On a Cost Allocation Problem Arising from a Star-Star Capacitated Concentrator Location Problem

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We analyze a cost allocation problem associated with the Star-Star Capacitated Concentrator Location (SSCCL) problem. The problem is formulated as a cost cooperative game in characteristic function form to be referred to as the SSCCL game. The characterization and computation of game theoretic solution concepts associated with this game are investigated. We show that, in general, the core of this cooperative game may be empty. However, we provide a polynomial representation of the core of the SSCCL game. In case of nonemptiness of the core we provide an efficient method to find the nucleolus. For the case when the core is empty, we propose the least weighted ε-core as a concept for fair cost allocation for the SSCCL problem and give its polynomial characterization. Moreover, certain 'central' point of the least weighted ε-core is also efficiently characterized.

Keywords: Capacitated Concentrator Location, Cost Allocation, Game Theory

1. Introduction

An important computer network design problem is how to connect several remote users (computers or terminals) to a central site, which could be a node of a backbone network or a processing site. This is often accomplished by a well known design method that uses concentrators (see for example Mirzaian (1985) and Pirkul (1987)). Some relatively local sites are connected to a concentrator and the concentrator is connected via a high speed line or a satellite to the central site. Each concentrator has a capacity limit on the traffic it can handle. The total design cost consists of the cost of opening capacitated concentrators and the cost of communication links used to connect users to concentrators. The objective of the design problem is then to determine the number and location of concentrators and to connect network nodes to these concentrators at minimum cost, while satisfying the capacity constraints.

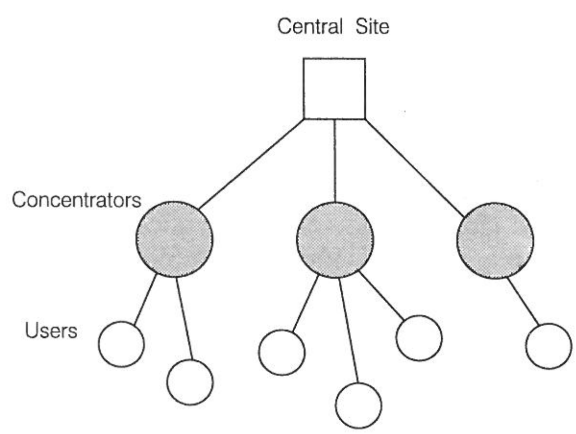

Fig. 1. Example of a SSCCL design

Formally, let $G = (N, E)$ be an underlying connected undirected network with a set of nodes N and a set of arcs E, and let O be a central site. Both the set of users and the potential concentrators' sites are represented by nodes in G. Each user j, $j \in N$ has a demand d_j for service that can be produced by concentrators yet to be constructed. A concentrator can be opened at node i, at cost c_i, which also includes the cost of linking concentrator i to central site O. Each user j should receive all of its service from a single neighboring concentrator i. There is a cost $c(i, j) = c_{ij} \geq 0$, $(i, j) \in E$ if arc (i, j) is used to link user j to a concentrator i. The total demand satisfied by a single concentrator is limited by capacity U. The objective is to provide service to all customers at minimum cost. We will refer to the above optimization problem as the Star-Star Capacitated Concentrator Location (SSCCL) problem (for an illustration see Fig. 1).

With this problem naturally arises the problem of allocating the cost among customers. There is an incentive to allocate the cost among users in a 'fair' manner. Namely, we would like to allocate the cost in such a way that no subset of users would have incentive to secede and build their own network. Clearly, breaking up the network might destroy global optimality of the cost. Game theory approach to this problem leads to a definition of cooperative game in characteristic function form with user's nodes being players and characteristic function defined on all subsets (coalitions) of users. The value of the characteristic function for a certain subset would be the cost of optimal subnetwork delivering service to that subset. We will formulate the associated cost allocation problem as a cost cooperative game in characteristic function form referred to as the SSCCL-game.

In cooperative game theory several different approaches for fair cost allocation have been suggested (for a survey of these concepts see, for example, Young (1985) and Driessen (1988)). Some of the suggested concepts are the Shapley value, the core, the least ε-core , the least per capita ε-core, the nucleolus and the nucleolus per capita. However, note that there is no apparent method for choosing one solution concept over another (see Sharkey (1985)). Also note that most of these solution concepts suffer from the fact that the amount of data required to compute them is enormous. In this paper we will analyze the core, the nucleolus and the least weighted ε-core of the SSCCL-game.

The SSCCL-game properly generalizes concentrator covering game which was studied by (D. Skorin–Kapov (1993)). His concentrator covering problem is concerned with the location of concentrators in already existing local area network. Therein he analyzed some game theoretic solution concepts like the core, the nucleolus and the least per capita ε-core to allocate the cost of concentrators, while ignoring the costs of communication links between users sites and concentrators. The main objective of this paper is to extend the analyses to the 'more general and more realistic design case.

Note, that in general relatively little has been done on analyses of game theoretic cost allocation concepts for classes of capacitated network problems (see for example D. Skorin–Kapov (1993)). The main concern is that characterization of game theoretic concepts often requires exponential number of constraints and is often computationally prohibitive. Nevertheless, in this paper we show that in a content of a SSCCL problem, even in quite realistic situations, computations of the above solutions may be feasible.

2. Definitions and Preliminaries

In order to analyze the cost allocation problem associated with the SSCCL problem, we need to introduce the following game theoretic definitions and notation. Let $N = \{1, 2, ..., n\}$ be a finite set of *players* and let $c : 2^N \to \mathbf{R}$, with $c(\emptyset) = 0$, be a *characteristic function* defined over subsets of N referred to as *coalitions*. If $c(N)$ designates a cost that has to be shared by all the players, then the pair (N, c) is called a *(cost) cooperative game*, or simply a *game*. For $x \in \mathbf{R}^n$ and $S \subseteq N$, let $x(S) = \sum_{j \in S} x_j$. We can interpret $x(S)$ as the part of total cost paid by the coalition S. A cost allocation vector x in a game (N, c) satisfies $x(N) = c(N)$, and the solution theory of cooperative games is concerned with the selection of a reasonable subset of cost allocation vectors. The characteristic function c is *submodular* if $c(S) + c(T) \geq c(S \cup T) + c(S \cap T)$ for all $S, T \subseteq N$. If c is submodular (N, c) is said to be convex.

Central to the solution theory of cooperative games is the concept of solution referred to as the *core* of the game. The core of a game (N, c) consists of all vectors $x \in \mathbf{R}^n$ such that $x(S) \leq c(S)$ for all $S \subseteq N$ and $x(N) = c(N)$. Observe that the core consists of all allocation vectors x which provide no incentive for any coalition to secede. In general, the core of a game may be empty.

For a real number ε, the *ε-core* of a game (N, c) consists of all vectors $x \in \mathbf{R}^n$ such that

$x(S) + \varepsilon \leq c(S)$ for all $\emptyset \neq S \subset N$, and $c(N) = x(N)$. Clearly for ε small enough the ε-core of the game (N, c) is not empty. The *least ε-core* is the intersection of all nonempty ε-cores. Equivalently, let ε_0 be the largest ε such that the ε-core is not empty, then the least ε-core is the ε_0-core.

The nucleolus, introduced by Schmeidler (1969), is another well known concept of solution. Intuitively, the nucleolus is an allocation that makes the least-well-off coalition as well-off as possible. We say that coalition S is better-off than T, relative to an allocation x, if $c(S) - x(S) > c(T) - x(T)$. The quantity $e(x, S) = c(S) - x(S)$ is called the excess of S relative to x. Formally, the nucleolus can be presented as follows. For a game (N, c) and an associated cost allocation vector x, let $e(x)$ be a vector in $\mathbf{R}^{(2^n-2)}$ whose entries are $e(x, S)$, $\emptyset \neq S \subset N$, arranged in a nondecreasing order. The nucleolus is the vector that maximizes $e(x)$ lexicographically. This means that the value of the smallest excess is as large as possible and is attained on as few sets as possible, the next smallest excess is as large as possible and is attained on as few sets as possible, etc. In contrast to the core, the nucleolus always exists. Moreover, it is unique and it is contained in the core if the core is not empty. Intuitively, the nucleolus is the center of the core if the core is nonempty and it is the 'closest' point to the core if the core is empty.

Another reasonable approach is to define the excess of a coalition on a per capita basis: $\tilde{e}(x, S) = \frac{1}{|S|}(c(S) - x(S))$, where $|S|$ is the cardinality of a set S. Let $\tilde{e}(x, S)$ be a vector in $\mathbf{R}^{2^n-2}$, whose entries are $\tilde{e}(x, S)$, $\emptyset \notin S \subset N$, arranged in a nondecreasing order. The per capita nucleolus (Grotte (1970)) is the vector that maximizes $\tilde{e}(x)$ lexicographically.

The cost allocation problem associated with the SSCCL problem is concerned with the allocation of the cost incurred for satisfying the user's demand for service. Given a SSCCL problem defined on $G = (N, E)$, for S, $S \subseteq N$, we denote by SSCCL_S, SSCCL problem on G with a set of users as well as concentrators potential sites restricted to S.

SSCCL_S problem can be formulated as the following integer linear programming problem. For all $i \in S$ and $(i, j) \in E$, let y_i and y_{ij} be respectively 0–1 variables with the following interpretation: $y_i = 1$ if a concentrator is opened at node i and zero otherwise and $y_{ij} = 1$ if user j is serviced by concentrator i and zero otherwise. Let d_j be the demand for service at node j, let c_i be the cost to open a concentrator at node i (which also includes the cost of linking concentrator i to central site O), let c_{ij} be the cost of link (i, j) and assume that potential concentrators have capacity U. Then

$$c(S) = \min \sum_{i \in S} c_i y_i + \sum_{i,j \in S} c_{ij} y_{ij}$$

subject to:

$$\begin{aligned} &\sum_{i \in S} y_{ij} = 1, \quad \text{for every } j \in S \\ &\sum_{j \in S} d_i y_{ij} \leq U y_i, \quad \text{for every } i \in S \qquad (\text{SSCCL}_S) \\ &y_{ij}, y_i \in \{0, 1\}, \text{ for all } i \in S, \text{ and } j \in S \end{aligned}$$

The first set of constraints ensures that each node in S is serviced by exactly one neighboring concentrator in S and the second set of constraints are capacity constraints.

Then, the pair (N, c), where $c : 2^N \rightarrow \mathbf{R}$ is such that $c(\emptyset) = 0$ and for each $S \subseteq N$, $c(S)$ is the minimum objective function value of SSCCL_S, is a cooperative game in characteristic function form, to be referred to as the *Star-Star Capacitated Concentrator Location* (SSCCL) game.

3. The Core of the SSCCL-Game

First we show that the core of the SSCCL game may be empty. Consider, for example, the network $G = (N, E)$ consisting of a three node ring (Fig. 2) with $N = \{1, 2, 3\}$ and $E = \{(1, 2), (2, 3), (1, 3)\}$. Assume that $c_1 = c_2 = c_3 = 1$, $c_{12} = c_{23} = c_{13} = 0.2$, $d_1 = d_2 = d_3 = 1$ and each potential concentrator has a capacity $U = 2$.

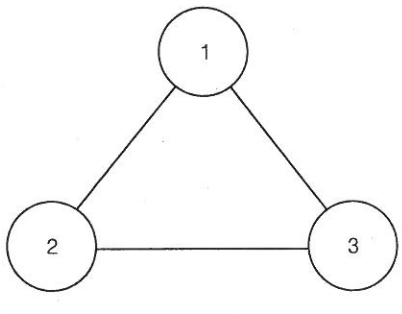

Fig. 2. Network $G = (N, E)$

Now, one can easily verify that the core constraints induced by the two-member coalitions are:

$$\begin{aligned} x_1 + x_2 &\le 1.2, \\ x_1 + x_3 &\le 1.2, \\ x_2 + x_3 &\le 1.2. \end{aligned} \tag{1}$$

The entire cost is

$$x_1 + x_2 + x_3 = 2.2. \tag{2}$$

If we sum up inequalities in (1) we get

$$x_1 + x_2 + x_3 \le 1.8 \tag{3}$$

which contradicts the total cost allocation (2). Thus, we conclude that the core of the SSCCL game associated with G is empty.

Consider now a general case of the SSCCL game. let $G = (N, E)$ be the underlying network with the set of users N, and assume that each concentrator can satisfy at most demand U. Clearly, the exponential number of core constraints, coupled with the fact that in general case the computation of $c(S)$, $S \subseteq N$ is strongly NP-complete (Mirzaian and Steiglitz (1981)), makes the core computations hard. Nevertheless, we provide an efficient representation of the core, which often enables us to test whether the core of a SSCCL game is empty, and generate core points (if they exist) in polynomial time.

Theorem 1. *Let $\mathscr{C}_i = \{S_i^l | S_i^l$ is a subset of N, such that an optimal solution to $SSCCL_{S_i^l}$ uses a single concentrator opened at node $i\}$, and let $\mathscr{C} = \bigcup_{i \in N} \mathscr{C}_i$. Then the core of a SSCCL game is given by all cost allocations $x \in \mathbf{R}^{|N|}$, satisfying:*

$$c(S) - x(s) \ge 0, \text{ for all } S \in \mathscr{C} \tag{4}$$

$$c(N) - x(N) = 0. \tag{5}$$

PROOF. We will show that all core constraints excluded from (4) and (5) are redundant. Let S be a proper subset of N which is not contained in $\mathscr{C}$, and let F, $F \subseteq N$, be the set of open facilities in the optimal solution to SSCCL$_S$. For each $i \in F$, let S_i be the subset of users which are serviced by a facility opened at node i in the optimal solution to SSCCL$_S$. Since $S_i \cap S_j = \emptyset$, for all $i, j \in F$ and $i \ne j$, we have that $S_i \in \mathscr{C}_i$ for each $i \in F$. Moreover $c(S) = \sum_{i \in F} c(S_i)$, and $x(S) = \sum_{i \in F} x(S_i)$. Then, it follows that for any $i \in F$:

$$\begin{aligned} c(S) - x(S) &= \sum_{i \in F} (c(S_i) - x(S_i)) \\ &\ge (c(S_i) - x(S_i)) \ge 0. \end{aligned} \tag{6}$$

Hence, the core constraint induced by S is redundant and the proof is complete. □

Note that the maximum cardinality of every set in $\mathscr{C}$ is bounded with capacity limit U. Depending on the structure of the underlying network G, sets S_i in $\mathscr{C}$, can often in practice be computed in polynomial time. Moreover, the family $\mathscr{C}$ used to characterize the core is often polynomial in size. We will discuss such cases in the concluding section of this paper.

4. The Nucleolus of a SSCCL Game

Even if the core of a SSCCL game is not empty, it may consist of many cost allocations which are not equally attractive. Consider a simple example of a SSCCL game in which the underlying network $G = (N, E)$ is a chain given in Fig. 3. Let $N = \{1, 2, 3\}$, $d_1 = d_2 = d_3 = 1$ and assume that facility costs are $c_1 = c_2 = c_3 = 2$, that link costs are $c_{12} = 0$, $c_{23} = 2$ and suppose that capacity of each potential concentrator is $U = 2$. It is easy to check that cost allocations $(x_1 = 2, x_2 = 0, x_3 = 2)$, $(x_1' = 1, x_2' = 1, x_3' = 2)$ are both in the core of the associated game (N, c). However, it is hard to believe that user 1 would accept the solution, in which he pays 2 and user 2 does not pay anything, as a fair solution.

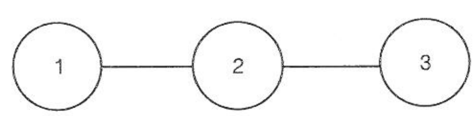

Fig. 3. Network $G = (N, E)$

For the case when the core of a SSCCL problem is not empty we will try to determine the most 'attractive' point in the core, namely its nucleolus.

A method for computing the nucleolus by solving a sequence of linear programming (LP) problems was implicitly suggested by Schmeidler (1969) and then further studied by Kopelowitz (1967), Keane (1969), Charnes and Kortanek (1967 and 1970) and, finally, by Maschler et. al. (1979). The k^{th} LP problem, LP_k, solved by this method is:

$$\max\{\varepsilon : \varepsilon_j = c(S) - x(S),\\ S \in P_j, j = 1, ..., k-1, \varepsilon \le c(S) - x(S),\\ S \notin \bigcup_{j=1}^{k-1} P_j,\ x(N) = c(N)\}, \quad (\text{LP}_k)$$

where for $j \ge 1$, ε_j and P_j are, respectively, the optimal value and the set of subsets whose corresponding inequality constraints are satisfied as equalities at an optimal solution of LP_j. The nucleolus is obtained at problem LP_i if the optimal solution to LP_i is unique. We will refer to this method as the Linear Programming (LP) procedure for computing the nucleolus. It is easy to show that SSCCL game is not necessarily convex which implies (Maschler et. al. (1979)) that the redundant core constraints may be needed to determine the nucleolus. Nevertheless, when the core of a SSCCL game is not empty, we show below, that the collection of constraints $\mathscr{C}$ used to characterize the core is sufficient to completely determine the nucleolus of a SSCCL game.

Theorem 2. *If the core of a SSCCL game is not empty then the nucleolus of a SSCCL game is completely determined by the collection of core constraints associated with subsets in* $\mathscr{C}$.

PROOF. The proof will follow if we show that all core constraints induced by subsets not in $\mathscr{C}$ are redundant in the LP procedure for computing the nucleolus. Let S be a subset of N which is not contained in $\mathscr{C}$. Let F be the set of open facilities in the optimal solution to SSCCL_S and again for each $i \in F$, let S_i be the set of customers serviced by a facility opened at node i in the optimal solution to SSCCL_S. Then by the same argument as in T.1 $c(S) = \sum_{i \in F} c(S_i)$ and $x(S) = \sum_{i \in F} x(S_i)$. Further, by nonemptyness of the core,

$$c(S) - x(S) \ge c(S_i) - x(S_i) \text{ for } i \in F. \quad (7)$$

Now if for Q, $Q \subseteq N$, $\text{LP}_{t(Q)}$ denotes the linear program in the LP procedure for computing the nucleolus of the SSCCL game in which the constraint $\varepsilon \le c(Q) - x(Q)$ corresponding to Q has been first satisfied as an equality at the corresponding optimal value $\varepsilon_{t(Q)}$, then

$$\varepsilon_{t(S_i)} = c(S_i) - x(S_i), \text{ for } i \in F \quad (8)$$

and

$$\varepsilon_{t(S)} = c(S) - x(S). \quad (9)$$

Observe that by (7) $\varepsilon_{t(S)} \ge \varepsilon_{t(S_i)}$, for all $i \in F$, which implies that the constraints (8) would be considered in the LP procedure for computing nucleolus prior to the constraint (9). Moreover, since $\varepsilon_{t(S)} = \sum_{i \in F} \varepsilon_{t(S_i)}$, constraints (8) completely determine the value of the total allocation to subset S and constraint (9) is redundant in the above procedure. □

The nucleolus is unique and exists even in the case when the core is empty. However, in this case, it appears to be difficult to characterize the nucleolus of the SSCCL game. Instead, in case of emptyness of the core, we propose the least weighted ε-core as a solution concept for the SSCCL game.

5. The Least Weighted ε-Core

For the case when the core of the SSCCL game (N, c) is empty, we propose the least weighted ε-core as a concept for a fair cost allocation associated with the SSCCL problem.

For each coalition S, $S \subseteq N$ let w_S be the weight associated with S. Then the weighted ε-core is a set of cost allocation vectors for which for all coalitions S, $S \subseteq N$: $c(S) - x(S) \ge w_S\varepsilon$ and $c(N) - x(N) = 0$.

The weighted ε-core can be interpreted as the set of 'attractive' cost allocations that can not be improved upon by any coalition S if forming a coalition entails a cost $-w_S\varepsilon$. Clearly, the weighted weighted ε-core is not empty if ε is sufficiently small. Let ε' be the largest ε for which the weighted ε-core is not empty. Then the weighted ε'-core is the least weighted

ε-core. It appears that ε' is particularly interesting when it is negative, i.e. when the core of a game (N, c) is empty.

We will show below that with the suitable choice of weights the least weighted ε-core of a SSCCL game can be characterized by our collection of constraints associated with the family of coalitions $\mathscr{C}$. We say that a set of weights $W = \{w_S \mid S \subseteq N\}$ is additive if for any two subsets $S_1, S_2 \subseteq N$ such that $S_1 \cap S_2 = \emptyset$, we have $w_{S_1} + w_{S_2} = w_{S_1 \cup S_2}$.

Theorem 3. *Let $W = \{w_S \mid S \subseteq N\}$ be the additive set of weights associated with coalitions in the partitive set of N. The least weighted ε-core of the SSCCL game (N, c) is characterized by the following LP problem:*

$$\max\{\varepsilon | (c(S) - x(S)) \geq w_S \varepsilon, \text{ for } S \in \mathscr{C} \text{ and } \emptyset \neq S \subseteq N, \text{ and } c(N) = x(N)\}. \quad (10)$$

PROOF. We will prove below that the weighted ε-core constraints corresponding to subsets which are not in the collection $\mathscr{C}$ are redundant. Let

(i) $\varepsilon' = max\{\varepsilon \mid c(S) - x(S) \geq w_S \varepsilon, S \in \mathscr{C}, \emptyset \neq S \subset N, \text{ and } c(N) = x(N)\}$

and

(ii) $\varepsilon'' = max\{\varepsilon \mid c(S) - x(S) \geq w_S \varepsilon, S \in 2^N, \emptyset \neq S \subset N, \text{ and } c(N) = x(N).\}$

We will show that $\varepsilon' = \varepsilon''$. Clearly $\varepsilon' \geq \varepsilon''$. It remains to be shown that $\varepsilon'' \geq \varepsilon'$. Let (x, ε') be an optimal solution to (i) and let S be any subset of N. Let F be the set of open facilities in an optimal solution of SSCCL_S and for $i \in F$, let S_i be the subset of users serviced by a concentrator i in that solution. Since, for each $i \in F$, S_i is in $\mathscr{C}$, we have:

$$\begin{aligned} c(S) - x(S) &= \sum_{i \in F} (c(S_i) - x(S_i)) \\ &\geq \sum_{i \in F} w_{S_i} \varepsilon' = w_S \varepsilon', \end{aligned} \quad (11)$$

which implies that $\varepsilon'' \geq \varepsilon'$. □

Note, that for a special case, when the above weights in T.3 are $w_i = 1$ for all $i \in N$, then the least weighted ε-core becomes the least per capita ε-core.

In order to further analyze the least weighted ε-core we introduce, for an arbitrary ε, the game (N, c_ε) whose characteristic function c_ε is defined as follows:

$$c_\varepsilon(S) = \begin{cases} c(S) - w_S \varepsilon, & \text{if } \emptyset \neq S \subset N, \\ c(S), & \text{if } S = N. \end{cases} \quad (12)$$

Observe that the core of the game (N, c_ε) coincides with the weighted ε-core of the game (N, c). Let ε' be the largest ε for which the game (N, c_ε) has a nonempty core. Then the core of a game $(N, c_{\varepsilon'})$ coincides with the least weighted ε-core of a game (N, c). For $i \in N$, we can interpret $-w_i \varepsilon'$ as the minimal weighted cost $w_i \varepsilon$, or cross-subsidy, for which the weighted ε-core is not empty.

A reasonable way of selecting a unique point from the least weighted ε-core is selecting the nucleolus of the game $(N, c_{\varepsilon'})$. Under the assumption that every user has agreed to participate in a cross-subsidy with additional cost $-w_i \varepsilon'$, such a cost allocation would lexicographically maximize minimal excess.

Next, we show that the nucleolus of the game $(N, c_{\varepsilon'})$ can also be characterized by the collection $\mathscr{C}$.

Theorem 4. *The nucleolus of a game $(N, c_{\varepsilon'})$ is completely determined by the collection of core constraints associated with subsets in $\mathscr{C}$.*

PROOF. The proof will follow if we show that all core constraints induced by subsets not in $\mathscr{C}$ are redundant in the LP procedure for computing the nucleolus.

Let S be a nonempty proper subset of N which is not contained in $\mathscr{C}$. Let F be the set of open facilities in the optimal solution to SSCCL_S. For each $i \in F$, let S_i be the subset of customers serviced by a facility opened at node i in the optimal solution to SSCCL_S. For $S \subseteq N$, let $e_{\varepsilon'}(x, S) = c_{\varepsilon'}(S) - x(S)$ be the excess of S relative to x. Since the nucleolus of the game $(N, c_{\varepsilon'})$ is in the core of $(N, c_{\varepsilon'})$ we have that:

$$\begin{aligned} e_{\varepsilon'}(x, S) &= c_{\varepsilon'}(S) - x(S) \\ &= c(S) - x(S) - w_S \varepsilon' \\ &= \sum_{i \in F} (c(S_i) - x(S_i) - w_{S_i} \varepsilon') \\ &\geq (c(S_i) - x(S_i) - w_{S_i} \varepsilon') \\ &= e_{\varepsilon'}(x, S_i), \text{ for each } i \in F. \end{aligned} \quad (13)$$

If, similarly to discussion in T.2, for $Q \subseteq N$, $\mathrm{LP}_{t(Q)}$ denotes the linear program in the LP procedure for computing the nucleolus of the game $(N, c_{\varepsilon'})$ in which the constraint $\varepsilon \leq c_{\varepsilon'}(Q) - x(Q)$ corresponding to Q has been first satisfied as an equality at the corresponding optimal value $\varepsilon_{t(Q)}$, then

$$\varepsilon_{t(S_i)} = c(S_i) - x(S_i) - w_{S_i}\varepsilon', \text{ for } i \in F \tag{14}$$

and

$$\varepsilon_{t(S)} = c(S) - x(S) - w_S\varepsilon'. \tag{15}$$

Clearly, by (13) $\varepsilon_{t(S)} \geq \varepsilon_{t(S_i)}$ for all $i \in F$, which implies that the constraints (14) would be considered in the LP procedure for computing nucleolus prior to the constraint (15). Moreover, since $\varepsilon_{t(S)} = \sum_{i \in F} \varepsilon_{t(S_i)}$, constraints (14) completely determine the value of the total allocation to subset S and constraint (15) is redundant in the above procedure. □

Observe that if $w_i = 1$, for all $i \in N$ and if the nucleolus of $(N, c_{\varepsilon'})$ is the unique solution to the first linear program in the LP procedure for computing the nucleolus, then the nucleolus of the game $(N, c_{\varepsilon'})$ coincides with the nucleolus per capita of the game (N, c).

Remark. With respect to choice of weights w_i, $i \in N$, we think that they should be decided upon, on a case by case basis. A reasonable practical choice could be a set of weights, w_i, $i \in N$, such that w_i is some additive function of i's demand d_i. A simple natural choice for $i \in N$ might be $w_i = d_i/D$, where $D = \sum_{i \in N} d_i$ represents the total demand for service of the entire network.

6. Conclusions and Applications

In T.1 — T.4 we characterized respectively, the core, the nucleolus, the least weighted ε-core and a certain 'central point' of the least weighted ε-core of the SSCCL game with the constraints corresponding to a family of coalitions $\mathscr{C}$. A game theoretic analysis proposed above addresses the questions of efficiency as well as fairness. Besides efficient characterization of the core and the nucleolus, we also suggested efficient weighted cross-subsidization for cases when a fair subsidy-free cost allocation does not exist. Let us now discuss how efficient are the above characterizations. First let us make a natural assumption that demands, d_i, $i \in N$, are such that they, together with concentrator's capacity limit U determine some fixed upper bound k on the number of nodes that can be served by a single concentrator. In such cases the cardinality of set $\mathscr{C}$ is bounded by a polynomial. Moreover, in this case we can compute all sets S_i and their characteristic function values $c(S_i)$ in polynomial time. In practical situations underlying network $G = (N, E)$ is rarely a complete graph and will often posses some structure. For example our network G may have bounded degree of each node (number of adjacent vertices) or G may be even highly structured like a tree, a ring, series-parallel, partial k-tree etc. Often, recognizing such structure enables us to explicitly compute $c(N)$. When the optimal solution $c(N)$ could not be found, we approximate it with the best known heuristic solution from the network design problem. In all such cases the verification of the core existence, the nucleolus (if the core is not empty) as well as the least weighted ε-core and its 'central point' can be computed in polynomial time.

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Received: June, 1993
Accepted: April, 1994

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