Statistics with Non-Precise Data

Reinhard Viertl

Department of Statistics and Probability Theory, Vienna University of Technology, Wien, Austria

In statistical analysis data are usually assumed to be numbers or vectors. But real measurement data of continuous quantities are not precise numbers but more or less non-precise. This imprecision is different from measurement errors and is also called fuzziness. Before analysing the data it is necessary to describe the imprecision of measurements quantitatively. This can be done using the concepts of fuzzy numbers and fuzzy vectors. Then statistical inference procedures are generalized to the more realistic situation of non-precise data.

Keywords: Data Analysis, Fuzzy Data, Fuzzy Numbers, Non-Precise Data, Statistics.

1. Introduction

Observing stochastic quantities one is facing different kinds of uncertainty. Three important types of uncertainty are

stochastic variation
errors in the data
imprecision of observations.

Stochastic variation is described by *stochastic models* $X \sim P_\theta,\ \theta \in \Theta$, where X is a stochastic quantity and $P_\theta,\ \theta \in \Theta$ is a family of probability distributions. The symbol "$\sim$" stands for "is distributed according to". The distribution P_θ has to be estimated from observations of X, called data.

Errors in the data are described by error models. Here it is assumed that not realizations x_i of X are observed but the observations y_i are translations of the "true" values x_i, i.e.

$$y_i = x_i + \epsilon$$

where ϵ is called *error*. The error term is usually modelled by a stochastic quantity $Z \sim N(0, \sigma^2)$, i.e. normally distributed errors. It should be noted that the results y_i are usually assumed to be precise numbers.

In standard statistical inference all observed quantities are assumed to be precise numbers. But this cannot be said for real data because measurements are more or less non-precise. This means that the result of a measurement is not a precise number x but a fuzzy number denoted by $x^\star$.

General considerations on modelling and statistical analysis of non-precise data are contained in BANDEMER (1993), DUBOIS & PRADE (1986), and KACPRZYK & FEDRIZZI (1988). Theoretical mathematical aspects are given in KRUSE & MEYER (1987), and VIERTL (1996).

To explain the fundamental ideas of statistical inference for non-precise data in the following observations without error term are considered.

The third type of observation uncertainty is the *imprecision* of single observations. This uncertainty is usually neglected in statistics. But continuous quantities cannot be measured precisely. Therefore such measurements are not just numbers but more complex objects.

2. Non-Precise Data and Fuzzy Numbers

Since observations are not just numbers but important figures they must be described quantitatively before carrying out statistical analysis.

Example 1: Reading on a digital instrument is a finite sequence of numbers. Therefore the result $x^\star$ of a measurement in this case is an interval $[a, b]$ in the most simple situation. This result can be described by the indicator function $I_{[a,b]}(\cdot)$ of $[a, b]$. This indicator function is characterizing the observation.

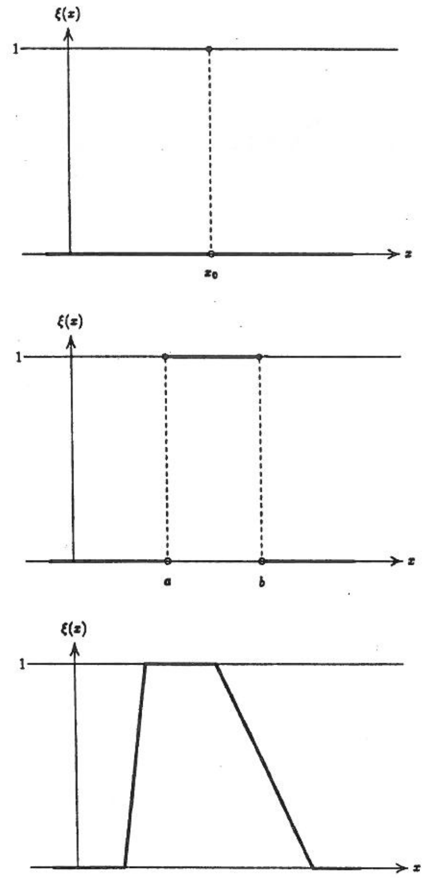

Fig. 1. Examples of characterizing functions

Example 2: Let the result of a one-dimensional measurement be a light point on the screen. This can, of course, not be identified with a precise real number. A reasonable description in generalization of an indicator function is a function derived from the light intensity $g(\cdot)$ of the point. This derived function is a real function $\xi(\cdot)$ defined by

$$\xi(x) := \frac{g(x)}{\max g(x)} \qquad \forall \quad x \in \mathbb{R}.$$

The function $\xi(\cdot)$ is characterizing the result of the measurement.

Remark 1: A precise observation $x_o \in \mathbb{R}$ is characterized by the indicator function $I_{\{x_o\}}(\cdot)$.

Results of observations, which are not precise numbers are called *fuzzy numbers*. Fuzzy numbers are denoted by $x^\star$. In generalization of precise numbers x the mathematical description of fuzzy numbers are so called *characterizing functions* which are special forms of membership functions from fuzzy set theory and generalizations of indicator functions.

Definition 1: A fuzzy number $x^\star$ is given by its characterizing function $\xi(\cdot)$ which is a real function obeying the following conditions:

(1) $\xi\colon \mathbb{R} \to [0,1]$

(2) $\exists\, x_0 \in \mathbb{R}\colon\ \xi(x_0) = 1$

(3) $\forall\ \alpha \in (0,1]$ the set $B_\alpha := \{x \in \mathbb{R} : \xi(x) \geq \alpha\} = [a_\alpha, b_\alpha]$ is a finite closed interval, called *α-cut* of $\xi(\cdot)$.

In figure 1 some characterizing functions are depicted.

Remark 2: Methods to obtain the characterizing functions of non-precise observations depend on the field of application. Some methodology can be recognized in the above examples. For more details compare the monograph VIERTL (1996).

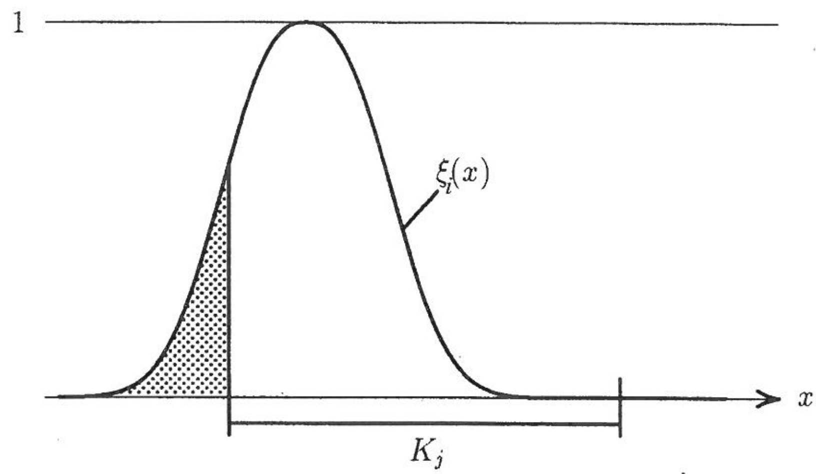

Fig. 2. Non-precise observation and class of a histogram

3. Histograms for Non-Precise Data

For non-precise data the construction of a histogram is not trivial because for fixed class K_j of a histogram it is generally not possible to decide if a non-precise observation $x_i^\star$ with characterizing function $\xi_i(\cdot)$ is in the class K_j or not. An example of this situation is given in figure 2.

Therefore it is necessary to generalize histograms to the situation of non-precise data. This is done by the concept of *fuzzy histograms*. In fuzzy histograms the height of the histogram over the classes $K_j = [a_j, b_j]$, $j = 1(1)k$, is a fuzzy number $h_n^\star(K_j)$ whose characterizing function $\eta_j(\cdot)$ is constructed in the following way.

First, by $\underline{n}_j$ we denote the number of those observations $x_i^\star$ which are certainly in the class K_j. Those are the observations whose characterizing functions $\xi_i(\cdot)$ fulfill

$$\xi_i(x) = 0 \qquad \text{for all} \quad x \notin K_j.$$

Next, by $\overline{n}_j$ we denote the number of those observations $x_i^\star$ for which

$$\exists\, x \in K_j \ : \ \xi_i(x) > 0.$$

Then the characterizing function $\eta_j(\cdot)$ is 0 outside the interval

$$\left[\frac{\underline{n}_j}{n}, \frac{\overline{n}_j}{n}\right].$$

In this interval the characterizing function $\eta_j(\cdot)$ is a polygon with

$$\eta_j\left(\frac{\underline{n}_j - 1}{n}\right) = 0 \quad \text{and} \quad \eta_j\left(\frac{\overline{n}_j + 1}{n}\right) = 0$$

and cutting points whose abszissa values are $\frac{\underline{n}_j}{n}\left(\frac{1}{n}\right)\frac{\overline{n}_j}{n}$.

To obtain the values of the ordinates at these points we consider

$$A_j\left(x_i^\star\right) := \int_{a_j}^{b_j} \xi_i(x)\, dx \quad \text{for} \quad i = 1(1)n.$$

and

$$B_j\left(x_i^\star\right) := \int_R \xi_i(x)\, dx - A_j\left(x_i^\star\right) \quad \text{for } i = 1(1)n$$

for those $x_i^\star$ with $A_j\left(x_i^\star\right) > 0$ and $B_j\left(x_i^\star\right) > 0$.

Denoting these observations by $x_{(\ell)}^\star$ for $\ell = 1, \cdots, \overline{n}_j - \underline{n}_j$ in order of increasing values of

$$B_j\left(x_{(\ell)}^\star\right)$$

where $\xi_{(\ell)}(\cdot)$ denotes the characterizing function of $x_{(\ell)}^\star$, the values of the characterizing function $\eta_j(\cdot)$ at the abszissa of the cutting points are given by

$$\eta_j\left(\frac{\underline{n}_j}{n}\right) = 1$$

$$\eta_j\left(\frac{\underline{n}_j+1}{n}\right) = 1 - \frac{B_j\left(x^{\star}_{(1)}\right)}{\sum\limits_{\ell=1}^{\overline{n}_j-\underline{n}_j} \int_R \xi_{(\ell)}(x)\,dx}$$

$$\vdots$$

$$\eta_j\left(\frac{\underline{n}_j+s}{n}\right) = 1 - \sum_{i=1}^{s} \frac{B_j\left(x^{\star}_{(i)}\right)}{\sum\limits_{\ell=1}^{\overline{n}_j-\underline{n}_j} \int_R \xi_{(\ell)}(x)\,dx}$$

$$\vdots$$

$$\eta\left(\frac{\overline{n}_j}{n}\right) = 1 - \sum_{i=1}^{\overline{n}_j-\underline{n}_j} \frac{B_j\left(x^{\star}_{(i)}\right)}{\sum\limits_{\ell=1}^{\overline{n}_j-\underline{n}_j} \int_R \xi_{(\ell)}(x)\,dx} .$$

In figure 3 an example is depicted.

Remark 3: Other concepts from descriptive statistics can be generalized too. For details compare the monograph VIERTL (1996).

In order to generalize statistical inference procedures it is necessary to look at functions of non-precise variables. This is done in the following section.

4. Functions of Non-Precise Samples

In order to generalize statistical functions $S = \mathcal{S}(X_1, \cdots, X_n)$ of samples from stochastic quantities X two points have to be noticed.

In case of precise data $x_1, \cdots, x_n$ with $x_i \in M$, where M denotes the *observation space* of X, the sample is considered as an n-tuple $(x_1, \cdots, x_n)$ which is an element of the *sample space* $M^n = M \times \cdots \times M$, the Cartesian product of n copies of the observation space.

Statistics S are defined by measurable functions $\mathcal{S} : M^n \to N$, where $(N, \mathcal{C})$ is some measurable space.

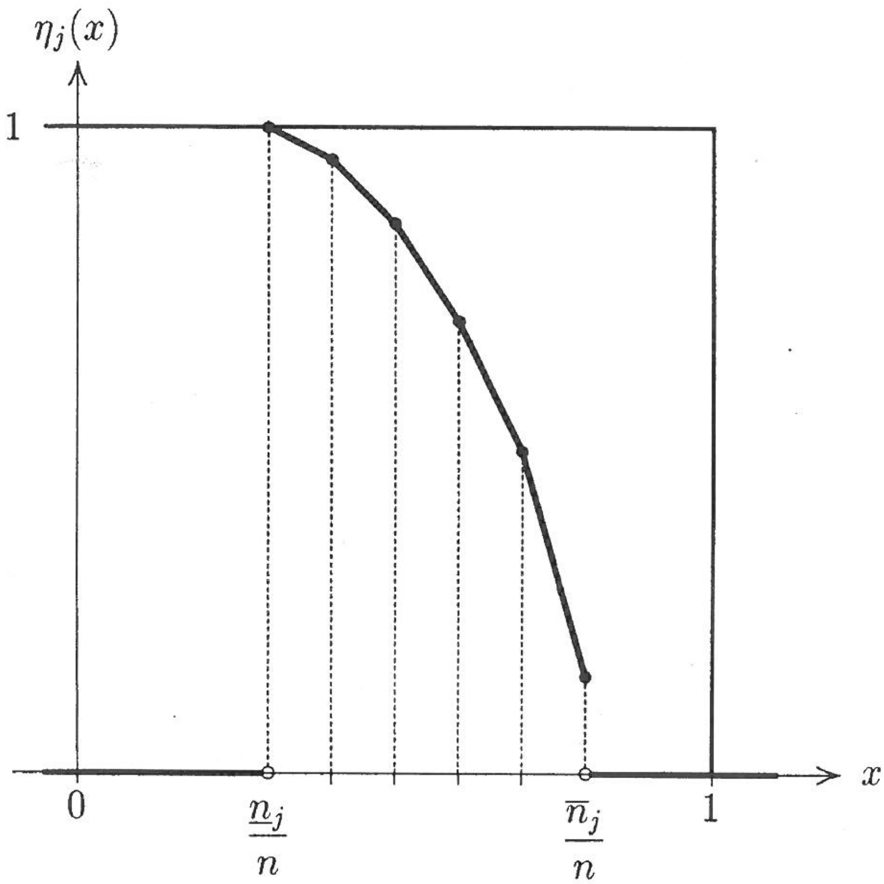

Fig. 3. Fuzzy height of a histogram

Remark 4: For parameter estimations the space N is the parameter space Θ from the introduction.

In case of non-precise samples two points are important:

I. The non-precise sample $x_1^\star, \cdots, x_n^\star$ has to be combined to form a fuzzy element $\underline{x}^\star$ of the sample space, called *non-precise combined sample element.*
II. Classical statistical functions $\mathcal{S}(\cdot)$ have to be generalized to the situation of fuzzy arguments.

4.1. Combination Rules

The first point above is realized via so-called *combination rules* $K(\cdot, \cdots, \cdot)$. These combination rules generate a so-called *fuzzy vector* which is a non-precise element of the sample space $M^n \subseteq \mathbb{R}^n$. These fuzzy vectors are defined in the following way.

Definition 2: An n-dimensional fuzzy vector $\underline{x}^\star$ is given by its n-dimensional characterizing function $\xi(\cdot, \cdots, \cdot)$ which is a real function of n real variables $x_1, \cdots, x_n$ with following properties with $\underline{x} = (x_1, \ldots, x_n)$:

(1) $\xi : \mathbb{R}^n \to [0, 1]$

(2) $\exists\, \underline{x}_0 \in \mathbb{R}^n : \xi(\underline{x}_0) = 1$

(3) $B_\alpha(\underline{x}^\star) := \{\underline{x} \in \mathbb{R}^n : \xi(\underline{x}) \geq \alpha\}$ is $\forall\ \alpha \in (0, 1]$ a star shaped and compact subset of $\mathbb{R}^n$.

Remark 5: There are different combination rules which generate a fuzzy vector $\underline{x}^\star$ from n fuzzy numbers $x_1^\star, \cdots, x_n^\star$. This combination is based on

$$\xi(\cdot, \cdots, \cdot) := K_n\Big(\xi_1(\cdot), \cdots, \xi_n(\cdot)\Big)$$

where the values $\xi(x_1, \cdots, x_n)$ are given by

$$\xi(x_1, \cdots, x_n) := K_n\Big(\xi_1(x_1), \cdots, \xi_n(x_n)\Big) \quad \forall \quad (x_1, \cdots, x_n) \in M^n.$$

The simplest combination rule is

$$\xi(x_1, \cdots, x_n) := \min_{i=1(1)n} \xi_i(x_i) \quad \forall \quad (x_1, \cdots, x_n) \in M^n,$$

called *minimum-rule.* For this combination rule the α-cuts $B_\alpha(\underline{x}^\star)$ of the so-called *non-precise combined sample element* $\underline{x}^\star$ are related to the α-cuts $B_\alpha(x_i^\star)$ of the non-precise data $x_1^\star, \cdots, x_n^\star$ by

$$B_\alpha(\underline{x}^\star) = \mathrm{X}_{i=1}^n B_\alpha(x_i^\star) \qquad \forall \quad \alpha \in (0, 1],$$

i.e. all $B_\alpha(\underline{x}^\star)$ are the Cartesian products of the α-cuts $B_\alpha(x_i^\star)$.

For the proof compare VIERTL (1996).

Another possible combination rule is the so-called *product-rule*

$$\xi(x_1, \cdots, x_n) := \prod_{i=1}^n \xi_i(x_i) \quad \forall\, (x_1, \cdots, x_n) \in M^n.$$

4.2. Extension Principle for Functions

With the construction of non-precise combined sample elements $\underline{x}^\star$ it is possible to model functions of n non-precise arguments. This can be done by an adaption of the so-called extension principle from fuzzy set theory.

Definition 3: Let $g : \mathbb{R}^n \to \mathbb{R}$ be a classical function whose argument values $\underline{x}$ are not known precisely but are fuzzy vectors $\underline{x}^\star \in \mathcal{F}(\mathbb{R}^n)$. For a non-precise argument value $\underline{x}^\star$ with corresponding characterizing function $\xi(\cdot)$ the non-precise value $g(\underline{x}^\star)$ of the function $g(\cdot)$ is defined by its characterizing function $\psi(\cdot)$ whose values are given by

$$\psi(y) := \begin{cases} \sup\{\xi(\underline{x}) : \ \underline{x} \in \mathbb{R}^n,\ g(\underline{x}) = y\} & \text{for } g^{-1}(\{y\}) \neq \emptyset \\ 0 & \text{for } g^{-1}(\{0\}) = \emptyset. \end{cases}$$

Remark 6: For continuous functions $g(\cdot)$ in definition 3 it can be proved that the function $\psi(\cdot)$ fulfills all properties of a characterizing function in definition 1.

5. Generalized Point Estimators based on Non-Precise Samples

Let $X \sim f(\cdot \mid \theta)$, $\theta \in \Theta$ be a stochastic model with observation space M. Furthermore let $\vartheta(X_1, \cdots, X_n)$ be a classical estimator for θ based on a sample $X_1, \cdots, X_n$ from X. For concrete observed precise data $x_1, \cdots, x_n$ a parameter value $\hat{\theta} = \vartheta(x_1, \cdots, x_n)$ from Θ is obtained as estimation for the true parameter θ_o.

If only non-precise observations $x_1^\star, \cdots, x_n^\star$ are available a fuzzy estimate $\hat{\theta}^\star$ for the true parameter θ_o can be obtained by applying definition 3 to the function $\vartheta(\cdot, \cdots, \cdot)$, based on the characterizing function $\xi(\cdot, \cdots, \cdot)$ of the non-precise combined sample element $\underline{x}^\star$ from section 4.1. Using the notation $\underline{x} = (x_1, \cdots, x_n) \in M^n$ the characterizing function $\psi(\cdot)$ of $\hat{\theta}^\star$ is given by its values

$$\psi(\theta) = \begin{cases} \sup\{\xi(\underline{x}) \colon \underline{x} \in \vartheta^{-1}(\theta)\} & \text{for} \quad \vartheta^{-1}(\theta) \neq \emptyset \\ 0 & \text{for} \quad \vartheta^{-1}(\theta) = \emptyset. \end{cases}$$

Example 3: Let the life time X of an electronic device be exponentially distributed with density

$$f(x \mid \theta) = \frac{1}{\theta} \exp\left(-\frac{x}{\theta}\right) I_{(0,\infty)}(x) \quad \text{with parameter} \quad \theta > 0.$$

The optimal point estimator $\hat{\theta}$ for the true parameter θ_o based on precise observed life times

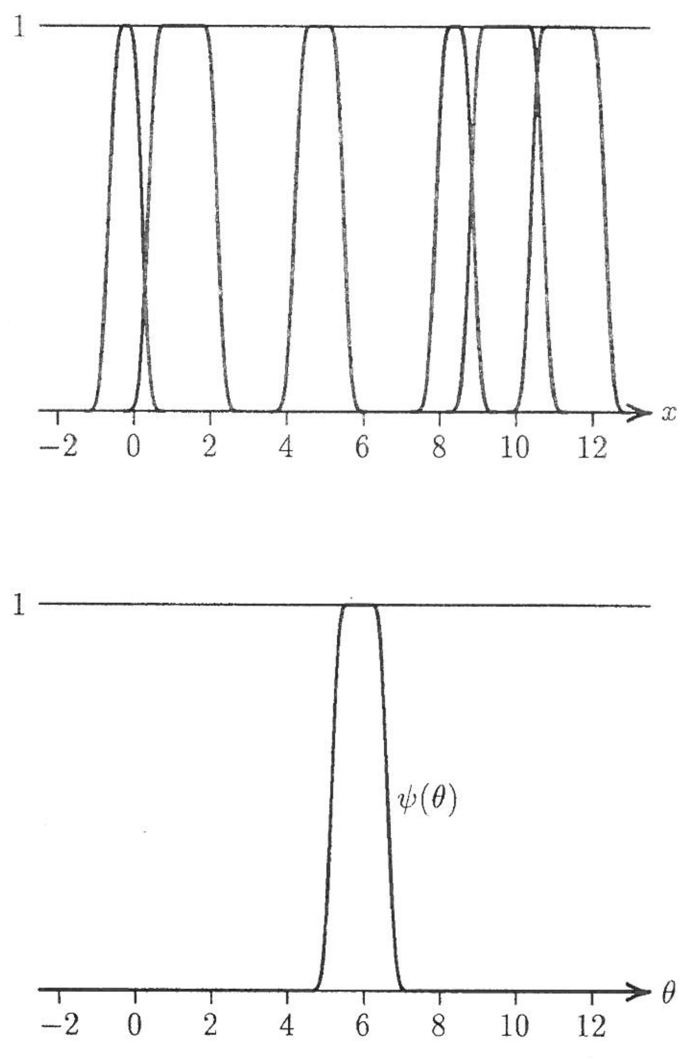

Fig. 4. Non-precise sample and fuzzy estimate $\hat{\theta}^\star$

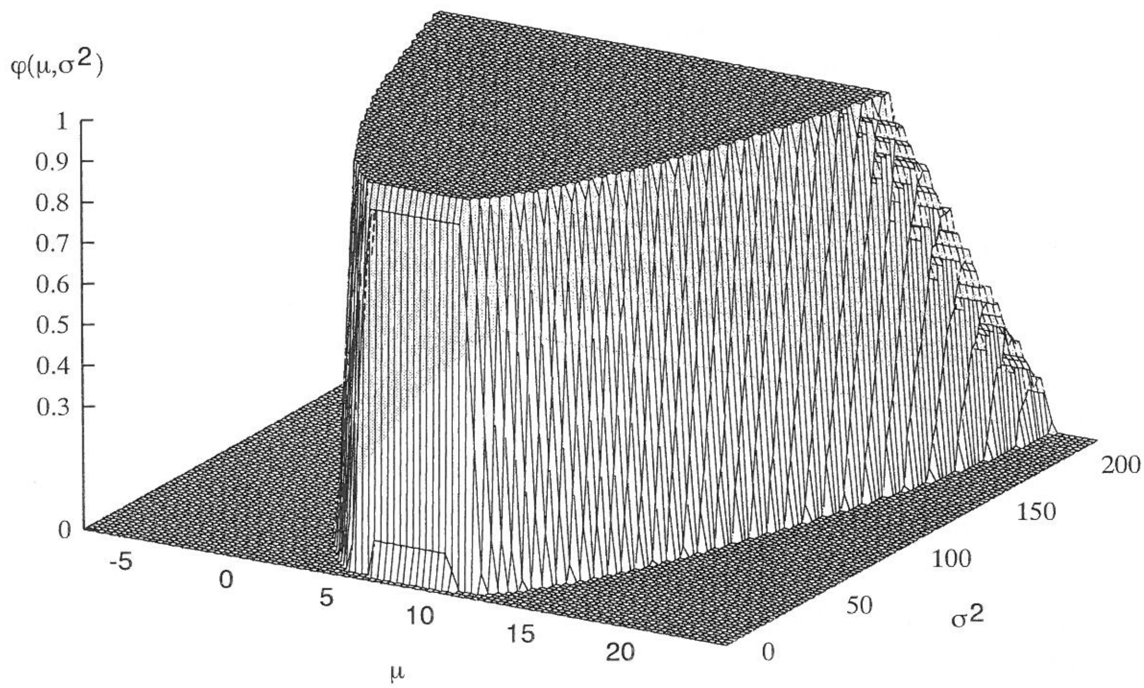

Fig. 5. Fuzzy confidence region

$x_1, \cdots, x_n$ is given by

$$\hat{\theta} = \vartheta(x_1, \cdots, x_n) = \bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

In case of non-precise observed life times $x_1^\star, \cdots, x_n^\star$ with corresponding characterizing functions $\xi_1(\cdot), \cdots, \xi_n(\cdot)$ the generalized fuzzy estimate $\hat{\theta}^\star$ is depicted – using the minimum combination rule – in figure 4.

An application for renewal problems is given in NICULESCU & VIERTL (1992)

6. Fuzzy Confidence Regions

It is assumed that the reader is familiar with the concept of confidence functions and confidence regions. Let $\kappa(X_1, \cdots, X_n)$ be a confidence function with confidence level $1 - \delta$ for the – possibly k-dimensional – parameter θ of a stochastic model. For precise concrete data $x_1, \cdots, x_n$ this yields a subset $\kappa(x_1, \cdots, x_n) \subseteq \Theta$, called *confidence set.*

For non-precise data with characterizing function $\xi(\cdot, \cdots, \cdot)$ of the non-precise combined sample element the generalized confidence region becomes a *fuzzy subset* $\Theta^\star$ of Θ. Using the vector notation $\underline{x} = (x_1, \cdots, x_n)$ the characterizing function $\varphi(\cdot)$ of $\Theta^\star$ is defined by its values

$$\varphi(\theta) := \begin{cases} \sup\{\xi(\underline{x}) : \ \theta \in \kappa(\underline{x})\} & \\ \qquad \text{if } \exists\, \underline{x} : \ \theta \in \kappa(\underline{x}) & \\ 0 & \\ \qquad \text{for all other} \quad \theta \in \Theta. & \end{cases}$$

Remark 7: $\varphi(\cdot)$ is a special form of the membership function from fuzzy set theory. This construction is a reasonable generalization because for precise data the resulting $\varphi(\cdot)$ is the indicator function of the corresponding classical confidence set.

7. Non-Precise Data and Statistical Tests

The problem of fuzzy data in connection with statistical tests was adressed in GIL & CORRAL (1988).

For non-precise data so called test statistics $T = \mathcal{T}(x_1^\star, \cdots, x_n^\star)$ become fuzzy quantities $t^\star$. This is a problem if the characterizing function $\eta(\cdot)$ of $t^\star$ has positive values in both the acceptance region and the rejection region of the test. In this case a decision is not possible. Then, similar to sequential procedures in statistics, more data have to be collected in order to obtain a decision.

There are some papers on this topic, but further research is necessary. For details compare RÖMER & KANDEL (1995).

8. Further Classical Statistical Methods for Non-Precise Data

Also the following inference procedures can be generalized to the situation of non-precise data:

Cumulative sums
Empirical distribution functions
Empirical fractiles
Correlation coefficients

For details compare the monograph VIERTL (1996).

9. Bayes' Theorem for Non-Precise Data

The fundamental concept for Bayesian inference in statistics is Bayes' theorem

$$\pi(\theta \mid x_1, \cdots, x_n) \propto \pi(\theta) \cdot \ell(\theta; x_1, \cdots, x_n),$$

where $\pi(\cdot)$ is the *a priori* density, $\ell(\cdot\,; x_1, \cdots, x_n)$ the likelihood function, $x_1, \cdots, x_n$ are the precise data, and $\pi(\cdot \mid x_1, \cdots, x_n)$ is the *a posteriori* density. The symbol $\propto$ denotes "proportional", since $\pi(\theta) \cdot \ell(\theta; x_1, \cdots, x_n)$ has generally to be normalized by a multiplicative constant in order to become a probability density function, i.e.

$$\int_\Theta \pi(\theta \mid x_1, \cdots, x_n)\, d\theta = 1.$$

For non-precise data Bayes' theorem has to be generalized. This is possible using the characterizing function $\xi(\cdot, \cdots, \cdot)$ of the non-precise combined sample element $\underline{x}^\star$. The outcome of generalized Bayes' theorem is an a posteriori density with fuzzy values, called *fuzzy a posteriori density* $\pi^\star(\cdot \mid \underline{x}^\star)$.

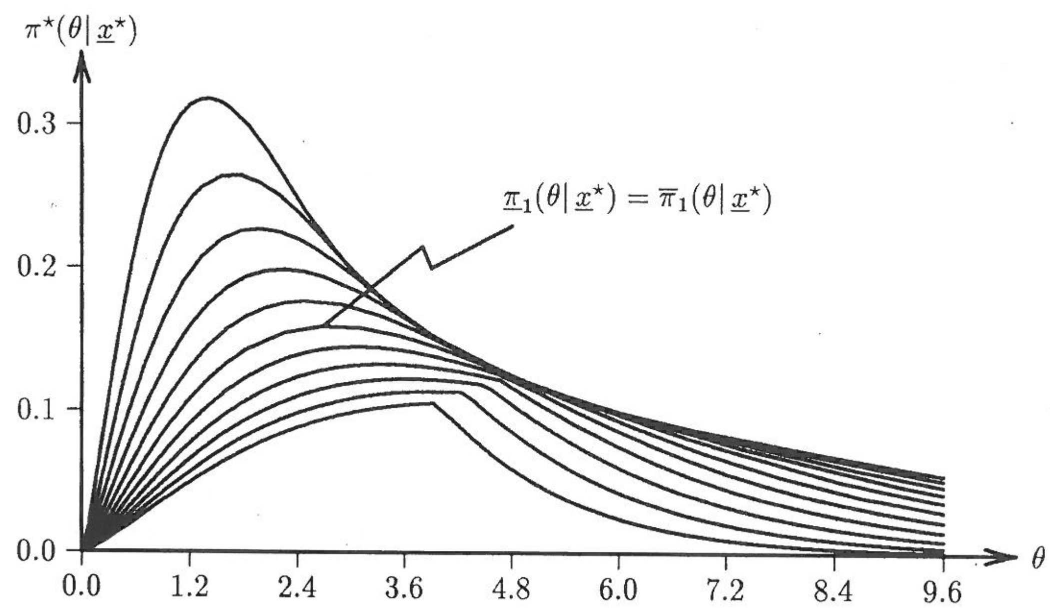

Fig. 6. Fuzzy a posteriori density

The characterizing function $\psi_\theta(\cdot)$ of the fuzzy value $\pi^\star(\theta \,|\, \underline{x}^\star)$ is given by its values

$$\psi_\theta(y) := \begin{cases} \sup\{\xi(\underline{x}) : \ \underline{x} \in M^n, \ \pi(\theta \mid \underline{x}) = y\} \\ \qquad\qquad \text{for} \quad \exists\, \underline{x} : \ \pi(\theta \mid \underline{x}) = y \\ 0 \\ \qquad\qquad \text{otherwise.} \end{cases}$$

The family $(\psi_\theta(\cdot); \theta \in \Theta)$ of fuzzy values of $\pi^\star(\theta \,|\, \underline{x}^\star)$ is called *fuzzy a posteriori density*

$$\pi^\star(\cdot \mid \underline{x}^\star) \mathrel{\widehat{=}} (\psi_\theta(\cdot); \theta \in \Theta).$$

Remark 8: Fuzzy a posteriori densities can be graphically represented by so-called *α-level curves* $\underline{\pi}_\alpha(\cdot \mid \underline{x}^\star)$ and $\overline{\pi}_\alpha(\cdot \mid \underline{x}^\star)$ respectively. These are deterministic functions which connect the endpoints of the α-cuts of $\psi_\theta(\cdot)$ for varying θ. In figure 6 an example of such a presentation is given.

A related paper is FRÜHWIRTH - SCHNATTER (1992).

10. Fuzzy Predictive Distributions

An important concept for stochastic predictions is the *predictive density*. For stochastic model $X \sim f(\cdot \,|\, \theta)$, $\theta \in \Theta$, a priori density $\pi(\cdot)$, precise data $x_1, \cdots, x_n$ the predictive density of X based on the data $\underline{x} = (x_1, \cdots, x_n)$, denoted by $p(\cdot \mid \underline{x})$ is defined by their values

$$p(x \mid \underline{x}) = \int_\Theta f(x \mid \theta)\, \pi(\theta \mid \underline{x})\, d\theta \quad \text{for all } \ x \in M.$$

Remark 9: As reference any introductory book on Bayesian statistics is suitable. A german language reference is VIERTL (1990).

In the situation of non-precise data with non-precise combined sample element $\underline{x}^\star$ and corresponding characterizing function $\xi(\cdot, \cdots, \cdot)$ the concept of predictive density can be generalized. The result is a generalized predictive density with fuzzy values $p^\star(x \,|\, \underline{x}^\star)$. The characterizing functions $\psi_x(\cdot)$ of these fuzzy values are defined by

$$\psi_x(y) = \begin{cases} \sup\{\xi(\underline{x}) : \ \underline{x} \in M^n, \ p(x|\underline{x}) = y\} \\ \qquad\qquad \text{if } \exists\, \underline{x} : \ p(x|\underline{x}) = y \\ 0 \\ \qquad\qquad \text{otherwise} \end{cases}$$

for x varying in the observation space M of the considered stochastic quantity X.

Remark 10: Also, predictive densities are graphically displayed using α-level curves. An example of a stochastic quantity with exponential distribution is given in figure 7.

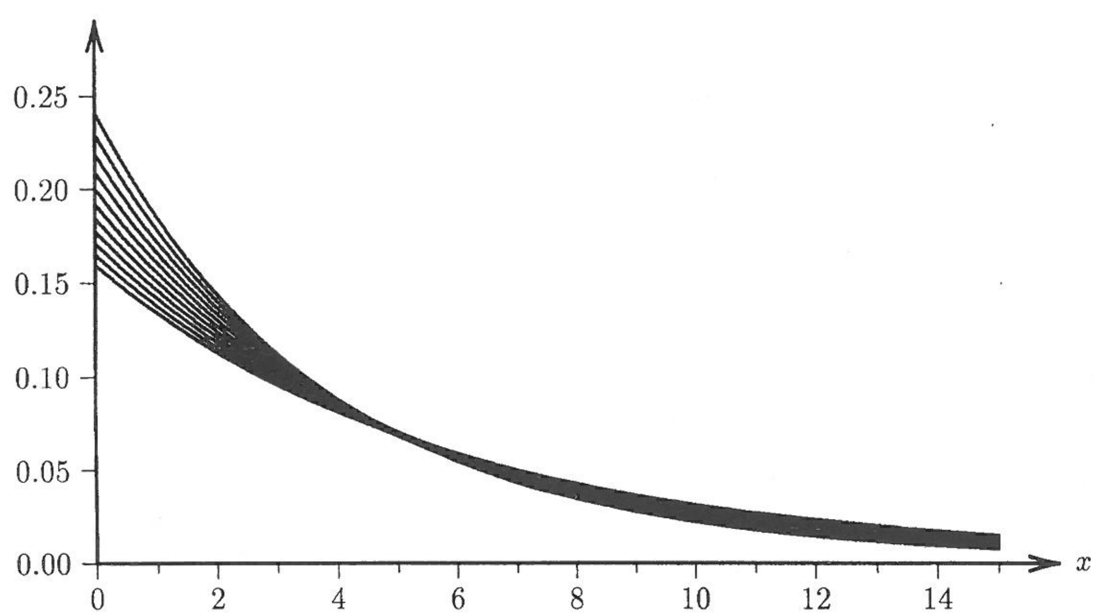

Fig. 7. Fuzzy predictive density

11. Further Bayesian Inference Methods for Non-Precise Information

Besides the above mentioned, the following Bayesian inference methods can be also generalized to the case of non-precise data:

Bayesian confidence regions
HPD-regions
Bayesian decisions under loss

Moreover, an important topic are non-precise a priori distributions and the generalization of Bayes' theorem to the situation of non-precise a priori densities and non-precise data. This is also possible.

For details on these topics see VIERTL (1995) and the monograph VIERTL (1996).

12. Conclusion

Before their statistical inference the formal description of non-precise data is necessary. Using the concepts of fuzzy numbers and fuzzy vectors to describe them it is possible to generalize classical statistical inference procedures as well as Bayesian methods to the situation of non-precise data. Moreover, it is also possible to model non-precise a priori information in Bayesian inference and to develop a corresponding methodology for related statistical inference. Basic concepts as well as hints for further readings are given in this paper.

Some problems need more research. For example statistical test procedures and general Bayesian decision rules in face of non-precise data and general fuzzy information. It would be interesting also to consider fuzzy utility functions which seem to be more realistic.

Remark 11: Calculations necessary to work out statistical inference for real non-precise data are only possible using computer programs. For some inference procedures such software is already available.

13. References

H. BANDEMER (Ed.) *Modelling Uncertain Data.* Akademie Verlag, Berlin, 1993.

D. DUBOIS, H. PRADE Fuzzy sets and statistical data. *European Journal of Operational Research* **25**(1986), 345-356.

S. FRÜHWIRTH-SCHNATTER On statistical inference for fuzzy data with applications to descriptive statistics. *Fuzzy Sets and Systems* **50**(1992), 143-165.

M.A. GIL, N. CORRAL, P. GIL The minimum inaccuracy estimates in χ^2 tests for goodness of fit with fuzzy observations. *Journal of Statistical Planning and Inference* **19**(1988), 95-115.

J. KACPRZYK, M. FEDRIZZI (Eds.) *Combining Fuzzy Imprecision with Probabilistic Uncertainty in Decision Making*, Lecture Notes in Economics and Mathematical Systems, Vol. 310. Springer-Verlag, Berlin, 1988.

R. KRUSE, K.D. MEYER, *Statistics with Vague Data.* D. Reidel, Dordrecht, 1987.

S.P. NICULESCU, R. VIERTL A comparison between two fuzzy estimators for the mean. *Fuzzy Sets and Systems* **48**(1992), 341-350.

C. RÖMER, A. KANDEL Statistical tests for fuzzy data. *Fuzzy Sets and Systems* **72**(1995), 1-26.

R. VIERTL, *Einführung in die Stochastik, mit Elementen der Bayes-Statistik und Ansätzen für die Analyse unscharfer Daten.* Springer-Verlag, Wien, 1990.

R. VIERTL Non-precise information in Bayesian inference. Presented at the *Proceedings of the international conference on Statistical and Bayesian Methods in Hydrological Sciences in honour of Jacques Bernier*, (1995) Paris, UNESCO.

R. VIERTL Real Data and their Mathematical Description for Parameter Estimation, *Mathematical Modelling of Systems* **2**(1996), 262-298.

R. VIERTL *Statistical Methods for Non-Precise DATA.* CRC Press, Boca Raton, Florida, 1996.

Received: October, 1996
Accepted: November, 1996

Contact address:
Department of Statistics
and Probability Theory
Vienna University of Technology
1040 Wien
Austria
Tel. (1) 58801-5425
Fax. (1) 5868093

REINHARD VIERTL was born 25 March 1946 at Hall in Tyrol, Austria. He received his degree in engineering mathematics in 1972 and the doctorate in engineering in 1974, and became appointed assistant at the Vienna University of Technology. Promotion to University Docent in 1979. He was research fellow and visiting lecturer at the University of California, Berkeley, from 1980 to 1981, and visiting lecturer at the University of Klagenfurt, Austria, from 1981 to 1982. Since 1982 full professor of applied statistics at the Vienna University of Technology. He was head of the Austrian Statistical Society from 1987 to 1995 and is a Fellow of the Royal Statistical Society, London, held the Max Kade fellowship in 1980, and is founder of the Austrian Bayesian Society.

Author of the following books: *Statistical Methods in Accelerated Life Testing* (1988), *Introduction to Stochastics*, published in German (1990), *Statistical Methods for Non-Precise Data* (1996). Editor of the books *Probability and Bayesian Statistics* (1987), *Contributions to Environmental Statistics* in German (1992), and co-editor of a book titled *Mathematical and Statistical Methods in Artificial Intelligence* (1995). Author of over 50 scientific papers in mathematics and statistics, most of which are in-depth examinations of algebra, probability theory, accelerated life testing, regional statistics, and statistics with non-precise data.