \varphi, t) d\rho, \quad \text{for } \alpha = 0, L.$$

5. Solvability of the Reynolds equation

Now, we study the solvability of the Reynolds-type equation (3.1) which determines the effective flow. Function $\hat{G}$ given by (4.5) satisfies the equations

$$\begin{cases} \tau\hat{G} - \mu\dfrac{\partial^2\hat{G}}{\partial\rho^2} = 1, \\ \hat{G}(0,\varphi,z,\tau) = 0, \\ \hat{G}(h,\varphi,z,\tau) = 0. \end{cases}$$

From these equations we get that

$$\begin{aligned} \int_0^h \overline{\hat{G}} d\rho &= \tau\int_0^h |\hat{G}|^2 d\rho + \mu\int_0^h |\frac{\partial\hat{G}}{\partial\rho}|^2 \\ &\geq \mu\int_0^h |\frac{\partial\hat{G}}{\partial\rho}|^2 \end{aligned}$$

for $Re\tau > 0$, as well as the estimate

$$1 \leq C(|\tau|+1)\sqrt{\int_0^h |\frac{\partial\hat{G}}{\partial\rho}|^2}.$$

As $Re\,\hat{\xi} = Re\int_0^h \overline{\hat{G}} d\rho$, for $Re\tau > 0$ we have the estimate

$$Re\,\hat{\xi} \geq \frac{C}{(|\tau|+1)^2},$$

where $C > 0$ is a constant. Therefore, for every $\tau \in \mathbf{C}$, $Re\,\tau > 0$, the bilinear form

$$a(u,v) = \int_\Omega \hat{\xi}\nabla_{\varphi,z}u\nabla_{\varphi,z}v d\varphi dz,$$

is continuous and elliptic on $[H^1(\Omega)/R] \times [H^1(\Omega)/R]$. Note that $\int_\Omega \hat{\eta} = 0$. For every $\tau \in \mathbf{C}$, $Re\,\tau > 0$, the compatibility condition

$$\int_0^{2\pi}\int_0^{h(\varphi,0)} \vec{e}_z \cdot \hat{g}_0(\rho,\varphi,\tau) d\rho d\varphi = \int_0^{2\pi}\int_0^{h(\varphi,L)} \vec{e}_z \cdot \hat{g}_L(\rho,\varphi,\tau) d\rho d\varphi$$

is satisfied. Functions $\hat{g}_0(.,\tau)$ and $\hat{g}_L(.,\tau)$ are in $H^{1/2}(S_1)$. Thus the problem

$$\begin{cases} \dfrac{1}{R}\dfrac{\partial}{\partial\varphi}(-\hat{\xi}\dfrac{\partial\hat{p}}{\partial\varphi} + \hat{\xi}w_\varphi^0) + R\dfrac{\partial}{\partial z}(-\hat{\xi}\dfrac{\partial\hat{p}}{\partial z} + \hat{\xi}w_z^0) = -\hat{\eta}\hat{\omega} \quad \text{in } \Omega, \\ \left[-\hat{\xi}\dfrac{\partial\hat{p}}{\partial z} + \hat{\xi}w_z^0\right]_{z=0} = \dfrac{1}{R}\displaystyle\int_0^{h(\varphi,0)} \vec{e}_z \cdot \hat{g}_0 d\rho, \\ \left[-\hat{\xi}\dfrac{\partial\hat{p}}{\partial z} + \hat{\xi}w_z^0\right]_{z=L} = \dfrac{1}{R}\displaystyle\int_0^{h(\varphi,L)} \vec{e}_z \cdot \hat{g}_L d\rho, \\ \hat{p} \text{ is } 2\pi\text{-periodic in variable } \varphi, \end{cases} \tag{5.1}$$

has a unique solution $\hat{p}(\tau) \in H^2(\Omega) \cap L^2_0(\Omega)$ for all $\tau \in \mathbf{C}$, $Re\,\tau > 0$. Applying the isomorphism theorem 2.1. in [7], pages 99-100 (for details see [3]), we get the existence of the unique solution $p \in \mathcal{D}'_+(t, H^1(\Omega) \cap L^2_0(\Omega))$ of the problem (3.1). For every $\tau \in \mathbf{C}$, $Re\,\tau > 0$, $\hat{p}(\tau) \in H^2(\Omega) \cap L^2_0(\Omega)$ thus we can conclude that the problem (3.1) has the unique solution $p \in H^2(\Omega) \cap L^2_0(\Omega)$ for (a.e.) $t \in]0, T[$.

Let us now study the regularity in time. We introduce the function

$$P(\varphi, z, t) = \int_0^t p(\varphi, z, \delta) d\delta.$$

Using the Laplace transform we get a problem for the Laplace transform $\hat{P}(\varphi, z, \tau)$ of the function P (instead of $\hat{P}(\varphi, z, \tau)$ we use abbreviated form $\hat{P}(\tau)$) :

$$(5.2) \qquad \begin{cases} \dfrac{1}{R}\dfrac{\partial}{\partial\varphi}(-\tau\hat{\xi}\dfrac{\partial\hat{P}}{\partial\varphi} + \hat{\xi}w^0_\varphi) + R\dfrac{\partial}{\partial z}(-\tau\hat{\xi}\dfrac{\partial\hat{P}}{\partial z} + \hat{\xi}w^0_z) = -\hat{\eta}\hat{\omega} \quad \text{in } \Omega, \\ \left[-\tau\hat{\xi}\dfrac{\partial\hat{P}}{\partial z} + \hat{\xi}w^0_z\right]_{z=0} = \dfrac{1}{R}\displaystyle\int_0^{h(\varphi,0)} \vec{e}_z \cdot \hat{g}_0 d\rho, \\ \left[-\tau\hat{\xi}\dfrac{\partial\hat{P}}{\partial z} + \hat{\xi}w^0_z\right]_{z=L} = \dfrac{1}{R}\displaystyle\int_0^{h(\varphi,L)} \vec{e}_z \cdot \hat{g}_L d\rho, \\ \hat{P} \text{ is } 2\pi\text{-periodic in } \varphi. \end{cases}$$

This problem has a unique solution $\hat{P}(\tau) \in H^2(\Omega) \cap L^2_0(\Omega)$ for every $\tau \in \mathbf{C}$, $Re\,\tau > 0$.

Note that $\tau\hat{\xi}$ is the analytic function in variable $\tau \in \mathbf{C}$ for $Re\,\tau > -\frac{\pi^2}{\beta_2^2}\mu$. Let $\sqrt{\frac{\tau}{\mu}} = a + ib$, where $Re\,\frac{\tau}{\mu} = a^2 - b^2 > 0$. As

$$\begin{aligned} \tau\hat{\xi} &= -2\,\mathrm{th}\,(\sqrt{\tau}\frac{h}{2})\frac{1}{\tau^{1/2}} + h \\ &= -2\left[\frac{\mathrm{sh}\,ah}{\mathrm{ch}\,ah + \cos bh} + i\frac{\sin bh}{\mathrm{ch}\,ah + \cos bh}\right]\frac{a - ib}{a^2 + b^2} + h, \end{aligned}$$

the real part of $\tau\hat{\xi}$ is equal

$$Re\,\tau\hat{\xi} = -2\frac{a\,\mathrm{sh}\,ah + b\sin bh}{(a^2 + b^2)(\mathrm{ch}\,ah + \cos bh)} + h.$$

We have

$$\frac{\partial}{\partial h}(Re\,\tau\hat{\xi}) = \frac{\mathrm{sh}^2 ah - \sin^2 bh}{(\mathrm{ch}\,ah + \cos bh)^2} > 0$$

and $Re\,\tau\hat{\xi} = 0$ for $h = 0$, therefore $Re\,\tau\hat{\xi} > 0$ for $Re\,\frac{\tau}{\mu} = a^2 - b^2 > 0$. Furthermore,

$$\begin{aligned} Re\,\tau\hat{\xi} \geq Re\,\tau\hat{\xi}(\beta_1) = -2\frac{a\,\mathrm{sh}\,a\beta_1 + b\sin b\beta_1}{(a^2+b^2)(\mathrm{ch}\,a\beta_1 + \cos b\beta_1)} + \beta_1 \\ = c(\beta_1, \tau) > 0. \end{aligned} \tag{5.3}$$

We notice that for $Re\,\tau > 0$, if $|\tau|$ is large enough, instead of a constant $c(\beta_1, \tau)$ in (5.3) we can take a constant C_1 which does not depend on τ. For $Re\,\tau > 0$, $|\tau|$ close to 0, a constant $c(\beta_1, \tau)$ depends on τ and is of the form $c(\beta_1, \tau) = C_2|\tau|$, $C_2 > 0$.

We need to know the behavior of $\hat{P}(\tau)$ when $\tau \to \infty$. We have $\hat{\xi} = O(\frac{1}{|\tau|})$ and $\hat{\eta} = O(\frac{1}{|\tau|})$ for a large $|\tau|$. It is easy to see that

$$\begin{aligned} &\lim_{\tau\to\infty} \tau\hat{\xi} = \xi(0) = h \quad \text{in } \ L^\infty(\Omega), \\ &\lim_{\tau\to\infty} \tau\hat{\eta} = \eta(0) = 0 \quad \text{in } \ L^\infty(\Omega), \\ &\lim_{\tau\to\infty} \tau\hat{\omega} = \omega(0) = 0, \\ &\lim_{\tau\to\infty} \tau\hat{g}_\alpha = g_\alpha(0) \quad \text{in } \ H^{1/2}(S_\alpha), \quad \text{for } \ \alpha = 0, L. \end{aligned}$$

Let $P(0)$ be a solution of the problem

$$\begin{cases} \mathrm{div}_{\varphi,z}(-h\nabla_{\varphi,z}P(0) + w^0) = 0 \\ \left[-h\dfrac{\partial P(0)}{\partial z} + w_z^0\right]_{z=0} = \dfrac{1}{R}\displaystyle\int_0^{h(\varphi,0)} \vec{e}_z \cdot g_0(\rho,\varphi,0)d\rho, \\ \left[-h\dfrac{\partial P(0)}{\partial z} + w_z^0\right]_{z=L} = \dfrac{1}{R}\displaystyle\int_0^{h(\varphi,L)} \vec{e}_z \cdot g_L(\rho,\varphi,0)d\rho, \\ P(0) \text{ is } 2\pi\text{-periodic in } \varphi. \end{cases} \tag{5.4}$$

We have $P(0) \in H^2(\Omega) \cap L^2_0(\Omega)$. Note that $P(0) = 0$ if and only if

$$\int_0^{h(\varphi,\alpha)} \vec{e}_z \cdot g_\alpha(\rho,\varphi,0)d\rho = w_z^0(\varphi,\alpha), \qquad \text{for } \ \alpha = 0, L.$$

For the difference $\nabla_{\varphi,z}(\hat{P}(\tau) - \frac{P(0)}{\tau})$ we have the following estimate:

Lemma 5.1.

$$\left|\nabla_{\varphi,z}(\hat{P}(\tau) - \frac{P(0)}{\tau})\right|_{L^2(\Omega)} \leq \frac{C}{|\tau|^2}$$

Proof. From (5.2) and (5.4) we get a problem for the difference $\hat{P}(\tau) - \frac{P(0)}{\tau}$:

(5.5)

$$\begin{cases} \operatorname{div}_{\varphi,z}\left[\hat{\xi}\nabla_{\varphi,z}(\hat{P}(\tau) - \dfrac{P(0)}{\tau})\right] \\ \qquad = \operatorname{div}_{\varphi,z}\left[\dfrac{1}{\tau}\hat{\eta}\hat{\omega} + \dfrac{1}{\tau}(\tau\hat{\xi} - h)\ \dfrac{w^0}{\tau} + \dfrac{1}{\tau}(h - \tau\hat{\xi})\ \dfrac{\nabla_{\varphi,z}P(0)}{\tau}\right] \\ \left[\hat{\xi}\nabla_{\varphi,z}(\hat{P}(\tau) - \dfrac{P(0)}{\tau}) - \dfrac{1}{\tau}\hat{\eta}\hat{\omega} - \dfrac{1}{\tau}(\tau\hat{\xi} - h)\ \dfrac{w^0}{\tau}\right. \\ \qquad \left. -\dfrac{1}{\tau}(h - \tau\hat{\xi})\ \dfrac{\nabla_{\varphi,z}P(0)}{\tau})\right] \cdot \nu = \dfrac{1}{\tau}(\hat{B}_\alpha - \dfrac{B_\alpha(0)}{\tau}), \quad \text{for} \quad \alpha = 0, L, \\ \hat{P}(\tau) - \dfrac{P(0)}{\tau} \text{ is } 2\pi\text{-periodic in } \varphi, \end{cases}$$

where $\hat{B}_\alpha = \frac{1}{R}\int_0^{h(\varphi,\alpha)} \vec{e}_z \cdot \hat{g}_\alpha d\rho$ for $\alpha = 0, L$. Multiplying the problem (5.5) with $\hat{P}(\tau) - \frac{P(0)}{\tau}$, after integration over Ω, for $|\tau| < \infty$ we get

$$\begin{aligned} &|\nabla_{\varphi,z}(\hat{P}(\tau) - \frac{P(0)}{\tau})|_{L^2(\Omega)} \leq \\ &\quad \leq \frac{C}{|\tau|^2}\Big(|\tau\hat{\eta}\hat{\omega}|_{L^2(\Omega)} + |\tau\hat{\xi} - h|_{L^2(\Omega)} + |h - \tau\hat{\xi}|_{L^2(\Omega)}|\nabla_{\varphi,z}P(0)|_{L^2(\Omega)} \\ &\quad\quad + |g_0(0) - \tau\hat{g}_0|_{H^{1/2}(S_0)} + |g_l(0) - \tau\hat{g}_L|_{H^{1/2}(S_L)}\Big) \\ &\quad \leq \frac{C}{|\tau|^2}, \end{aligned}$$

and in that case we have the result. For a large $|\tau|$ we have the estimate

$$\begin{aligned} &|\nabla_{\varphi,z}(\hat{P}(\tau) - \frac{P(0)}{\tau})|_{L^2(\Omega)} \leq \\ &\quad \leq \frac{C}{|\tau|}\Big(|\tau\hat{\eta}\hat{\omega}|_{L^2(\Omega)} + |\tau\hat{\xi} - h|_{L^2(\Omega)} + |h - \tau\hat{\xi}|_{L^2(\Omega)}|\nabla_{\varphi,z}P(0)|_{L^2(\Omega)} \\ &\quad\quad + |g_0(0) - \tau\hat{g}_0|_{H^{1/2}(S_0)} + |g_l(0) - \tau\hat{g}_l|_{H^{1/2}(S_L)}\Big). \end{aligned} \tag{5.6}$$

Note that $|\tau\hat{\xi} - h|_{L^2(\Omega)} \leq \frac{C}{|\tau|}$ for a large $|\tau|$. Analogous estimates are verified for other terms on the right side of the inequality (5.6). Therefore the claim of Lemma 5.1 follows for every τ, $Re\,\tau > 0$. ∎

For every τ, $Re\,\tau > 0$, $\tau\hat{P}(\tau) - P(0)$ is the element of the vector space $L^2_0(\Omega)$, therefore

$$|\tau\hat{P}(\tau) - P(0)|_{H^1(\Omega)} \leq C|\nabla_{\varphi,z}(\tau\hat{P}(\tau) - P(0))|_{L^2(\Omega)}.$$

From this inequality and Lemma 5.1 we have

$$\lim_{\tau\to\infty} \tau\hat{P}(\tau) = P(0) \quad \text{in} \ H^1(\Omega).$$

$\hat{P}(\tau) - \frac{P(0)}{\tau}$ is the Laplace transform of the function $P(t) - P(0)$. According to Lemma 5.1 $|\tau \hat{P}(\tau) - P(0)|_{H^1(\Omega)} \leq \frac{C}{|\tau|^2}$ and the representation formula for the inverse Laplace transform implies that P is a continuous function in variable t. Furthermore, $P \in C([0,T]; H^2(\Omega) \cap L_0^2(\Omega))$ and consequently $p = \frac{\partial P}{\partial t} \in H^{-1}(0,T; H^2(\Omega) \cap L_0^2(\Omega))$.

6. A priori estimates

Lemma 6.1. *Let $\phi^\varepsilon(t) \in H^1(\mathcal{C}_\varepsilon)$, for (a.e.) $t \geq 0$, be a solution of the problem*

$$\begin{cases} div\ \phi^\varepsilon(t) = 0, \\ \phi^\varepsilon(t) = \omega(t)\vec{e}_\varphi, \quad for \;\; r = R, \\ \phi^\varepsilon(t) = 0, \quad for \;\; r = R + \varepsilon h, \\ \phi^\varepsilon(t) = g_0(\dfrac{r-R}{\varepsilon}, \varphi, t), \quad for \;\; z = 0, \\ \phi^\varepsilon(t) = g_L(\dfrac{r-R}{\varepsilon}, \varphi, t), \quad for \;\; z = L. \end{cases}$$

Then we have

$$|\phi^\varepsilon(t)|_{H^1(\mathcal{C}_\varepsilon)} \leq \frac{C}{\sqrt{\varepsilon}}(|g_0(t)|_{H^{1/2}(\mathcal{S}_0)} + |g_0(t)|_{H^{1/2}(\mathcal{S}_L)} + |\omega(t)|)$$

and

$$|\frac{\partial}{\partial t}\phi^\varepsilon(t)|_{H^1(\mathcal{C}_\varepsilon)} \leq \frac{C}{\sqrt{\varepsilon}}(|\frac{\partial}{\partial t}g_0(t)|_{H^{1/2}(\mathcal{S}_0)} + |\frac{\partial}{\partial t}g_0(t)|_{H^{1/2}(\mathcal{S}_L)} + |\frac{\partial}{\partial t}\omega(t)|).$$

Proof. As compatibility conditions are satisfied, conclusion follows from Lemma 4 in [2] and continuous dependence of solution on given data.

Furthermore, $\phi^\varepsilon \in W^{1,\infty}_{loc}(0,\infty; H^1(\mathcal{C}_\varepsilon))$. ∎

Proposition 6.2. *Let u^ε be a weak solution for (2.1). Then we have*

$$\begin{aligned} &|u^\varepsilon|_{L^2(Q_{\varepsilon T})} \leq C\sqrt{\varepsilon}, \\ &|\nabla u^\varepsilon|_{L^2(Q_{\varepsilon T})} \leq \frac{C}{\sqrt{\varepsilon}}, \qquad (6.1) \\ &|u^\varepsilon|_{L^\infty(0,T;L^2(\mathcal{C}_\varepsilon))} \leq C\sqrt{\varepsilon}, \end{aligned}$$

where $Q_{\varepsilon T} = \mathcal{C}_\varepsilon \times (0,T)$.

Proof. Let ϕ^ε be a function from Lemma 6.1 and $v^\varepsilon = u^\varepsilon - \phi^\varepsilon$. Then $(v^\varepsilon, p^\varepsilon)$ is a solution of the problem

$$\begin{cases} \dfrac{\partial v^\varepsilon}{\partial t} - \mu\varepsilon^2 \Delta v^\varepsilon + \nabla p^\varepsilon = -\dfrac{\partial \phi^\varepsilon}{\partial t} + \mu\varepsilon^2\Delta\phi^\varepsilon \;\; \text{in } Q_{\varepsilon T}, \\ \text{div } v^\varepsilon = 0 \;\; \text{in } Q_{\varepsilon T}, \\ v^\varepsilon(x,0) = w^0(x) - \phi^\varepsilon(x,0) \;\; \text{in } \mathcal{C}_\varepsilon, \\ v^\varepsilon = 0 \;\; \text{on } \partial\mathcal{C}_\varepsilon \times (0,T). \end{cases} \qquad (6.2)$$

The energy inequality that corresponds to this problem (see Temam [11], page 291) is

$$\int_{\mathcal{C}_\varepsilon} |v^\varepsilon(t)|^2 + 2\mu\varepsilon^2 \int_0^t \int_{\mathcal{C}_\varepsilon} |\nabla v^\varepsilon|^2$$
$$\leq \int_{\mathcal{C}_\varepsilon} |v^\varepsilon(0)|^2 + 2\int_0^t \int_{\mathcal{C}_\varepsilon} (-\frac{\partial \phi^\varepsilon}{\partial t} + \mu\varepsilon^2 |\Delta\phi^\varepsilon|^2) v^\varepsilon,$$

for (a.e.) $t \in [0,T]$. The inequalities

$$2\int_0^t \int_{\mathcal{C}_\varepsilon} \frac{\partial \phi^\varepsilon}{\partial t} v^\varepsilon \leq 2\int_0^t \int_{\mathcal{C}_\varepsilon} |\frac{\partial \phi^\varepsilon}{\partial t}|^2 + \frac{1}{2}\int_0^t \int_{\mathcal{C}_\varepsilon} |v^\varepsilon|^2,$$
$$2\mu \int_0^t \int_{\mathcal{C}_\varepsilon} \varepsilon^2 \Delta\phi^\varepsilon v^\varepsilon \leq 2\varepsilon^2\mu \int_0^t \int_{\mathcal{C}_\varepsilon} |\nabla\phi^\varepsilon|^2 + \frac{1}{2}\mu\varepsilon^2 \int_0^t \int_{\mathcal{C}_\varepsilon} |\nabla v^\varepsilon|^2,$$

and the energy inequality give

$$\int_{\mathcal{C}_\varepsilon} |v^\varepsilon(t)|^2 + \mu\varepsilon^2 \int_0^t \int_{\mathcal{C}_\varepsilon} |\nabla v^\varepsilon|^2$$
$$\leq \int_{\mathcal{C}_\varepsilon} |v^\varepsilon(0)|^2 + 2\int_0^t \int_{\mathcal{C}_\varepsilon} |\frac{\partial \phi^\varepsilon}{\partial t}|^2 + 2\mu\varepsilon^2 \int_0^t \int_{\mathcal{C}_\varepsilon} |\nabla\phi^\varepsilon|^2.$$

Backward substitution, $u^\varepsilon = v^\varepsilon + \phi^\varepsilon$, implies the following inequality for u^ε

$$\int_{\mathcal{C}_\varepsilon} |u^\varepsilon(t)|^2 + \mu\varepsilon^2 \int_0^t \int_{\mathcal{C}_\varepsilon} |\nabla u^\varepsilon|^2 \leq \int_{\mathcal{C}_\varepsilon} |u^\varepsilon(0)|^2 + \int_{\mathcal{C}_\varepsilon} |\phi^\varepsilon(0)|^2 + \int_{\mathcal{C}_\varepsilon} |\phi^\varepsilon(t)|^2$$
$$+ 2\int_0^t \int_{\mathcal{C}_\varepsilon} |\frac{\partial \phi^\varepsilon}{\partial t}|^2 + 3\mu\varepsilon^2 \int_0^t \int_{\mathcal{C}_\varepsilon} |\nabla\phi^\varepsilon|^2.$$

We use estimates for ϕ^ε given in Lemma 6.1 and conclude that

$$\int_{\mathcal{C}_\varepsilon} |u^\varepsilon(t)|^2 + \mu\varepsilon^2 \int_0^t \int_{\mathcal{C}_\varepsilon} |\nabla u^\varepsilon|^2 \leq C\varepsilon$$

for (a.e.) $t \in [0,T]$. The conclusion follows. $\square$

In the next step we introduce the pressure. Following the idea from Temam [11], page 307, we introduce the function

$$U^\varepsilon(t) = \int_0^t u^\varepsilon(s)ds.$$

Let u^ε be a weak solution of the problem (2.1), then $U^\varepsilon \in C([0,T], H^1(\mathcal{C}_\varepsilon))$, div $U^\varepsilon = 0$. According to De Rham's lemma (see Temam [11]), there exists $P^\varepsilon \in C([0,T], L_0^2(\mathcal{C}_\varepsilon))$, $\nabla P^\varepsilon \in C([0,T], H^{-1}(\mathcal{C}_\varepsilon))$, such that

$$\nabla P^\varepsilon(t) + u^\varepsilon(t) - w^0 - \mu\varepsilon^2 \Delta U^\varepsilon(t) = 0 \quad \text{in } H^{-1}(\mathcal{C}_\varepsilon), \tag{6.3}$$

for every $t \in [0,T]$.

LEMMA 6.3. *P^ε satisfies the following estimate*

$$|P^\varepsilon|_{C([0,T],L_0^2(\mathcal{C}_\varepsilon))} \leq C\varepsilon^{1/2}. \tag{6.4}$$

PROOF. We suppose that $\int_{\mathcal{C}_\varepsilon} P^\varepsilon(t) = 0$. Let z_ε be a solution of the problem

$$\begin{cases} \operatorname{div} z_\varepsilon = P^\varepsilon(t) \text{ in } \mathcal{C}_\varepsilon, \\ z_\varepsilon = 0 \text{ on } \partial\mathcal{C}_\varepsilon. \end{cases}$$

From Lemma 6.1 follows

$$|z_\varepsilon|_{H^1(\mathcal{C}_\varepsilon)} \leq \frac{C}{\varepsilon}|P^\varepsilon(t)|_{L^2(\mathcal{C}_\varepsilon)}.$$

We take z_ε as a test function in (6.3) and get

$$\begin{aligned} |P^\varepsilon(t)|^2_{L^2(\mathcal{C}_\varepsilon)} &= \mu\varepsilon^2 \int_{\mathcal{C}_\varepsilon} \nabla U^\varepsilon(t)\nabla z_\varepsilon - \int_{\mathcal{C}_\varepsilon} u^\varepsilon(t) z_\varepsilon + \int_{\mathcal{C}_\varepsilon} w^0 z_\varepsilon \\ &\leq C(\mu\varepsilon^2|\nabla u^\varepsilon|_{L^2(Q_{\varepsilon T})} + \varepsilon|u^\varepsilon(t)|_{L^2(\mathcal{C}_\varepsilon)} + \varepsilon|w^0|_{L^2(\mathcal{C}_\varepsilon)})|\nabla z_\varepsilon|_{L^2(\mathcal{C}_\varepsilon)} \\ &\leq C\varepsilon^{1/2}|P^\varepsilon(t)|_{L^2(\mathcal{C}_\varepsilon)}. \end{aligned}$$

□

To prove a convergence it is convenient for sequences of functions $u^\varepsilon, U^\varepsilon, P^\varepsilon$ defined on $\mathcal{C}_\varepsilon$ to introduce rescaling functions

$$\begin{aligned} u(\varepsilon)(\rho,\varphi,z,t) &= u^\varepsilon(r,\varphi,z,t), \\ U(\varepsilon)(\rho,\varphi,z,t) &= U^\varepsilon(r,\varphi,z,t), \\ P(\varepsilon)(\rho,\varphi,z,t) &= P^\varepsilon(r,\varphi,z,t), \end{aligned}$$

where $\rho = \frac{r-R}{\varepsilon}$. Functions $u(\varepsilon)$, $U(\varepsilon)$ and $P(\varepsilon)$ are defined on a domain $Q_T = \mathcal{C} \times (0,T)$ which does not depend on ε, where $\mathcal{C} = \{(\rho,\varphi,z) \in \mathbf{R}^3; \varphi \in]0,2\pi[, z \in]0,L[, 0 < \rho < h(\varphi,z)\}$. The following estimates are obtained from Proposition 6.2 and preceding lemma by simply changing the variable.

LEMMA 6.4. *For $u(\varepsilon)$ we have the following estimates*

$$\begin{aligned} &|u(\varepsilon)|_{L^2(Q_T)} \leq C, \\ &|u(\varepsilon)|_{L^\infty(0,T;L^2(\mathcal{C}))} \leq C, \\ &|\frac{\partial u(\varepsilon)}{\partial\rho}|_{L^2(Q_T)} \leq C, \\ &|\nabla_\varepsilon^{\varphi,z} u(\varepsilon)|_{L^2(Q_T)} \leq C\varepsilon^{-1}, \\ &|\frac{\partial u_r(\varepsilon)}{\partial\rho}|_{L^2(Q_T)} \leq C\varepsilon, \end{aligned}$$

where

$$\nabla_\varepsilon^{\varphi,z} u = \begin{bmatrix} \frac{1}{R+\varepsilon\rho}(\frac{\partial u_r}{\partial \varphi} - u_\varphi) & \frac{\partial u_r}{\partial z} \\ \frac{1}{R+\varepsilon\rho}(\frac{\partial u_\varphi}{\partial \varphi} + u_r) & \frac{\partial u_\varphi}{\partial z} \\ \frac{1}{R+\varepsilon\rho}\frac{\partial u_z}{\partial \varphi} & \frac{\partial u_z}{\partial z} \end{bmatrix}.$$

LEMMA 6.5. *Function $U(\varepsilon)$ satisfies the following estimates*

$$\begin{aligned} &|U(\varepsilon)|_{C([0,T],L^2(\mathcal{C}))} \le C, \\ &|\frac{\partial U(\varepsilon)}{\partial \rho}|_{C([0,T],L^2(\mathcal{C}))} \le C, \\ &|\nabla_\varepsilon^{\varphi,z} U(\varepsilon)|_{C([0,T],L^2(\mathcal{C}))} \le C\varepsilon^{-1}, \\ &|\frac{\partial U_r(\varepsilon)}{\partial \rho}|_{C([0,T],L^2(\mathcal{C}))} \le C\varepsilon. \end{aligned} \tag{6.5}$$

LEMMA 6.6. *Function $P(\varepsilon)$ satisfies estimates*

$$\begin{aligned} &|P(\varepsilon)|_{C([0,T],L_0^2(\mathcal{C}))} \le C \\ &|\nabla_\varepsilon^{\varphi,z} P(\varepsilon)|_{C([0,T],H^{-1}(\mathcal{C}))} \le C, \\ &|\frac{\partial P(\varepsilon)}{\partial \rho}|_{C([0,T],H^{-1}(\mathcal{C}))} \le C\varepsilon. \end{aligned} \tag{6.6}$$

where

$$\nabla_\varepsilon^{\varphi,z} P(\varepsilon) = \begin{bmatrix} \frac{1}{R+\varepsilon\rho}\frac{\partial P(\varepsilon)}{\partial \varphi} \\ \frac{\partial P(\varepsilon)}{\partial z} \end{bmatrix}.$$

7. CONVERGENCE

We introduce functional spaces

$$W_T = \{\phi \in L^2(Q_T)^2 \ : \frac{\partial \phi}{\partial \rho} \in L^2(Q_T)^2\},$$

and

$$Y_T = \{\phi \in C([0,T], L^2(\mathcal{C})^2) \ : \frac{\partial \phi}{\partial \rho} \in C([0,T], L^2(\mathcal{C})^2)\}.$$

In the following, for any function $u = (u_r, u_\varphi, u_z)$ we denote by $\tilde{u} = (u_\varphi, u_z)$.

PROPOSITION 7.1. *There exist subsequences of $u(\varepsilon)$, $U(\varepsilon)$ and $P(\varepsilon)$ (again denoted by $u(\varepsilon)$, $U(\varepsilon)$ and $P(\varepsilon)$) and functions $u \in W_T$, $U \in Y_T$,*

$P \in L^\infty(0,T;L_0^2(\mathcal{C}))$ and $p = \frac{\partial}{\partial t} P \in H^{-1}(0,T;L_0^2(\mathcal{C}))$, such that

$$\begin{aligned}
&\tilde{u}(\varepsilon) \rightharpoonup u \quad \& \quad \frac{\partial \tilde{u}(\varepsilon)}{\partial \rho} \rightharpoonup \frac{\partial u}{\partial \rho} \text{ weakly in } L^2(Q_T),\\
&u_r(\varepsilon) \rightharpoonup 0 \quad \& \quad \frac{\partial u_r(\varepsilon)}{\partial \rho} \rightharpoonup 0 \text{ weakly in } L^2(Q_T),\\
&\int_0^{h(\varphi,z)} \tilde{u}(\varepsilon)(\rho,\varphi,z)d\rho \rightharpoonup \int_0^{h(\varphi,z)} u(\rho,\varphi,z)d\rho \text{ weakly in } L^2((0,T)\times\Omega),\\
&\tilde{U}(\varepsilon) \rightharpoonup U \quad \& \quad \frac{\partial \tilde{U}(\varepsilon)}{\partial \rho} \rightharpoonup \frac{\partial U}{\partial \rho} \text{ weak* in } L^\infty(0,T;L^2(\mathcal{C})),\\
&U_r(\varepsilon) \rightharpoonup 0 \quad \& \quad \frac{\partial U_r(\varepsilon)}{\partial \rho} \rightharpoonup 0 \text{ weak* in } L^\infty(0,T;L^2(\mathcal{C})),\\
&P(\varepsilon) \rightharpoonup P \text{ weak* in } L^\infty(0,T;L_0^2(\Omega)),\\
&\frac{\partial}{\partial t} P(\varepsilon) \rightharpoonup p \text{ weakly in } H^{-1}(0,T;L_0^2(\Omega)),
\end{aligned}$$

where $\Omega = \{(\varphi,z) \in \mathbf{R^2}; \varphi \in]0,2\pi[, z \in]0,L[\}$. Further, $P = P(\varphi,z,t)$ and $p = p(\varphi,z,t)$. For (a.e.) $t \in (0,T)$, function u satisfies the equations

$$\begin{cases}
div_{\varphi,z} \displaystyle\int_0^{h(\varphi,z)} u(\rho,\varphi,z,t)d\rho = 0 \quad in \quad \Omega,\\
\nu \cdot \displaystyle\int_0^{h(\varphi,0)} u(\rho,\varphi,0,t)d\rho = \int_0^{h(\varphi,0)} \vec{e}_z \cdot g_0(\rho,\varphi,t)d\rho,\\
\nu \cdot \displaystyle\int_0^{h(\varphi,L)} u(\rho,\varphi,L,t)d\rho = \int_0^{h(\varphi,L)} \vec{e}_z \cdot g_L(\rho,\varphi,t)d\rho,\\
u(0,\varphi,z,t) = \omega(t)\vec{e}_\varphi, \quad u(h,\varphi,z,t) = 0,\\
u \;\; is \; 2\pi\text{-}periodic \; in \; variable \; \varphi.
\end{cases} \tag{7.1}$$

Function U satisfies the equations

$$\begin{cases}
div_{\varphi,z} \displaystyle\int_0^{h(\varphi,z)} U(\rho,\varphi,z,t)d\rho = 0 \quad in \quad \Omega,\\
\nu \cdot \displaystyle\int_0^{h(\varphi,0)} U(\rho,\varphi,0,t)d\rho = \int_0^{h(\varphi,0)} \vec{e}_z \cdot \int_0^t g_0(\rho,\varphi,s)dsd\rho,\\
\nu \cdot \displaystyle\int_0^{h(\varphi,L)} U(\rho,\varphi,L,t)d\rho = \int_0^{h(\varphi,L)} \vec{e}_z \cdot \int_0^t g_L(\rho,\varphi,s)dsd\rho,\\
U(0,\varphi,z,t) = \displaystyle\int_0^t \omega(t)dt\vec{e}_\varphi, \quad U(h,\varphi,z,t) = 0,\\
U \;\; is \; 2\pi\text{-}periodic \; in \; variable \; \varphi,
\end{cases} \tag{7.2}$$

for every $t \in [0,T]$ *and*

$$U(t) = \int_0^t u(\rho,\varphi,z,s)ds. \tag{7.3}$$

Proof. Weak convergence is a direct consequence, according to the weak compactness, of the estimates given by the previous lemmas. The estimate (6.6) implies that the pressure P is independent of variable ρ. From (6.5) we deduce $\frac{\partial U_r}{\partial \rho} = 0$. To prove that $U_r = 0$, we choose the test function $q = \theta(t,\varphi,z)\rho$ in $\operatorname{div}_\varepsilon U(\varepsilon) = 0$, where $\theta \in \mathcal{D}((0,T)\times(0,L);\mathcal{D}_{per}(0,2\pi))$. Using the Green formula, the boundary conditions on $\partial\mathcal{C}$ imply

$$\int_0^T \int_{\mathcal{C}} \nabla_\varepsilon(\theta(t,\varphi,z)\rho)\cdot U(\varepsilon) = 0.$$

We get

$$\int_0^T \int_{\mathcal{C}} \theta U_r(\varepsilon) = -\varepsilon \int_0^T \int_{\mathcal{C}} (\frac{1}{R+\varepsilon\rho}\frac{\partial\theta}{\partial\varphi}U_\varphi(\varepsilon) + \frac{\partial\theta}{\partial z}U_z(\varepsilon)).$$

When ε tends to zero we get $\int_0^T\int_{\mathcal{C}} \theta U_r = 0$. This implies $U_r = 0$.

It is easy to get (7.3). As $\frac{dU}{dt} = u \in L^2(Q_T)$, U is an element of the space Y_T.

On the other side,

$$\begin{aligned} 0 &= \int_0^T \int_{\mathcal{C}} \theta \operatorname{div}_\varepsilon U(\varepsilon) = \int_0^T\int_\Omega \theta \int_0^{h(\varphi,z)} \operatorname{div}_\varepsilon U(\varepsilon) d\rho\, d\varphi dz dt \\ &= \int_0^T\int_\Omega \theta \int_0^{h(\varphi,z)} [\operatorname{div}_{\varphi,z}\tilde{U}(\varepsilon) + \frac{\partial(\rho U_r(\varepsilon))}{\partial\rho} + \frac{R}{\varepsilon}\frac{\partial U_r(\varepsilon)}{\partial\rho} \\ &\qquad + \varepsilon\rho\frac{\partial U_z(\varepsilon)}{\partial z}] d\rho\, d\varphi dz dt. \end{aligned}$$

Due to the boundary conditions on $U(\varepsilon)$,

$$\begin{aligned} &\int_0^T\int_\Omega \theta \operatorname{div}_{\varphi,z}\int_0^{h(\varphi,z)} \tilde{U}(\varepsilon) d\rho\, d\varphi dz dt = \\ &\qquad = -\varepsilon \int_0^T\int_\Omega \theta\frac{\partial}{\partial z}\int_0^{h(\varphi,z)} \rho U_z(\varepsilon) d\rho\, d\varphi dz dt, \end{aligned}$$

thus we have

$$\int_0^T\int_{\mathcal{C}} \nabla_{\varphi,z}\theta\, \tilde{U}(\varepsilon) d\rho d\varphi dz dt = \varepsilon\int_0^T\int_\Omega \frac{\partial\theta}{\partial z}\int_0^{h(\varphi,z)} \rho U_z(\varepsilon) d\rho\, d\varphi dz dt.$$

In the limit, when ε tends to zero, we get

$$\int_0^T\int_{\mathcal{C}} \nabla_{\varphi,z}\theta\, U d\rho d\varphi dz dt = 0,$$

which implies

$$\int_0^T \int_\Omega \theta \,\mathrm{div}_{\varphi,z} \int_0^{h(\varphi,z)} U d\rho = 0.$$

From this we deduce that $\mathrm{div}_{\varphi,z} \int_0^{h(\varphi,z)} U(\rho,\varphi,z,t)d\rho = 0$ for every $t \in [0,T]$.

$$\mathrm{div}_{\varphi,z} \int_0^{h(\varphi,z)} \tilde{U}(\varepsilon)(\rho,\varphi,z,t)d\rho$$

is a bounded sequence in $C([0,T],L^2(\Omega))$ and

$$\int_0^{h(\varphi,z)} \tilde{U}(\varepsilon)(\rho,\varphi,z,t)d\rho$$

is a bounded sequence in $C([0,T],L^2(\Omega))$, therefore

$$\nu \cdot \int_0^{h(\varphi,z)} \tilde{U}(\varepsilon)(\rho,\varphi,z,t)d\rho, \qquad \text{for } z = 0, L$$

are bounded sequences in $C([0,T],H^{-1/2}(0,2\pi))$. Taking a limit when ε tends to zero we get $\nu \cdot \int_0^{h(\varphi,\alpha)} U(\rho,\varphi,\alpha,t)d\rho = \int_0^{h(\varphi,\alpha)} \vec{e}_z \cdot \int_0^t g_\alpha(\rho,\varphi,s)dsd\rho$ for $\alpha = 0, L$. In the same way, working with the spaces $L^2((0,T)\times\Omega)$ and $L^2(0,T;H^{-1/2}(0,2\pi))$ instead of $C([0,T],L^2(\Omega))$ and $C([0,T],H^{-1/2}(0,2\pi))$, we can see that u satisfies (7.1). □

Proposition 7.2. *Let u and p be defined in previous lemma. Then u verifies (3.2) and p is a solution of the Reynolds equation (3.1).*

Proof. Multiplying the equation

$$\nabla_\varepsilon P(\varepsilon) + u(\varepsilon) - w^0 - \mu\varepsilon^2 \Delta_\varepsilon U(\varepsilon) = 0 \quad \text{ in } H^{-1}(\mathcal{C})$$

with $\phi \in C_0^\infty(Q_T)$ we get

$$\mu\varepsilon^2 \int_0^T \int_{\mathcal{C}} \nabla_\varepsilon U(\varepsilon)\nabla_\varepsilon\phi + \int_0^T \int_{\mathcal{C}} u(\varepsilon)\phi + \int_0^T \int_{\mathcal{C}} w^0\phi + \int_0^T \int_{\mathcal{C}} P(\varepsilon) div_\varepsilon \phi = 0.$$

As $U_r = 0$ and $u_r = 0$, we can take $\phi = (0,\tilde{\phi})$. Taking a limit when ε tends to zero gives

$$\mu \int_0^T \int_{\mathcal{C}} \frac{\partial U}{\partial \rho}\frac{\partial \tilde{\phi}}{\partial \rho} + \int_0^T \int_{\mathcal{C}} u\tilde{\phi} + \int_0^T \int_{\mathcal{C}} w^0\tilde{\phi} - \int_0^T \int_{\mathcal{C}} P\nabla_{\varphi,z}\tilde{\phi} = 0,$$

and then

$$-\mu\frac{\partial^2 U}{\partial \rho^2} + u + w^0 = -\nabla_{\varphi,z}P.$$

Deriving this equation with respect to t gives the following equations for u:

$$\frac{\partial u}{\partial t} - \mu\frac{\partial^2 u}{\partial \rho^2} = -\nabla_{\varphi,z}p, \tag{7.4}$$

$$u(0) = w^0. \tag{7.5}$$

Solving these equations under conditions

$$u(0,\varphi,z,t)=\omega(t)\vec{e}_\varphi,\quad u(h,\varphi,z,t)=0,\quad u \text{ is } 2\pi\text{-periodic u } \varphi$$

we find that u as a function of $\nabla_{\varphi,z}p$ is given by (3.2). The other equations of (7.1) lead to the Reynolds equation (3.1) for p. ∎

Remark 7.3. Theorem 3.1 is a direct consequnce of Propositions 7.1 and 7.2. It is easy to see that these results hold true if the initial condition $u^\varepsilon(x,0)=w^0(\varphi,z)$ in the problem (2.1) is replaced by

$$u^\varepsilon(x,0)=w^{0\epsilon}=w^0(\frac{r-R}{\varepsilon},\varphi,z),$$

where $w^0\in H^1(\mathcal{C})$ is 2π-periodic function in variable φ. In that case the function u is given by (3.2) and p is the solution of the problem

$$\begin{cases}\dfrac{1}{R}\dfrac{\partial}{\partial\varphi}(-\xi*\dfrac{\partial p}{\partial\varphi}+\chi(w^0_\varphi))+R\dfrac{\partial}{\partial z}(-\xi*\dfrac{\partial p}{\partial z}+\chi(w^0_z))=-\eta*\omega,\\ \left[-\xi*\dfrac{\partial p}{\partial z}+\chi(w^0_z)\right]_{z=0}=\dfrac{1}{R}\displaystyle\int_0^{h(\varphi,0)}\vec{e}_z\cdot g_0(\rho,\varphi,t)d\rho,\\ \left[-\xi*\dfrac{\partial p}{\partial z}+\chi(w^0_z)\right]_{z=L}=\dfrac{1}{R}\displaystyle\int_0^{h(\varphi,L)}\vec{e}_z\cdot g_L(\rho,\varphi,t)d\rho,\\ p \text{ is } 2\pi\text{-periodic in } \varphi,\end{cases}\tag{7.6}$$

where

$$\chi(w^0_\varphi)=\int_0^{h(\varphi,z)}G(\rho,\varphi,z,t)w^0_\varphi(\rho,\varphi,z)d\rho,\tag{7.7}$$

$$\chi(w^z_\varphi)=\int_0^{h(\varphi,z)}G(\rho,\varphi,z,t)w^0_z(\rho,\varphi,z)d\rho.\tag{7.8}$$

8. Proofs of Theorems 3.3 and 3.4

In this section we prove theorems characterizing the behaviour of problem 2.1 for large t.

Proof of Theorem 3.3. If $\lim_{\tau\to 0}\tau\hat{p}$ exists, it is equal $\lim_{t\to\infty}p(t)$. In the same way, $\lim_{\tau\to 0}\tau\hat{u}_\varphi=\lim_{t\to\infty}u_\varphi(t)$ and $\lim_{\tau\to 0}\tau\hat{u}_z=\lim_{t\to\infty}u_z(t)$ if these limits exist.

We use the equations

$$\begin{aligned}\tau\hat{u}_\varphi &= \hat{G}\left[-\frac{1}{R}\tau\frac{\partial\hat{p}}{\partial\varphi}+\tau w^0_\varphi\right]+\tau\hat{g}\hat{\omega},\\ \tau\hat{u}_z &= \hat{G}\left[-\tau\frac{\partial\hat{p}}{\partial z}+\tau w^0_z\right].\end{aligned}$$

and the problem for $\hat{p}$

$$
(8.1)\qquad \begin{cases} \dfrac{1}{R}\dfrac{\partial}{\partial\varphi}(-\hat{\xi}\dfrac{\partial\hat{p}}{\partial\varphi}+\hat{\xi}w_\varphi^0)+R\dfrac{\partial}{\partial z}(-\hat{\xi}\dfrac{\partial\hat{p}}{\partial z}+\hat{\xi}w_z^0)=-\hat{\eta}\hat{\omega} \quad \text{in } \Omega \\ \left[-\hat{\xi}\dfrac{\partial\hat{p}}{\partial z}+\hat{\xi}w_z^0\right]_{z=0}=\dfrac{1}{R}\displaystyle\int_0^{h(\varphi,0)}\vec{e}_z\cdot\hat{g}_0 d\rho, \\ \left[-\hat{\xi}\tau\dfrac{\partial\hat{p}}{\partial z}+\hat{\xi}\tau w_z^0\right]_{z=L}=\dfrac{1}{R}\displaystyle\int_0^{h(\varphi,L)}\vec{e}_z\cdot\hat{g}_L d\rho, \\ \hat{p} \text{ is } 2\pi\text{-periodic in } \varphi. \end{cases}
$$

Let us first show that $\lim_{\tau\to 0}\tau\hat{p}=p^\infty$. The difference $\hat{\pi}=\frac{p^\infty}{\tau}-\hat{p}$ is the solution of the problem

$$
\begin{cases} \operatorname{div}_{\varphi,z}\left[\hat{\xi}\nabla_{\varphi,z}\hat{\pi}\right]=\operatorname{div}_{\varphi,z}\left[\dfrac{1}{\tau}(\hat{\xi}-\dfrac{1}{12}h^3)\nabla_{\varphi,z}\hat{p}^\infty-\hat{\xi w^0}\right]+\hat{\eta}\hat{\omega}-\dfrac{1}{2}\dfrac{\partial h}{\partial\varphi}\mu\dfrac{\omega^\infty}{\tau}, \\ \left[\hat{\xi}\nabla_{\varphi,z}\hat{\pi}-\dfrac{1}{\tau}(\hat{\xi}-\dfrac{1}{12}h^3)\nabla_{\varphi,z}\hat{p}^\infty+\hat{\xi w^0}\right]\cdot\nu=(\hat{B}_\alpha-\dfrac{B_\alpha(0)}{\tau}), \text{ for } \alpha=0,L, \\ \hat{\pi} \text{ is } 2\pi\text{-periodic in } \varphi. \end{cases}
$$

The estimate

$$
\begin{aligned} |\nabla_{\varphi,z}\hat{\pi}|_{L^2(\Omega)} \quad\le\quad & C(|\frac{1}{\tau}(\hat{\xi}-\frac{1}{12}h^3)\nabla_{\varphi,z}\hat{p}^\infty|_{L^2(\Omega)}+|\hat{\xi}w^0|_{L^2(\Omega)} \\ & +|\hat{\eta}\hat{\omega}-\frac{1}{2}\frac{\partial h}{\partial\varphi}\mu\frac{\omega^\infty}{\tau}|_{L^2(\Omega)}+|\hat{g}_0-\frac{g_0^\infty}{\tau}|_{H^{1/2}(S_0)} \\ & +|\hat{g}_L-\frac{g_L^\infty}{\tau}|_{H^{1/2}(S_L)}). \end{aligned}
$$

is valid for $\hat{\pi}$. From the last result we see that for $\tau\hat{\pi}=p^\infty-\tau\hat{p}$ holds the estimate

$$
\begin{aligned} |\nabla_{\varphi,z}\tau\hat{\pi}|_{L^2(\Omega)} \quad\le\quad & C(|(\hat{\xi}-\frac{1}{12}h^3)\nabla_{\varphi,z}\hat{p}^\infty|_{L^2(\Omega)}+|\tau\hat{\xi}w^0|_{L^2(\Omega)} \\ & +|\tau\hat{\eta}\hat{\omega}-\frac{1}{2}\frac{\partial h}{\partial\varphi}\mu\omega^\infty|_{L^2(\Omega)}+|\tau\hat{g}_0-g_0^\infty|_{H^{1/2}(S_0)} \\ & +|\tau\hat{g}_L-g_L^\infty|_{H^{1/2}(S_L)}). \end{aligned}
$$

As $\lim_{\tau\to 0}\hat{\xi}=\frac{1}{12}h^3$ and $\lim_{\tau\to 0}\hat{\eta}=\frac{1}{2}\frac{\partial h}{\partial\varphi}\mu$ in $L^\infty(\Omega)$, we get that $\lim_{\tau\to 0}\nabla_{\varphi,z}(\tau\hat{p}-p^\infty)=0$ in $L^2(\Omega)$. This implies that $\lim_{t\to\infty}p(t)=p^\infty$ in $H^1(\Omega)\cap L_0^2(\Omega)$.

We consider now the difference $|\tau\hat{u}_\varphi-u_\varphi^\infty|_{L^2(C)}$. As $\lim_{\tau\to 0}\hat{G}=\frac{1}{2}(\rho-h)\rho$ and $\lim_{\tau\to 0}\hat{g}=1-\frac{\rho}{h}$ u $L^\infty(\mathcal{C})$, we can conclude that $\lim_{t\to\infty}u(t)=u^\infty$ in $L^2(\mathcal{C})$. ∎

Proof of Theorem 3.4. We consider the difference π between p^∞ and $p(t)$, $\pi=p^\infty-p(t)$. From the proof of the previous theorem for $\hat{\pi}=\frac{p^\infty}{\tau}-\hat{p}$

we have the following estimate

$$
\begin{aligned}
|\nabla_{\varphi,z}\hat{\pi}|_{L^2(\Omega)} \quad \le \quad & C(|\frac{1}{\tau}(\hat{\xi}-\frac{1}{12}h^3)\nabla_{\varphi,z}\hat{p}^\infty|_{L^2(\Omega)}+|\hat{\xi}w^0|_{L^2(\Omega)} \\
& +|\hat{\eta}\hat{\omega}-\frac{1}{2}\frac{\partial h}{\partial\varphi}\mu\frac{\omega^\infty}{\tau}|_{L^2(\Omega)}+|\hat{g}_0-\frac{g_0^\infty}{\tau}|_{H^{1/2}(S_0)} \\
& +|\hat{g}_L-\frac{g_L^\infty}{\tau}|_{H^{1/2}(S_L)}).
\end{aligned}
$$

As

$$
Re\,\hat{\xi} = -\frac{bh+bh\cos bh-2\sin bh}{2b^3\cos\frac{bh}{2}} \ge -\frac{b\beta_1+b\beta_1\cos b\beta_1-2\sin b\beta_1}{2b^3\cos\frac{b\beta_1}{2}} > 0,
$$

for $\frac{\tau}{\mu}=-b^2>-\frac{\pi^2}{\beta_2^2}$, $b\in R$, we can conclude that $\hat{p}(\tau-\lambda)$ is defined for τ close to 0 and

$$
\begin{aligned}
& \lim_{t\to\infty}\exp(\lambda t)|\nabla_{\varphi,z}(p(t)-p^\infty)|_{L^2(\Omega)} = \lim_{\tau\to 0+}\tau\left|\nabla_{\varphi,z}(\hat{p}(\tau-\lambda)-\frac{p^\infty}{\tau-\lambda})\right|_{L^2(\Omega)} \\
\le \quad & C\lim_{\tau\to 0+}\tau\Bigg(|\frac{1}{\tau-\lambda}(\hat{\xi}(\tau-\lambda)-\frac{1}{12}h^3)\nabla_{\varphi,z}\hat{p}^\infty|_{L^2(\Omega)} \\
& +|\hat{\xi}(\tau-\lambda)w^0|_{L^2(\Omega)}+|\hat{\eta}(\tau-\lambda)\hat{\omega}(\tau-\lambda)-\frac{1}{2}\frac{\partial h}{\partial\varphi}\mu\frac{\omega^\infty}{\tau-\lambda}|_{L^2(\Omega)} \\
& +|\hat{g}_0(\tau-\lambda)-\frac{g_0^\infty}{\tau-\lambda}|_{H^{1/2}(S_0)}+|\hat{g}_L(\tau-\lambda)-\frac{g_L^\infty}{\tau-\lambda}|_{H^{1/2}(S_L)}\Bigg) \\
\le \quad & C\lim_{\tau\to 0+}\Bigg(|\frac{\tau}{\tau-\lambda}(\hat{\xi}(\tau-\lambda)-\frac{1}{12}h^3)|_{L^2(\Omega)}+|\tau\hat{\xi}(\tau-\lambda)|_{L^2(\Omega)} \\
& +\tau|\hat{\eta}(\tau-\lambda)\hat{\omega}(\tau-\lambda)-\frac{1}{2}\frac{\partial h}{\partial\varphi}\mu\frac{\omega^\infty}{\tau-\lambda}|_{L^2(\Omega)}\Bigg) \\
& +C\lim_{t\to\infty}\exp(\lambda t)\left(|g_0(t)-g_0^\infty|_{H^{1/2}(S_0)}+|g_L(t)-g_L^\infty|_{H^{1/2}(S_L)}\right) \\
= \quad & 0,
\end{aligned}
$$

for $\lambda<\lambda_1, \frac{\pi^2}{\beta_2^2}\mu$. This proves the assertion of Theorem 3.4. □

At the end we give a regularity result for the limit problem.

8.1. *Regularity in time.* The regularity in time for the limit velocity and the limit pressure is given by the following theorem.

THEOREM 8.1. *Let (u,p) be a solution to the problem (3.1)– (3.2). Then we have*

$$tp\in C([0,T];H^1(\Omega)\cap L_0^2(\Omega)), \tag{8.2}$$

$$tu\in C([0,T];L^2(\mathcal{C})). \tag{8.3}$$

Moreover,

$$\lim_{t\to 0} t|p(t)|_{H^1(\Omega)\cap L^2_0(\Omega)} = \lim_{t\to 0} t|u(t)|_{L^2(\Omega)} = 0.$$

PROOF. Using the finite difference in τ, from the problem (5.5) we get the problem for $\frac{\partial}{\partial\tau}(\hat{P}-\frac{P(0)}{\tau})$. Multiplying that problem by $\frac{\partial}{\partial\tau}(\hat{P}-\frac{P(0)}{\tau})$, after integrating over po Ω, for a large $|\tau|$ we get the estimate

$$\begin{aligned}
&|\nabla_{\varphi,z}\frac{\partial}{\partial\tau}(\hat{P}-\frac{P(0)}{\tau})|_{L^2(\Omega)} \le \\
&\quad \le C\left(|(\hat{\xi}+\tau\frac{\partial\hat{\xi}}{\partial\tau})\nabla_{\varphi,z}(\hat{P}-\frac{P(0)}{\tau})|_{L^2(\Omega)}\right.\\
&\qquad \left.+|\frac{\partial}{\partial\tau}(\hat{g}_0-\frac{g_0(0)}{\tau})|_{H^{1/2}(S_0)}+|\frac{\partial}{\partial\tau}(\hat{g}_L-\frac{g_L(0)}{\tau})|_{H^{1/2}(S_L)}\right)\\
&\quad \le C\left(\frac{1}{|\tau|^2}|\nabla_{\varphi,z}(\hat{P}-\frac{P(0)}{\tau})|_{L^2(\Omega)}+|\frac{\partial\hat{g}_0}{\partial\tau}+\frac{g_0(0)}{\tau^2}|_{H^{1/2}(S_0)}\right.\\
&\qquad \left.+|\frac{\partial\hat{g}_L}{\partial\tau}+\frac{g_L(0)}{\tau^2}|_{H^{1/2}(S_L)}\right)\\
&\quad \le C(\frac{1}{|\tau|^4}+\frac{1}{|\tau|^3})\le\frac{C}{|\tau|^3}.
\end{aligned}$$

For $\tau<\infty$, we have the estimate

$$\begin{aligned}
&|\nabla_{\varphi,z}\frac{\partial}{\partial\tau}(\hat{P}-\frac{P(0)}{\tau})|_{L^2(\Omega)} \le \\
&\quad \le \frac{C}{|\tau|}\left(|(\hat{\xi}+\tau\frac{\partial\hat{\xi}}{\partial\tau})\nabla_{\varphi,z}(\hat{P}-\frac{P(0)}{\tau})|_{L^2(\Omega)}+|\frac{\partial\hat{g}_0}{\partial\tau}+\frac{g_0(0)}{\tau^2}|_{H^{1/2}(S_0)}\right.\\
&\qquad \left.+|\frac{\partial\hat{g}_L}{\partial\tau}+\frac{g_l(0)}{\tau^2}|_{H^{1/2}(S_l)}\right)\\
&\quad \le \frac{C}{|\tau|}\left(\frac{1}{|\tau|^2}+\frac{1}{|\tau|^2}|\tau^2\frac{\partial\hat{g}_0}{\partial\tau}+g_0(0)|_{H^{1/2}(S_0)}+\frac{1}{|\tau|^2}|\tau^2\frac{\partial\hat{g}_L}{\partial\tau}+g_L(0)|_{H^{1/2}(S_L)}\right)\\
&\quad \le \frac{C}{|\tau|^3}.
\end{aligned}$$

As $\frac{\partial\hat{p}}{\partial\tau}=\hat{P}+\tau\frac{\partial\hat{P}}{\partial\tau}=\hat{P}-\frac{P(0)}{\tau}+\tau\frac{\partial}{\partial\tau}(\hat{P}-\frac{P(0)}{\tau})$, for $\frac{\partial\hat{p}}{\partial\tau}$ we can infer the following estimate

$$|\nabla_{\varphi,z}\frac{\partial\hat{p}}{\partial\tau}|_{L^2(\Omega)}\le\frac{C}{|\tau|^2}.$$

$\frac{\partial\hat{p}}{\partial\tau}$ is the Laplace transform of $t\,p$. Using the representation formula for the inverse Laplace transform we can conclude that $tp\in C([0,T];H^1(\Omega)\cap L^2_0(\Omega))$. As

$$\frac{\partial\hat{u}}{\partial\tau}=\frac{\partial\hat{G}}{\partial\tau}\nabla_{\varphi,z}\hat{p}+\hat{G}\nabla_{\varphi,z}\frac{\partial\hat{p}}{\partial\tau}+\frac{\partial}{\partial\tau}\begin{bmatrix}\hat{g}\,\hat{\omega}\\ 0\end{bmatrix},$$

$\frac{\partial \hat{G}}{\partial \tau} = O(\frac{1}{|\tau|^2})$ and $\frac{\partial}{\partial \tau}(\hat{g}\,\hat{\omega}) = O(\frac{1}{|\tau|^2})$ for a large $|\tau|$, with the same argument, we get the assertion (8.3). $\square$

Roughly speaking this theorem says that p is singular at $t = 0$. For $t > 0$, p is regular in time, with the space regularity depending on the functions g_α, $\alpha = 0, L$.

References

[1] G. Bayada and M. Chambat, *The Transition Between the Stokes Equations and the Reynolds Equation: A Mathematical Proof*, Appl. Math. Optim. **14** (1986), 73–93.

[2] A. Duvnjak, *Derivation of non–linear Reynolds–type Problem for Lubrication of a Rotating Shaft*, ZAMM **82** (2002), 317–333.

[3] A. Duvnjak, *Neki niždimenzionalni modeli za opisivanje toka fluida kroz tanko područje*, Thesis, University of Zagreb, 1999.

[4] A. Duvnjak and E. Marušić-Paloka, *Derivation of the Reynolds Equation for Lubrication of a Rotating Shaft*, Arch. Math. **36** (2000), 239–253.

[5] A. Duvnjak and E. Marušić-Paloka, *Correctors for Reynolds Equation Describing the Process of Lubrication of a Rotating Shaft*, Proceedings of STAMM 98, Nica, 1998, Chapman and Hall Monographs and Surveys in Pure and Applied Mathematics **106** (2000), 272–281.

[6] H. Le Dret, Problémes variationnels dans les multidomaines, R.M.A. 19, Masson, Paris, 1991.

[7] J. L. Lions, *Problémes aux Limites en Théorie des Distributions*, Acta Mathematica **94** (1955), 13–153.

[8] A. Mikelić, *Mathematical Derivation of the Darcy–type Law with Memory Effects, Governing Transient Flow Through Porous Media*, Glasnik Matematički **29(49)** (1994), 57–77.

[9] S. A. Nazarov, *Asimptotic Solution of the Navier–Stokes Problem on the Flow of a Thin Layer of Fluid*, Siberian Math. J. **31** (1990), 296–307.

[10] Szeri, A. Z., Fluid Film Lubrication, Theory and Design, Cambridge University Press, 1998.

[11] R. Temam, Navier–Stokes Equations, Theory and Numerical Analysis, North–Holland, Amsterdam, 1984.

[12] V. S. Vladimirov, Equations of Mathematical Physics, Mir Publishers, Moscow, 1984.

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