GLASNIK MATEMATIČKI
Vol. 39(59)(2004), 139 – 144

A GENERALIZATION OF THE 0-NUMERICAL RANGE

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Abstract. Let H be a complex Hilbert space. Given a bounded linear operator A on H, we describe the set $R^n(A) = \{V^*AW : V, W : \mathbf{C}^n \to H, V^*V = W^*W = I_n, V^*W = 0\}$. It is shown that the closed matricial convex hull of $R^n(A)$ is a closed ball of radius $\min\{\|A - \lambda I\| : \lambda \in \mathbf{C}\}$ centered at the origin.

1. Introduction

Throughout this paper H will denote a complex Hilbert space with an inner product $(\cdot\,,\cdot)$. By $B(H)$ we denote the algebra of all bounded linear operators on H.

In [15] E. L. Stolov showed that the 0-numerical range of a linear operator A acting on a finite dimensional Hilbert space H (i.e., the set $W_0(A) = \{(Ax, y) : x, y \in H, (x, x) = (y, y) = 1, (x, y) = 0\}$) is a circular disc with center at the origin and with radius $\min\{\|A - \lambda I\| : \lambda \in \mathbf{C}\}$. The infinite dimensional analogue of this theorem was given in [8, Proposition 2.11].

In this paper we will consider the matricial generalization of the 0-numerical range of $A \in B(H)$. More precisely, our aim is to provide for $R^n(A) = \{V^*AW : V, W : \mathbf{C}^n \to H, V^*V = W^*W = I_n, V^*W = 0\}$ a theorem analogous to the theorem of E. L. Stolov.

One obvious consequence of Stolov's theorem is that $\sup\{|\lambda| : \lambda \in W_0(A)\}$ is equal to $\min\{\|A - \lambda I\| : \lambda \in \mathbf{C}\}$. (For hermitian $A \in B(H)$ this result was first obtained by Mirsky ([11]).) As it will be seen, the same assertion is valid for the set $R^n(A)$.

2000 *Mathematics Subject Classification.* 47A12.
Key words and phrases. q-numerical range of an operator.

2. Main result

Definition 2.1. *For an operator $T \in B(H)$ we define the set*

$$R^n(A) = \{V^*AW : V, W : \mathbf{C}^n \to H, V^*V = W^*W = I_n, V^*W = 0\}.$$

Remark 2.2. Observe that the operators V and W from the above definition are isometries from $\mathbf{C}^n$ to H with orthogonal ranges. Therefore, to avoid the trivial case $R^n(A) = \emptyset$, we shall assume that the dimension of H is greater than or equal to $2n$.

Remark 2.3. Note that x and y are orthogonal unit vectors of H if and only if $V, W : \mathbf{C} \to H$, where $V(1) = y$ and $W(1) = x$, are isometries with orthogonal ranges. Then (identifying $B(\mathbf{C})$ with $\mathbf{C}$) we have $V^*AW = (Ax, y)$. So, in the case $n = 1$ the set $R^1(A)$ coincides to the 0-numerical range of an operator A. (For the definition and more details see [8, 10, 15, 16]).

Remark 2.4. Similar concept to the set $R^n(A)$ is the spatial matricial range of $A \in B(H)$ defined by $V^n(A) = \{V^*AV : V : \mathbf{C}^n \to H, V^*V = I_n\}$. When $n = 1$ this set reduces to the classical numerical range of A, i.e., $W(A) = \{(Ax, x) : x \in H, \|x\| = 1\}$. However, the set $V^n(A)$ lacks an important property of $W(A)$: it need not be convex if $n > 1$ ([4, p. 142]). The closure of $W(A)$, known as the numerical range of A, is the set of all $\phi(A)$, where ϕ ranges over all norm-one positive linear functionals on $B(H)$. Using completely positive maps, W. B. Arveson ([1]) generalized the concept of numerical range in defining matricial range. J. Bunce and N. Salinas proved in [5, Theorem 3.5] that the matricial convex hull of $V^n(A)$ has the matricial range of A as its closure. Basic references for the numerical and matricial ranges are [1, 3, 4, 5, 6, 7, 13, 14].

One other familiar concept is the set $\{V^*AW : V, W : \mathbf{C}^n \to H, V^*V = W^*W = I_n\}$ where H is a finite dimensional space which dimension is greater than or equal to n. In [9] the authors examine the conditions on A under which this set is convex or starshaped.

Remark 2.5. If H is a finite dimensional space then $R^n(A)$ is a compact set. Indeed, let us take an arbitrary sequence $(V_i^*AW_i)_i$ in $R^n(A)$. Since (V_i) and (W_i) are the bounded sequences of isometries in the finite dimensional space $B(\mathbf{C}^n, H)$ of all linear operators from $\mathbf{C}^n$ to H such that $V_i^*W_i = 0$ they have the subsequences which converge to some isometries in $B(\mathbf{C}^n, H)$ with orthogonal ranges. Therefore, $(V_i^*AW_i)_i$ must also have a subsequence that converges in $R^n(A)$. Hence, $R^n(A)$ is compact.

Before stating our results we introduce some notation.

The matricial convex hull of a subset S of $B(\mathbf{C}^n)$, denoted by $\mathrm{mconv}(S)$, is the set of all finite sums of the form $\sum_i T_i^* A_i T_i$, where $A_i \in S$ and where the operators $T_i \in B(\mathbf{C}^n)$ are such that $\sum_i T_i^* T_i = I_n$.

We denote by S^- the topological closure of a set S.

The result which follows resembles those obtained by E. L. Stolov ([15]) and by C. K. Li, P. P. Mehta and L. Rodman ([8, Proposition 2.11]).

THEOREM 2.6. *Let $A \in B(H)$. Then*

$$\mathrm{mconv}(R^n(A)^-) = (\mathrm{mconv}(R^n(A)))^- = \{L \in B(\mathbf{C}^n) : \|L\| \leq r\},$$

where $r = \min\{\|A - \lambda I\| : \lambda \in \mathbf{C}\}$. Particularly, if H is finite dimensional then

$$\mathrm{mconv}(R^n(A)) = \{L \in B(\mathbf{C}^n) : \|L\| \leq r\}.$$

PROOF. The first equality follows by [6, Corollary 2.5] since $R^n(A)$ is a bounded subset of $B(\mathbf{C}^n)$ and $\mathbf{C}^n$ is finite dimensional.

Take any $V^*AW \in R^n(A)$. Since $V^*W = 0$, for every $\lambda \in \mathbf{C}$ we have

$$\|V^*AW\| = \|V^*(A - \lambda I)W\| \leq \|V^*\|\|A - \lambda I\|\|W\| = \|A - \lambda I\|.$$

Hence, $\|V^*AW\| \leq \min\{\|A - \lambda I\| : \lambda \in \mathbf{C}\} = r$. We conclude that $R^n(A) \subseteq \{L \in B(\mathbf{C}^n) : \|L\| \leq r\}$. Since $\{L \in B(\mathbf{C}^n) : \|L\| \leq r\}$ is a compact matricially convex set, it follows that $(\mathrm{mconv}(R^n(A)))^- \subseteq \{L \in B(\mathbf{C}^n) : \|L\| \leq r\}$.

Recall that the unit ball in $B(\mathbf{C}^n)$ is the closed convex hull of the set of all unitary operators of $B(\mathbf{C}^n)$ ([12, Proposition 1.1.12]). Therefore, for the opposite inclusion it is enough to show that $(\mathrm{mconv}(R^n(A)))^-$ contains every normal operator in $B(\mathbf{C}^n)$ whose norm is less than or equal to r. Hence, let L be a normal operator in $B(\mathbf{C}^n)$ with $\|L\| \leq r$. Denote by $\{e_1, \ldots, e_n\}$ an orthonormal basis of $\mathbf{C}^n$ consisting of eigenvectors of L. Let λ_i be the eigenvalue of L corresponding to e_i and let $P_i \in B(\mathbf{C}^n)$ be the orthogonal projection on the subspace spanned by e_i, $i = 1, \ldots, n$. Clearly, $L = \sum_{i=1}^n \lambda_i P_i$ and $\sum_{i=1}^n P_i = I_n$. Given $0 < \varepsilon < 1$ we get $|\lambda_i - \varepsilon\lambda_i| = (1-\varepsilon)|\lambda_i| \leq (1-\varepsilon)\|L\| \leq (1 - \varepsilon)r < r$, so by [15] (i.e. [8, Proposition 2.11]) there exist two orthogonal unit vectors $x_i, y_i \in H$ such that

$$\lambda_i - \varepsilon\lambda_i = (Ax_i, y_i)$$

for $i = 1, \ldots, n$. Now, for $x_i, y_i \in H$ and a unit vector e_i one can find two isometries $V_i, W_i : \mathbf{C}^n \to H$ with orthogonal ranges such that $V_i e_i = y_i$ and

$W_i e_i = x_i$, $i = 1, \ldots, n$. From this we have $(V_i^* A W_i e_i, e_i) = (Ax_i, y_i)$, so $P_i V_i^* A W_i P_i = (Ax_i, y_i) P_i$. Therefore,

$$L = \sum_{i=1}^{n} \lambda_i P_i = \sum_{i=1}^{n} (Ax_i, y_i) P_i + \sum_{i=1}^{n} \varepsilon \lambda_i P_i = \sum_{i=1}^{n} P_i V_i^* A W_i P_i + \varepsilon L,$$

so we obtain

$$\|L - \sum_{i=1}^{n} P_i V_i^* A W_i P_i\| = \|\varepsilon L\| \le \varepsilon r.$$

Hence, the arbitrariness of $0 < \varepsilon < 1$ implies $L \in (\mathrm{mconv}(R^n(A)))^-$.

The second assertion follows from the first one and Remark 2.5. ∎

Given a bounded linear operator A defined on a complex Hilbert space H, Mirsky's constant of A ([11]), i.e.,

$$\sup\{|(Ax, y)| : x, y \in H, (x, x) = (y, y) = 1, (x, y) = 0\}$$

is equal to $\min\{\|A - \lambda I\| : \lambda \in \mathbf{C}\}$, which is an obvious consequence of the result of [15] (see also [8, Proposition 2.11]). In what follows we shall see that an analogous assertion holds for the set $R^n(A)$.

THEOREM 2.7. *Let $A \in B(H)$. Then*

$$\sup\{\|V^* A W\| : V, W : \mathbf{C}^n \to H, \|V\| = \|W\| = 1, V^* W = 0\} =$$
$$= \sup\{\|L\| : L \in R^n(A)\} = \min\{\|A - \lambda I\| : \lambda \in \mathbf{C}\}.$$

PROOF. Let us denote

$$m_1(A) = \{\|V^* A W\| : V, W : \mathbf{C}^n \to H, \|V\| = \|W\| = 1, V^* W = 0\}$$
$$m_2(A) = \{\|L\| : L \in R^n(A)\}$$
$$r = \min\{\|A - \lambda I\| : \lambda \in \mathbf{C}\}.$$

Since $V^* V = W^* W = I_n$ implies $\|V\| = \|W\| = 1$, it follows that $m_2(A) \subseteq m_1(A)$. Further, for $V, W : \mathbf{C}^n \to H$, $\|V\| = \|W\| = 1$, $V^* W = 0$ we have

$$\|V^* A W\| = \|V^* (A - \lambda I) W\| \le \|V^*\| \|A - \lambda I\| \|W\| = \|A - \lambda I\|$$

for every $\lambda \in \mathbf{C}$, so $\|V^* A W\| \le r$. Hence,

$$\sup m_2(A) \le \sup m_1(A) \le r. \tag{2.1}$$

If $r = 0$ we are done. So assume that $r > 0$. By [15] (i.e. [8, Proposition 2.11]) we conclude that for an arbitrary $0 < \varepsilon \le r$ there exist $x_\varepsilon, y_\varepsilon \in H$ such that $(x_\varepsilon, x_\varepsilon) = (y_\varepsilon, y_\varepsilon) = 1$, $(x_\varepsilon, y_\varepsilon) = 0$, $|(Ax_\varepsilon, y_\varepsilon)| = r - \varepsilon$. Let $V_\varepsilon, W_\varepsilon : \mathbf{C}^n \to H$ be two isometries with mutually orthogonal ranges such that $V_\varepsilon e = y_\varepsilon$ and $W_\varepsilon e = x_\varepsilon$, where $e \in \mathbf{C}^n$ is an arbitrary unit vector. Then we obtain

$$r - \varepsilon = |(Ax_\varepsilon, y_\varepsilon)| = |(AW_\varepsilon e, V_\varepsilon e)| = |(V_\varepsilon^* A W_\varepsilon e, e)| \le \|V_\varepsilon^* A W_\varepsilon\|,$$

so $r = \sup m_2(A)$. To complete the proof, it remains to apply (2.1). ∎

REMARK 2.8. In the original manuscript a concept of a generalized numerical range equivalent to the one introduced in Definition 2.1 was described for operators on Hilbert C^*-modules. As it was pointed out by the referee this reduces to the case of Hilbert space operators (after representing a Hilbert C^*-module as a concrete space of operators). However, in our subsequent paper we shall present some results concerned with the generalized numerical ranges for operators on Hilbert C^*-modules that can be obtained by the methods based on the results of [2].

ACKNOWLEDGEMENTS.
The author would like to thank Professor Damir Bakić for his useful comments and helpful advices. Thanks are also due to the referee for his/her valuable suggestions.

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Received: 11.12.2002
Revised: 05.03.2003