

ON DIRECTLY ADDITIVE SELECTORS FOR CONVEX AND STAR BODIES

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ABSTRACT. The paper deals with direct additivity of selectors, as well for convex as for star bodies. Sections 3 and 4 concern selectors for convex bodies. In Section 5 two methods of extending a selector for convex bodies to a selector for star bodies are considered, both preserving direct additivity.

1. INTRODUCTION

As it was proved by Schneider in 1971 ([10]), among all the selectors for compact convex subsets of \mathbb{R}^n which are continuous with respect to the Hausdorff metric and equivariant under the isometries, only the Steiner point map is Minkowski additive.

We are interested in direct additivity of selectors, that is, roughly speaking, additivity with respect to direct sum.

In Sections 3 we give three examples of directly additive selectors for convex bodies: the Steiner point, the gravity centre, and the Čebyšev point. Section 4 concerns G -pseudo-centres, which are studied separately because in this case the situation is more complicated.

In Section 5 we consider two methods of extending selectors for convex bodies to those for star bodies.

Our motivation for studying selectors for star bodies is the following (compare [7], p.132). Various constructions for star bodies are defined in terms of radial function, which depends on the position of 0 in a star set considered. For instance, the intersection body IA , whose radial function is defined by

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the formula

$$\varrho_{IA}(u) := V_{n-1}(A \cap u^\perp)$$

for $u \in S^{n-1}$, (see [3]), depends on the choice of the origin. Let us recall that the notion of intersection body plays an essential role in geometric tomography (see [5]).

2. PRELIMINARIES

We follow, in principle, terminology and notation used in [12]. In particular, bd, int, cl, conv, lin, and aff are, respectively, boundary, interior, closure, convex hull, linear hull, and affine hull. For $v \in \mathbb{R}^n \setminus \{0\}$,

$$\text{pos } v := \{\lambda v \mid \lambda \geq 0\}.$$

The k -dimensional Lebesgue measure in Euclidean k -dimensional space is λ_k , the k -dimensional spherical measure in a k -dimensional unit sphere is σ_k (or simply σ , if it does not lead to a confusion). The k -dimensional Hausdorff measure is \mathcal{H}^k .

B^n and S^{n-1} are the unit ball and the unit sphere in \mathbb{R}^n , respectively, and \mathcal{K}^n is the class of compact convex subsets of \mathbb{R}^n . However, a convex body in \mathbb{R}^n is assumed to have the non-empty interior, while Schneider in [12] refers to any compact convex set as a convex body; \mathcal{K}_0^n is the class of convex bodies in \mathbb{R}^n . More generally, for any Euclidean space E , the class of all compact convex subsets of E and the class of all convex bodies in E are $\mathcal{K}(E)$ and $\mathcal{K}_0(E)$, respectively.

For any $A \in \mathcal{K}^n$, the intrinsic volume of order i of A is $V_i(A)$. Let us recall that $V_0(A) = 1$ for every A .

Let $\mathcal{F} \subset \mathcal{K}^n$. A map $s : \mathcal{F} \rightarrow \mathbb{R}^n$ is called a selector for \mathcal{F} if $s(A) \in A$ for every $A \in \mathcal{F}$.

We shall also consider selectors for star sets. Let $A \subset \mathbb{R}^n$. The kernel of A is defined by the formula

$$\ker A := \{a \in A \mid \forall x \in A \Delta(a, x) \subset A\},$$

where $\Delta(a, x)$ is the segment with endpoints a, x .

A subset A of \mathbb{R}^n is called a star set if $\ker A \neq \emptyset$; a compact star set A in \mathbb{R}^n is called a star body if $\text{cl int } A = A$.

Let \mathcal{S}^n be the class of compact star sets in \mathbb{R}^n and \mathcal{S}_0^n the class of compact star bodies in \mathbb{R}^n . Similarly as we do for convex bodies, we use the symbols $\mathcal{S}(E)$ and $\mathcal{S}_0(E)$ for the classes of suitable star sets in any Euclidean space E .

Let $\mathcal{F} \subset \mathcal{S}^n$. A map $s : \mathcal{F} \rightarrow \mathbb{R}^n$ is a selector for \mathcal{F} if and only if $s(A) \in \ker A$ for every $A \in \mathcal{F}$ (see [7]).

A selector $s : \mathcal{F} \rightarrow \mathbb{R}^n$ is said to be equivariant under an isometry f of \mathbb{R}^n onto itself if for every $A \in \mathcal{F}$

$$s(f(A)) = f(s(A)). \tag{2.1}$$

For any $m \leq n$, let \mathcal{E}_m^n be the family of m -dimensional affine subspaces of \mathbb{R}^n , and let

$$\mathcal{E}^n := \bigcup_{m \leq n} \mathcal{E}_m^n.$$

LEMMA 2.1. *Let $E \in \mathcal{E}_m^n$ for some $m \leq n$ and let $f_1, f_2 : E \rightarrow \mathbb{R}^m$ be isometries. If $A \in \mathcal{K}_0(E)$ and a selector $s : \mathcal{K}_0^m \rightarrow \mathbb{R}^m$ is equivariant under the isometries of \mathbb{R}^m , then*

$$f_1^{-1}(s(f_1(A))) = f_2^{-1}(s(f_2(A))). \quad (2.2)$$

PROOF. Let $f := f_2 \circ f_1^{-1}$. Then f is an isometry of \mathbb{R}^m mapping $f_1(A)$ on $f_2(A)$. Thus, by (2.1),

$$f(s(f_1(A))) = s(f_2(A)),$$

which is equivalent to (2.2). \square

Let now \oplus be the direct sum: if $\mathbb{R}^n = E_1 \oplus E_2$ for two affine subspaces E_1, E_2 of \mathbb{R}^n with positive dimensions, then for every $X_i \subset E_i$, $i = 1, 2$,

$$X_1 \oplus X_2 := X_1 + X_2.$$

Thus, direct sum operation is a restriction of the Minkowski addition.

For any $A \in \mathcal{K}^n$, let $\xi_A : \mathbb{R}^n \rightarrow A$ be the metric projection (i.e., the nearest point map) of \mathbb{R}^n on A . We shall need the following property of this map.

LEMMA 2.2. *If $\mathbb{R}^n = E_1 \oplus E_2$ for E_1, E_2 orthogonal, $a_i \in E_i$, and $X_i \in \mathcal{K}(E_i)$ for $i = 1, 2$, then*

$$\xi_{X_1 \oplus X_2}(a_1 + a_2) = \xi_{X_1}(a_1) + \xi_{X_2}(a_2).$$

PROOF. Let $a = a_1 + a_2$. Then

$$\begin{aligned} \inf\{\|x - a\| \mid x \in X_1 \oplus X_2\} &= \\ &= \inf\{\sqrt{\|x_1 - a_1\|^2 + \|x_2 - a_2\|^2} \mid x_i \in X_i, \ i = 1, 2\}. \end{aligned}$$

Thus, if $x_i \in E_i$ and $x = x_1 + x_2$, then $x = \xi_{X_1 \oplus X_2}(a_1 + a_2)$ if and only if $x_i = \xi_{X_i}(a_i)$ for $i = 1, 2$. This completes the proof. \square

3. DIRECTLY ADDITIVE SELECTORS FOR CONVEX BODIES

We begin with selectors for \mathcal{K}^n . For our purpose it is reasonable to consider a sequence of selectors, $(s^{(m)} : \mathcal{K}^m \rightarrow \mathbb{R}^m)_{m \leq n}$ instead of an individual selector for a fixed n .

In view of Lemma 2.1, for a given n , every sequence $(s^{(m)} : \mathcal{K}^m \rightarrow \mathbb{R}^m)_{m \leq n}$ of equivariant selectors determines the family of selectors,

$$\mathbf{s} := (s_E : \mathcal{K}(E) \rightarrow E)_{E \in \mathcal{E}^n},$$

defined as follows: for every $E \in \mathcal{E}_m^n$ and $A \in \mathcal{K}(E)$,

$$s_E(A) := f^{-1}s^{(m)}f(A), \quad (3.1)$$

where $f : E \rightarrow \mathbb{R}^m$ is an isometry.

The following is evident.

PROPOSITION 3.1. *Let \mathbf{s} be the family determined by a sequence of equivariant selectors, defined by (3.1). If $E, E' \in \mathcal{E}_m^n$ for some $m \leq n$, then for every isometry $f : E \rightarrow E'$ and $A \in \mathcal{K}(E)$*

$$f(s_E(A)) = s_{E'}(f(A)).$$

In particular, every member of the family \mathbf{s} is equivariant under the isometries of the corresponding affine subspace.

Let us consider three examples:

For $A \in \mathcal{K}_0^n$, let $c^{(n)}(A)$ be the centroid of A (i.e. the gravity centre with respect to λ_n) :

$$c^{(n)}(A) := \frac{1}{\lambda_n(A)} \int_A x d\lambda_n(x). \quad (3.2)$$

Further, for any $A \in \mathcal{K}^n$, let $p^{(n)}(A)$ and $\check{c}^{(n)}(A)$ be, respectively, the Steiner point and the Čebyšev point of A :

$$p^{(n)}(A) := \frac{1}{\kappa_n} \int_{S^{n-1}} u h_A(u) d\sigma(u), \quad (3.3)$$

and $\check{c}^{(n)}(A)$ is the centre of the smallest ball in \mathbb{R}^n containing A .

As is well known, all these selectors are equivariant under the isometries.

For $E \in \mathcal{E}^n$ the points $c_E(A)$, $p_E(A)$, and $\check{c}_E(A)$ are the *centroid*, the *Steiner point*, and the *Čebyšev point of A with respect to E* .

We now define direct additivity as follows.

DEFINITION 3.2. *Let $\mathbf{s} = (s_E)_{E \in \mathcal{E}^n}$ be the family determined by a sequence $(s^{(m)})_{m \leq n}$ of equivariant selectors (see (3.1)). The family \mathbf{s} is directly additive (or, the sequence $(s^{(m)})_{m \leq n}$ is directly additive) if and only if*

$$s^{(n)}(A_1 \oplus A_2) = s_{E_1}(A_1) + s_{E_2}(A_2) \quad (3.4)$$

whenever $E_i = \text{aff } A_i$ and $\mathbb{R}^n = E_1 \oplus E_2$ with E_1, E_2 orthogonal.

THEOREM 3.3. *The family of centroids, \mathbf{c} , of the Steiner point maps, \mathbf{p} , and the family of Čebyšev point maps, $\check{\mathbf{c}}$, are directly additive.*

PROOF. Since the Lebesgue measure is multiplicative, that is, for every pair of orthogonal subspaces E_1, E_2 of \mathbb{R}^n with $\dim E_i = n_i$ and every Borel sets X_1, X_2 with $X_i \subset E_i$

$$\lambda_n(X_1 \oplus X_2) = \lambda_{n_1}(X_1) \cdot \lambda_{n_2}(X_2),$$

from (3.2) it follows that \mathbf{c} is directly additive.

Since the Steiner point is independent of dimension of the ambient space, that is, for every subspace E of \mathbb{R}^n with $\dim E < n$ and every $A \in \mathcal{K}(E)$

$$p_E(A) = p^{(n)}(A),$$

the Minkowski additivity of p implies direct additivity of \mathbf{p} .

It remains to prove direct additivity of \check{c} .

Let $\mathbb{R}^n = E_1 \oplus E_2$ for some orthogonal affine subspaces E_1, E_2 with $\dim E_i \geq 1$; let aff $A_i = E_i$, and $A_i \in \mathcal{K}_0(E_i)$ for $i=1,2$.

Let $a_i = \check{c}_{E_i}(A_i)$, and let $B(a_i, r_i)$ be the Čebyšev ball of A_i in E_i for $i = 1, 2$ (i.e. the smallest ball in E_i containing A_i).

Let $X := B(a_1, r_1) \oplus B(a_2, r_2)$; then $A_1 \oplus A_2 \subset X$. Let $a = a_1 + a_2$. Since, evidently, a is the symmetry centre of X , it follows that $a = \check{c}^{(n)}(X)$.

Let \check{B} be the Čebyšev ball of X . Then $\check{B} = B(a, r)$, where

$$r = \sqrt{(r_1)^2 + (r_2)^2}.$$

Suppose that \check{B} is not the Čebyšev ball of $A_1 \oplus A_2$. Thus, there exists a ball B_0 containing $A_1 \oplus A_2$, with a radius $r_0 < r$. Let $B_{0,i} := B_0 \cap E_i$ for $i = 1, 2$. Then the ball $B_{0,i}$ in E_i contains A_i , whence its radius $r_{0,i}$ is at least r_i . But

$$r_0 = \sqrt{(r_{0,1})^2 + (r_{0,2})^2} < r,$$

a contradiction. Hence \check{B} is the Čebyšev ball of $A_1 \oplus A_2$, and thus $a = \check{c}^{(n)}(A_1 \oplus A_2)$, i.e.,

$$\check{c}^{(n)}(A_1 \oplus A_2) = \check{c}_{E_1}(A_1) + \check{c}_{E_2}(A_2).$$

□

Let us notice that direct additivity of \mathbf{p} is a particular case of much more general fact (Theorem 3.5).

Let us recall that every $A \in \mathcal{K}^n$ determines a sequence of curvature measures $\Phi_k(A, \cdot)$, $k = 0, \dots, n$ (see [11]). This notion was originally introduced and studied by H. Federer in [2] for arbitrary set A with positive reach in \mathbb{R}^n ; we shall need the following version of his Theorem 5.14 (4) in [2] p. 422.

THEOREM 3.4. *Let $A_i \in \mathcal{K}_0(E_i)$ and $A_i \subset E_i$ for orthogonal affine subspaces E_1, E_2 of \mathbb{R}^n , with $\dim E_i = n_i$, where $n_1 + n_2 = n$, and let $k \in \{0, \dots, n\}$. Then for every Borel sets X_i in E_i , $i = 1, 2$,*

$$\Phi_k(A_1 \oplus A_2, X_1 \oplus X_2) = \sum_{i_1+i_2=k} \Phi_{i_1}(A_1, X_1) \cdot \Phi_{i_2}(A_2, X_2).$$

Let us consider the curvature centroids, i.e. the gravity centres with respect to curvature measures:

$$p_i(A) := c_{\Phi_{n-i}(A, \cdot)}(A), \tag{3.7}$$

In particular, p_n is the Steiner point map, i.e., $p_n = p^{(n)}$. (see [12]). As a consequence of the Federer Theorem 3.4, we obtain the following result concerning a relationship between the selectors p_0, \dots, p_n .

THEOREM 3.5. *Let E_1, E_2 be two orthogonal affine subspaces of \mathbb{R}^n with $\dim E_k = n_k$ and $n_1 + n_2 = n$. Let $A_1, A_2 \in \mathcal{K}^n$ and $A_k \subset E_k$. Then, for every $i \in \{0, \dots, n\}$*

$$p_{n-i}(A_1 \oplus A_2) = \sum_{i_1+i_2=i} \alpha_{i_1, i_2} \cdot ((p_{n_1-i_1})_{E_1}(A_1) + (p_{n_2-i_2})_{E_2}(A_2)),$$

where, for $i_k \leq n_k$, $k = 1, 2$,

$$\alpha_{i_1, i_2} := \frac{V_{i_1}(A_1)V_{i_2}(A_2)}{V_i(A_1 \oplus A_2)}. \quad (3.8)$$

PROOF. By Theorem 3.4 combined with the Fubini theorem,

$$\begin{aligned} \int_{A_1 \oplus A_2} x d\Phi_i(A_1 \oplus A_2, x) &= \sum_{i_1+i_2=i} \int_{A_1} \int_{A_2} (x_1 + x_2) d\Phi_{i_1}(A_1, x_1) d\Phi_{i_2}(A_2, x_2) \\ &= \sum_{i_1+i_2=i} V_{i_1}(A_1)V_{i_2}(A_2)((p_{n_1-i_1})_{E_1}(A_1) + (p_{n_2-i_2})_{E_2}(A_2)). \end{aligned}$$

To complete the proof, it now remains to use (3.7) and (3.8). \square

Finally, let us mention that direct additivity of centre of the minimal ring was studied in [6]; it was proved that this selector was not directly additive.

4. PROBLEM OF DIRECT ADDITIVITY FOR G -PSEUDO-CENTRES

In [8], for any subgroup G of $O(n)$, the notion of G -pseudo-centre of a convex body in \mathbb{R}^n was defined:

for $A \in \mathcal{K}_0^n$, the set $P_G(A)$ of G -pseudo-centres of A consists of maximizers of the function

$$A \ni x \mapsto V_n\left(\bigcap_{g \in G} g(A - x)\right).$$

Generally, the set $P_G(A)$ does not have to be a singleton. However, if G is generated by the central symmetry σ_0^n of \mathbb{R}^n , then every A has a unique G -pseudo-centre, $p_G(A)$ (compare [1])¹; moreover, $\langle \sigma_0^n \rangle$ is the only group with this property.

Further, for any $G \subset O(n)$, if $\text{fix}G = \{0\}$, then every strictly convex A has a unique G -pseudo-centre (see [8, 9]).

We start with the following

DEFINITION 4.1. *Let (E_1, E_2) be a pair of orthogonal linear subspaces of \mathbb{R}^n , with $\mathbb{R}^n = E_1 \oplus E_2$, and let G_{E_i} be a group of linear isometries of E_i for $i = 1, 2$.*

¹In [1] this point is called *quasi-centre* of A

(i) For every $g_i \in G_i$, $i = 1, 2$, the map $g_1 \star g_2 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is defined by

$$(g_1 \star g_2)(x_1 + x_2) := g_1(x_1) + g_2(x_2) \text{ for every } x_i \in E_i, i = 1, 2.$$

(ii) $G_{E_1} \star G_{E_2} := \{g_1 \star g_2 \mid g_i \in G_{E_i}\}$.

REMARK 4.2. Under the assumptions of 4.1, the group $G_{E_1} \star G_{E_2}$ is isomorphic to the Cartesian product of these two groups.

We shall now prove the following (compare 3.7 in [8]).

THEOREM 4.3. Let E_1, E_2 be orthogonal linear subspaces of \mathbb{R}^n with $\dim E_i = n_i$ and $n_1 + n_2 = n$, and let G_i be a group of linear isometries of E_i . Then, for every $A_i \in \mathcal{K}(E_i)$, $i = 1, 2$,

$$P_{G_1 \star G_2}(A_1 \oplus A_2) = P_{G_1}(A_1) \oplus P_{G_2}(A_2).$$

If, in particular, $P_{G_i}(A_i)$ is a singleton for $i = 1, 2$, then $P_{G_1 \star G_2}(A_1 \oplus A_2)$ is a singleton too, and

$$p_{G_1 \star G_2}(A_1 \oplus A_2) = p_{G_1}(A_1) + p_{G_2}(A_2).$$

PROOF. Let $A = A_1 \oplus A_2$ and let $x \in A$. Then $x = x_1 + x_2$ for some $x_i \in A_i$, $i = 1, 2$, and

$$\begin{aligned} \bigcap_{g \in G_1 \star G_2} g(A - x) &= \bigcap_{g_1 \in G_1} \bigcap_{g_2 \in G_2} g_1(A_1 - x_1) \oplus g_2(A_2 - x_2) \\ &= \left(\bigcap_{g_1 \in G_1} g_1(A_1 - x_1) \right) \oplus \left(\bigcap_{g_2 \in G_2} g_2(A_2 - x_2) \right). \end{aligned}$$

Thus

$$x \in P_{G_1 \star G_2}(A) \iff x_i \in P_{G_i}(A_i) \text{ for } i = 1, 2. \quad (4.1)$$

□

Let us notice that the property of G -pseudo-centres described by Theorem 4.3 is weaker than direct additivity, because G_i depends on E_i . To define the suitable family \mathbf{p}_G of selectors for all the \mathcal{E}^n , we need equivariance with respect to the isometries (see Lemma 1.1). As was proved in [9] (see (2.2) in Example 2.2 and Example 2.5), G -pseudo-centre p_G is equivariant with respect to all isometries if and only if $G = \langle \sigma_0 \rangle$.

COROLLARY 4.4. The sequence $(p_{G_n} : \mathcal{K}^n \rightarrow \mathbb{R}^n)_{n \in \mathbb{N}}$ of G_n -pseudo-centres induces the family \mathbf{p}_G if and only if G_n is generated by σ_0^n for every n . This unique family is directly additive.

5. EXTENSIONS OF SELECTORS FROM CONVEX TO STAR BODIES

Let us recall two methods of extending selectors for \mathcal{K}_0^n to selectors for \mathcal{S}_0^n (compare [7]). The first method can be applied to an arbitrary selector $s : \mathcal{K}_0^n \rightarrow \mathbb{R}^n$: its extension $\bar{s} : \mathcal{S}_0^n \rightarrow \mathbb{R}^n$ is defined by the formula

$$\bar{s}(A) := \xi_{\ker A}(s(\text{conv } A)) \quad \text{for every } A \in \mathcal{S}_0^n. \quad (5.1)$$

The second method can be applied to a selector s which satisfies the following condition:

- (*) There exists a family $(\Phi_A : A \rightarrow \mathbb{R})_{A \in \mathcal{K}^n}$ of functions such that
- (i) for every $A \in \mathcal{K}_0^n$, the point $s(A)$ is the unique maximizer (minimizer) of Φ_A ,
 - (ii) for every $A \in \mathcal{K}_0^n$ and $X \in \mathcal{K}^n$ with $X \subset A$, the function $\Phi_A|_X$ has the unique maximizer (minimizer), $s(A, X)$.

If s satisfies (*), then its extension $\tilde{s} : \mathcal{S}_0^n \rightarrow \mathbb{R}^n$ is defined by the formula

$$\tilde{s}(A) := s(\text{conv } A, \ker A). \quad (5.2)$$

In [7] the second method was introduced ² for s being the radial centre map $r_\phi : \mathcal{K}_0^n \rightarrow \mathbb{R}^n$ associated with a function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. If $n \geq 2$, the function Φ_A for which $r_\phi(A)$ is the maximizer is then defined by

$$\Phi_A(x) := \int_{S^{n-1}} \phi \varrho_{A-x}(u) d\sigma(u) \quad \text{for every } x \in A. \quad (5.3)$$

The problem of direct additivity of radial centre maps is discussed in [4].

If $\mathbf{s} = (s_E)_{E \in \mathcal{E}^n}$ is the family defined by (3.1) for a given sequence $(s^{(n)})$ of selectors, then the corresponding families for extended selectors will be denoted by $\bar{\mathbf{s}}$ and $\tilde{\mathbf{s}}$.

We shall now prove that the extensions $s \mapsto \bar{s}$ and $s \mapsto \tilde{s}$ preserve direct additivity (Theorems 5.3 and 5.4). Let us start with the following.

PROPOSITION 5.1. (i) For every subsets A_1, A_2 of \mathbb{R}^n

$$\ker(A_1 + A_2) \supset \ker A_1 + \ker A_2.$$

- (ii) If $\mathbb{R}^n = E_1 \oplus E_2$ (for E_1, E_2 non-necessarily orthogonal) and $A_i \subset E_i$ for $i = 1, 2$, then

$$\ker(A_1 \oplus A_2) = \ker A_1 \oplus \ker A_2.$$

PROOF. (i) : Let $x_i \in \ker A_i$ for $i = 1, 2$ and $x = x_1 + x_2$. Then for every $a \in A_1 + A_2$ there exist $a_i \in A_i$ for $i = 1, 2$ such that $a = a_1 + a_2$ and $\Delta(a_i, x_i) \subset A_i$. Since

$$\forall t \in [0, 1] \quad (1-t)x + ta = ((1-t)x_1 + ta_1) + ((1-t)x_2 + ta_2),$$

²On the p. 139 of [7] the restriction to \mathcal{T}^n is not needed; \mathcal{T}^n should be replaced by \mathcal{S}_0^n .

³In [7] this selector was denoted by M_ϕ .

it follows that

$$\Delta(x, a) \subset \Delta(x_1, a_1) + \Delta(x_2, a_2) \subset A_1 + A_2.$$

Thus $x \in \ker(A_1 + A_2)$.

- (ii) : Let now $x \in \ker(A_1 \oplus A_2)$. Then $x = x_1 + x_2$ for a unique pair $(x_1, x_2) \in A_1 \times A_2$. Take $a_i \in A_i$ for $i = 1, 2$ and let $a = a_1 + a_2$; then $\Delta(x, a) \subset A_1 \oplus A_2$, i.e.

$$\forall t \in [0, 1] \quad (1-t)x + ta \in A_1 \oplus A_2.$$

Thus $(1-t)x_i + ta_i \in A_i$ for every $t \in [0, 1]$, whence $x_i \in \ker A_i$ for $i = 1, 2$. Therefore

$$x \in \ker A_1 \oplus \ker A_2. \quad \square$$

As a direct consequence of 5.1.(i), we obtain

COROLLARY 5.2. *If A_1 and A_2 are star sets in \mathbb{R}^n , then so is $A_1 + A_2$.*

We shall now prove the following.

THEOREM 5.3. *If \mathbf{s} is a directly additive family of selectors for \mathcal{K}_0^n , then $\bar{\mathbf{s}}$ is a directly additive family of selectors for \mathcal{S}_0^n .*

PROOF. By (5.1), $\bar{\mathbf{s}}^{(n)}(A) = \xi_{\ker A}(s^{(n)}(\text{conv } A))$. Let $A = A_1 \oplus A_2$ with $A_i \in \mathcal{K}(E_i)$ for $i = 1, 2$. Since \mathbf{s} is directly additive and $\text{conv } A = \text{conv } A_1 \oplus \text{conv } A_2$, by Proposition 5.1(ii) combined with Lemma 2.2 it follows that

$$\bar{\mathbf{s}}^{(n)}(A) = \xi_{\ker A_1}(s_{E_1}(\text{conv } A_1)) + \xi_{\ker A_2}(s_{E_2}(\text{conv } A_2)) = \bar{\mathbf{s}}_{E_1}(A_1) + \bar{\mathbf{s}}_{E_2}(A_2). \quad \square$$

By 3.3, Theorem 5.3 applies to the centroid, the Steiner point, and the Čebyšev point; by 4.4, it applies to the $\langle \sigma_0 \rangle$ -pseudo-centre:

COROLLARY 5.4. *The families $\bar{\mathbf{c}}$, $\bar{\mathbf{p}}$, $\bar{\mathbf{c}}$, and $\bar{\mathbf{p}}_{\langle \sigma_0^n \rangle}$ are directly additive.*

Let us now pass to the second method of extending selectors from convex to star bodies.

THEOREM 5.5. *For every $n \geq 2$, let $s^{(n)} : \mathcal{K}_0^n \rightarrow \mathbb{R}^n$ be a selector satisfying condition (*). If for any $A_i \in \mathcal{K}_0(E_i)$ and $X_i \in \mathcal{K}(E_i)$ with $X_i \subset A_i$ for $i = 1, 2$*

$$s^{(n)}(A_1 \oplus A_2, X_1 \oplus X_2) = s_{E_1}(A_1, X_1) + s_{E_2}(A_2, X_2), \quad (5.4)$$

then $\tilde{\mathbf{s}}$ is directly additive for \mathcal{S}_0^n .

PROOF. Since $\text{conv}(A_1 \oplus A_2) = \text{conv } A_1 \oplus \text{conv } A_2$ and, by 5.1(ii), $\ker(A_1 \oplus A_2) = \ker A_1 \oplus \ker A_2$, it follows that for every $A_i \in \mathcal{S}_0(E_i)$, $i = 1, 2$,

$$\begin{aligned} \tilde{\mathbf{s}}^{(n)}(A_1 \oplus A_2) &= \mathbf{s}(\text{conv}(A_1 \oplus A_2), \ker(A_1 \oplus A_2)) \\ &= s_{E_1}(\text{conv } A_1, \ker A_1) + s_{E_2}(\text{conv } A_2, \ker A_2) \\ &= \tilde{\mathbf{s}}_{E_1}(A_1) + \tilde{\mathbf{s}}_{E_2}(A_2). \quad \square \end{aligned}$$

EXAMPLE 5.6. The family of Čebyšev point maps, \check{c} , is directly additive (see 3.3). Moreover, $\check{c}^{(n)}$ satisfies condition (*); indeed, for every $A \in \mathcal{K}^n$, the point $\check{c}^{(n)}(A)$ is the unique minimizer of the function $\Psi_A : A \rightarrow \mathbb{R}$ defined by the formula

$$\Psi_A(x) := \inf\{r > 0 \mid B(x, r) \supset A\},$$

thus $\check{c}^{(n)}$ satisfies the first part of (*); it is easy to verify that it also satisfies the second part. Moreover, (5.4) holds. Hence \check{c} is directly additive.

EXAMPLE 5.7. By 4.4 combined with 5.5, the family $\tilde{\mathbf{p}}_{\langle \sigma_0^n \rangle}$ is directly additive, because the sequence of $\langle \sigma_0^n \rangle$ -pseudo-centres satisfies condition (*). Indeed, the first part of condition (*) is obvious; it is also easy to prove the second part, because the proof of the uniqueness in [1] (compare Satz 1) is based on Hilfsatz 1, p.209, which can be applied in our situation as well.

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