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SPANS OF CONTINUA RELATED TO INDENTED CIRCLES

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Abstract. Let X be a special type of simple closed curve in the plane known as an indented circle. Let Y be a continuum which is contained in $X \cup V$ where V is the bounded component of $R^2 - X$. We show that $\tau(Y) \leq \tau(X)$ where τ is the span σ, surjective span σ^*, semispan σ_0, surjective semispan σ_0^*, symmetric span s, or the surjective symmetric span s^*.

1. Introduction

The span of a metric continuum was originally defined by A. Lelek (see [L1], p. 209). Later variations of the span were defined (cf [L2, L3, D]). In general it is difficult to calculate the spans of a particular geometric object. Also, it is not clear how the various spans of related objects compare to each other. The following question on this topic was asked by H. Cook[C].

If X_1 is a plane simple closed curve and X_2 is a simple closed curve which is contained in the bounded component of $R^2 - X_1$ then is $\sigma(X_2) < \sigma(X_1)$?

There have been various partial results on this question (cf [W1, W2, W3, T1, T2, DF]). In this paper we show the following:

If X is a particular type of a simple closed curve known as an indented circle and Y is any continuum contained in $X \cup V$ where V is the bounded component of $R^2 - X$, then $\tau(Y) \leq \tau(X)$ where τ is any of the various spans.

2. Preliminaries

The standard projections $p_1, p_2 : X \times X \to X$ are mappings defined by $p_1(x, y) = x$ and $p_2(x, y) = y$ for $(x, y) \in X \times X$.

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Let X be a nonempty metric space. The *surjective span* $\sigma^*(X)$ of X is the least upper bound of real number α such that there exist nonempty connected sets $C_\alpha \subset X \times X$ with $d(x, y) \geq \alpha$ for $(x, y) \in C_\alpha$ and

$$(\sigma^*) \qquad p_1(C_\alpha) = p_2(C_\alpha) = X.$$

Relaxing condition (σ^*) to the conditions

$$(\sigma) \qquad p_1(C_\alpha) = p_2(C_\alpha),$$

$$(\sigma_0^*) \qquad p_2(C_\alpha) = X,$$

$$(\sigma_0) \qquad p_1(C_\alpha) \subset p_2(C_\alpha),$$

we obtain the definitions of the span $\sigma(X)$, the surjective semispan $\sigma_0^*(X)$, and the semispan $\sigma_0(X)$ of X, respectively.

If to condition (σ^*) we add the condition that $C^* = (C^*)^{-1}$ we get $s^*(X)$ the surjective symmetric span. If to condition (σ) we add the condition that $C^* = (C^*)^{-1}$ we get $s(X)$ the symmetric span.

In [W1] we defined a particular type of closed curve which we called an indented circle. The construction is given below.

We start with a circle S in the complex plane of radius r and center the origin O. Also, we will consider X as a subset of the real plane whenever this will simplify the exposition.

We choose angles $\theta_1, \ldots, \theta_n$ such that

$$0 < \theta_1 < \theta_2 < \cdots < \theta_n < \pi.$$

We choose $4n$ more angles $\theta_j^1, \theta_j^2, \theta_j^3, \theta_j^4$, for $j = 1, 2, \ldots, n$ such that

$$0 \leq \theta_1^1 \leq \theta_1 \leq \theta_1^2 \leq \cdots \leq \theta_n^1 \leq \theta_n \leq \theta_n^2 \leq \pi,$$
$$\pi \leq \theta_1^3 \leq \theta_1 + \pi \leq \theta_1^4 \leq \cdots \leq \theta_n^3 \leq \theta_n + \pi \leq \theta_n^4 \leq 2\pi,$$
$$\text{either } \theta_j^1 = \theta_j = \theta_j^2 \text{ or } \theta_j^1 < \theta_j < \theta_j^2 \text{ for } j = 1, 2, \ldots, n,$$
$$\text{either } \theta_j^3 = \theta_j + \pi = \theta_j^4 \text{ or } \theta_j^3 < \theta_j + \pi < \theta_j^4 \text{ for } j = 1, 2, \ldots, n,$$
$$\theta_j + \alpha_j^2 \leq \theta_{j+1} - \alpha_{j+1}^1 \text{ for } j = 1, 2, \ldots, n-1,$$

where $\alpha_j^1 = \text{Max}\{\theta_j - \theta_j^1, \theta_j + \pi - \theta_j^3\}$, $\alpha_j^2 = \text{Max}\{\theta_j^4 - (\theta_j + \pi), \theta_j^2 - \theta_j\}$.

Let $r_j = re^{i\theta_j}$, $q_j = re^{i(\theta_j + \pi)}$, $x_j = re^{i\theta_j^1}$, $y_j = re^{i\theta_j^2}$, $s_j = re^{i\theta_j^3}$, and $t_j = re^{i\theta_j^4}$ for $j = 1, 2, \ldots, n$.

We represent the straight line interval in the plane with endpoints a and b by $\overline{ab}$. Pick points $v_j, w_j \neq O$ where $v_j \in \overline{Or_j}$ and $w_j \in \overline{Oq_j}$ for $j = 1, 2, \ldots, n$. We must choose v_j and w_j such that the following restrictions are satisfied for $j = 1, 2, \ldots, n$. If $\theta_j^1 = \theta_j^2$, then $v_j = r_j$. If $\theta_j^3 = \theta_j^4$, then $w_j = q_j$.

Otherwise, we must choose v_j and w_j so that the following conditions are satisfied. If $\theta_j^1 \neq \theta_j^2$, then the smaller angles formed by the following pairs of line intervals, the pair $\overline{x_j v_j}$ and $\overline{v_j r_j}$, and the pair $\overline{r_j v_j}$ and $\overline{v_j y_j}$ must be no greater than 90°. If $\theta_j^3 \neq \theta_j^4$, then the smaller angles formed by the following pairs of line intervals, the pair $\overline{s_j w_j}$ and $\overline{w_j q_j}$, and the pair $\overline{q_j w_j}$ and $\overline{w_j t_j}$ must be no greater than 90°. We will refer to these conditions as the angle conditions.

For each j, when $\theta_j^1 \neq \theta_j^2$, the shorter arc on S with endpoints x_j and y_j is replaced by $\overline{x_j v_j} \cup \overline{v_j y_j}$ and when $\theta_j^3 \neq \theta_j^4$, the shorter arc on S with endpoints s_j and t_j is replaced by $\overline{s_j w_j} \cup \overline{w_j t_j}$.

We refer to both $\overline{x_j v_j} \cup \overline{v_j y_j}$ and $\overline{s_j w_j} \cup \overline{w_j t_j}$ as indentations of X for $j = 1, 2, \ldots, n$. We refer to v_j and w_j as the vertices of the corresponding indentations. The space X consists of the remaining points of S and the added indentations.

From the construction of X, we see that it is a simple closed curve. We call each such simple closed curve X an indented circle (see Fig. 1).

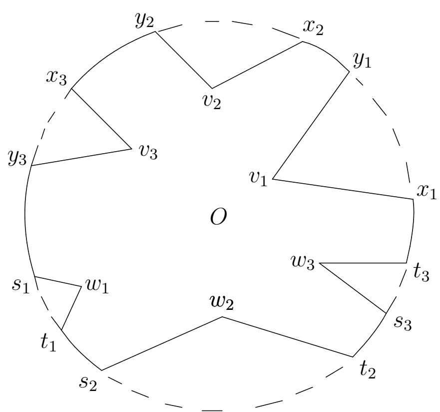

FIGURE 1

Let d_j be the point on $\overline{x_j v_j}$ closet to t_j, c_j the point on $\overline{v_j y_j}$ closest to s_j, b_j the point on $\overline{s_j w_j}$ closest to y_j, and a_j be the point on $\overline{w_j t_j}$ closest to x_j, for $j = 1, 2, \ldots, n$.

Let $d'_j = d(d_j, t_j), c'_j = d(c_j, s_j), b'_j = d(b_j, y_j)$, and $a'_j = d(a_j, x_j)$, for $j = 1, 2, \ldots, n$. We call the number

$$s_X = \text{Min}\{\text{Max}\{\text{Min}\{a'_j, d'_j\}, \text{Min}\{b'_j, c'_j\}\} : j = 1, 2, \ldots, n\}$$

the indentation spread of the indented circle X.

In [W1] we proved the following:

THEOREM 2.1. *If X is an indented circle and s_X is the indentation spread of X, then*

$$\sigma(X) = \sigma_0(X) = \sigma^*(X) = \sigma_0^*(X) = s_X.$$

Though it was not stated in this theorem, the proof also gives us that $s(X) = s^*(X)$, since the continuum $C \subset X \times X$ constructed in the proof of Theorem 2.1 is such that $C = C^{-1}$ and $p_1(C) = p_2(C) = X$.

3. Main Result

Theorem 3.1. *If X is an indented circle, V is the bounded component of $R^2 - X$, and Y is a continuum such that $Y \subset X \cup V$ then $\tau(Y) \leq \tau(X)$ where $\tau = \sigma, \sigma_0, \sigma^*, \sigma_0^*, s, s^*$.*

Proof. Suppose that X^* is an indented circle that has n indentations. We know from [W1, Th2.1] that

$$\begin{aligned}\sigma(X^*) = \sigma_0(X^*) = \sigma_0^*(X^*) = \sigma^*(X^*) = s_{X^*}\\ = \text{Min}\{\text{Max}\{\text{Min}\{a'_j, d'_j\}, \text{Min}\{c'_j, d'_j\} : j = 1, 2, \ldots, n\}\}\end{aligned}$$

where s_{X^*} is the indentation spread of X^*. For some j,

$$s_{X^*} = \text{Max}\{\text{Min}\{a'_j, d'_j\}, \text{Min}\{c'_j, b'_j\}\}.$$

Let $r : R^2 \to R^2$ be the function that rotates the plane by an angle of $\frac{\pi}{2} - \theta$ about the origin; so,

$$r(v_j) = r_v e^{i\frac{\pi}{2}}$$

where

$$v_j = r_v e^{i\theta_j}$$

and

$$r(w_j) = r_w e^{i\frac{3\pi}{2}}$$

where

$$w_j = r_w e^{i(\theta_j + \pi)}.$$

Let

$$\begin{aligned}r(x_j) = x, r(y_j) = y, r(s_j) = s, r(t_j) = t,\\ r(a_j) = a, r(b_j) = b, r(c_j) = c, \text{ and } r(d_j) = d.\end{aligned}$$

Let

$$d(x, a) = a', d(t, d) = d', d(s, c) = c' \text{ and } d(y, d) = d'.$$

Let

$$X = \overline{xv} \cup \overline{vy} \cup \overline{sw} \cup \overline{wt} \cup \{re^{i\theta} | \theta \in [0, \theta_x] \cup [\theta_y, \theta_s] \cup [\theta_t, 2\pi]\}$$

where $0 \leq \theta_x \leq \theta_y \leq \theta_s \leq \theta_t \leq 2\pi$ and $x = re^{i\theta_x}$, $y = re^{i\theta_y}$, $s = re^{i\theta_s}$ and $t = re^{i\theta_t}$. From [W1, Th2.1] we see that

$$s_{X^*} = \tau(X^*) = \tau(X) = s_X \text{ for } \tau = \sigma, \sigma_0, \sigma^*, \sigma_0^*.$$

From the proof of the theorem we also see that

$$s_{X^*} = \tau(X^*) = \tau(X) = s_X \text{ for } \tau = \sigma, \sigma_0, \sigma^*, \sigma_0^*$$

where $\tau = s$ or s^*. Also, if Y is a continuum such that $Y \subset X^* \cup V^*$ where V^* is the bounded component of $R^2 - X^*$ then $Y \subseteq X \cup V$ where V is the bounded component of $R^2 - X$. So, without loss of generality we can assume that our indented circle is X rather than X^*. Note that either

a) $0 < \theta_x < \frac{\pi}{2} < \theta_y < \pi < \theta_s < \frac{3\pi}{2} < \theta_t < 2\pi$,
b) $0 < \theta_x = \frac{\pi}{2} = \theta_y < \pi < \theta_s < \frac{3\pi}{2} < \theta_t < 2\pi$, or
c) $0 < \theta_x < \frac{\pi}{2} < \theta_y < \pi < \theta_s = \frac{3\pi}{2} = \theta_t < 2\pi$.

We first consider the situation in a) we have sixteen cases to consider.

A1
$$s_X = \max\{a', b'\}$$
$$a \neq w \neq b$$

A2
$$s_X = \max\{c', d'\}$$
$$c \neq v \neq d$$

If we rotate X by 180° in R^2 about the origin then case $A2$ is comparable to case $A1$.

B1
$$s_X = \max\{a', c'\}$$
$$a \neq w, c \neq v$$

B2
$$s_X = \max\{d', b'\}$$
$$d \neq v, b \neq w$$

If we rotate X by 180° in R^2 about the y-axis then case $B2$ is comparable to case $B1$.

C1
$$s_X = \max\{a', b'\}$$
$$a = b = w$$

C2
$$s_X = \max\{c, d\}$$
$$c = v = d$$

If we rotate X by 180° in R^2 about the origin then case $C2$ is comparable to case $C1$.

D1
$$s_X = \max\{a', c'\}$$
$$a = w, c = v$$

D2
$$s_X = \max\{d', b'\}$$
$$d = v, b = w$$

If we rotate X by 180° in R^2 about the y-axis then case $D2$ is comparable to case $D1$.

E1
$$s_X = \max\{a', b'\}$$
$$a = w, b \neq w$$

E2

$$s_X = \max\{a', b'\}$$
$$a \neq w, b = w$$

If we rotate X by 180° in R^2 about the y-axis then case $E2$ is comparable to case $E1$.

E3

$$s_X = \max\{d', c'\}$$
$$d = v, c \neq v$$

If we rotate X by 180° in R^2 about the x-axis then case $E3$ is comparable to case $E1$.

E4

$$s_X = \max\{d', c'\}$$
$$d \neq v, c = v$$

If we rotate X by 180° in R^2 about the origin then case $E4$ is comparable to case $E1$.

F1

$$s_X = \max\{a', c'\}$$
$$a = w, c \neq v$$

F2

$$s_X = \max\{a', c'\}$$
$$a \neq w, c = v$$

If we rotate X by 180° in R^2 about the origin then case $F2$ is comparable to case $F1$.

F3

$$s_X = \max\{d', b'\}$$
$$d = v, b \neq w$$

If we rotate X by 180° in R^2 about the x-axis then case $F3$ is comparable to case $F1$

F4

$$s_X = \max\{d', b'\}$$
$$d \neq v, b = w$$

If we rotate X by 180° in R^4 about the y-axis then case $F4$ is comparable to case $F1$.

Now we consider the situations in b) and c).

G1

$$s_X = d(v, w)$$
$$v = re^{i\frac{\pi}{2}}$$

G2

$$s_X = d(v, w)$$
$$w = re^{i\frac{3\pi}{2}}$$

If we rotate X by $180°$ in R^2 about the origin then case $G2$ is comparable to case $G1$. So, in order to prove the theorem we just need to examine cases $A1, B1, C1, D1, E1, F1$ and $G1$.

In order to do this we first define functions p_ε and q_ε under various conditions. We define continuous functions p_ε and q_ε where

$$p_\varepsilon : R \to \overrightarrow{wt}$$

$$q_\varepsilon : L \to \overrightarrow{ws}$$

$$R = \{(x_1, y_1) \in X \cup V | x_1 \geq 0\},$$

and

$$L = \{(x_1, y_1) \in X \cup V | x_1 \leq 0\}.$$

First we define p_ε in two different cases.

p_ε CASE 1: $a \neq w$

We define p_ε for ε where $0 < \varepsilon < \frac{1}{4}\min\{d(w,a), d(w,v)\}$. Pick $m \in \overline{wa}$ such that $0 < d(w,m) < \varepsilon$. Let $n \in \overline{vx}$ such that $\overline{mn}$ is perpendicular to $\overline{wt}$. Let P_1 be the portion of the plane which is bound by

$$B_1 = \overline{tm} \cup \overline{mn} \cup \overline{nx} \cup \{re^{i\theta} | 0 \leq \theta \leq \theta_x, \theta_t \leq \theta \leq 2\pi\}$$

together with its boundary B_1.

For $0 \leq t \leq 1$, let $n_t = tn + (1-t)v$, $m_t = tm + (1-t)w$, and $R_t = \overline{m_t n_t}$. We define $p_\varepsilon : R \to \overrightarrow{wt}$ as follows:

a) p_ε / P_1 is the perpendicular projection of P_1 into $\overrightarrow{wt}$,
b) p_ε / R_t is the constant function which sends each point of R_t to m_t for $0 \leq t \leq 1$.

OBSERVATION 1: If x_1 and $x_2 \in P_1$ where $\overline{x_1 x_2}$ is perpendicular to $\overleftrightarrow{wt}$ then $d(x_1, x_2) \leq a'$.

PROOF. To see this, let L_x be the line through x which is parallel to $\overleftrightarrow{wt}$. Note that

$$P_1 \subseteq L_x \cup \overleftrightarrow{wt} \cup V(L_x, \overleftrightarrow{wt})$$

where $V(L_x, \overleftrightarrow{wt})$ is the portion of the plane bound by L_x and $\overleftrightarrow{wt}$. Consequently, if x_1 and $x_2 \in P_1$ where $\overline{x_1 x_2}$ is perpendicular to $\overleftrightarrow{wt}$ then $d(x_1, x_2) \leq d(a, x) = a'$. ∎

OBSERVATION 2: If $x_1, x_2 \in R_t$ then $d(x_1, x_2) \leq a' + 2\varepsilon$.

Proof. Note that

$$\begin{aligned} d(x_1,x_2) &\le d(m_t,n_t) \\ &\le d(m_t,m)+d(m,n_t) \\ &\le \varepsilon + d(m,n_t) \\ &\le \varepsilon + \max\{d(m,v),d(m,n)\} \\ &\le \varepsilon + \max\{a', d(v,w)+\varepsilon\} \\ &\le \varepsilon + \max\{a', a'+\varepsilon\} \\ &= a'+2\varepsilon. \end{aligned}$$

□

(*) So we see that for $y' \in p_\varepsilon(R)$, $\operatorname{diam}(p_\varepsilon^{-1}\{y'\}) \le a' + 2\varepsilon$.

p_ε Case 2: $a = w$

We define p_ε for ε where $0 \le \varepsilon \le \frac{1}{4}\min\{d(v,w),d(w,t)\}$. Pick $m \in \overline{wt}$ such that $0 < d(w,m) < \varepsilon$. Pick $m_1 \in \overline{mt}$ such that $0 < d(m_1,m_2) < \varepsilon$. Let $m_2 \in X$ such that $\overline{m_1m_2}$ is perpenducular to $\overline{wt}$. Let $m_2 = re^{i\theta_{m_2}}$. Either $0 \le \theta_{m_2} < \theta_x$ or $\theta_t < \theta_{m_2} < 2\pi$.

Let

$$B_1 = \overline{m_1t} \cup \overline{m_1m_2} \cup X_{tm_2}$$

where $X_{tm_2} = \{re^{i\theta} | \theta \in [\theta_t, \theta_{m_2}]$ if $\frac{3}{2} < \theta_{m_2} < 2\pi$, or $\theta \in [\theta_t, 2\pi) \cup [0, \theta_{m_2}]$ if $0 \le \theta_{m_2} < \frac{\pi}{2}\}$.

Let P_1 be the portion of the plane bound by B_1 together with its boundary B_1. Let

$$\begin{aligned} X_{m_2x} \quad = \quad & \{re^{i\theta} | \theta \in [\theta_{m_2}, \theta_x] \text{ if } 0 \le \theta_{m_2} < \frac{\pi}{2} \text{or} \\ & \theta \in [\theta_{m_2}, 2\pi) \cup [0, \theta_x] \text{ if } \frac{3\pi}{2} < \theta_{m_2} < 2\pi\}. \end{aligned}$$

Let $r : [0,1] \to X_{m_2x}$ be a continuous surjective function where $r(0) = m_2$ and $r(1) = x$. Let

$$m_{1t} = (1-t)m_1 + tm \text{ and}$$
$$M_t = \overline{m_{1t}r(t)}.$$

For $0 \le t \le 1$ let

$$n_t = tx + (1-t)v,$$
$$m_t = tm + (1-t)w, \text{ and}$$
$$R_t = \overline{m_tn_t}.$$

We define $p_\varepsilon : R \to \overrightarrow{wt}$ as follows:

a) p_ε/P_1 is the perpendicular projection of P_1 into $\overrightarrow{m_1t}$,
b) p_ε/M_t is the constant function which sends each point of M_t to the point m_{1t},

c) p_ε/R_t is the constant function which sends each point of R_t to the point m_t.

OBSERVATION 3: If $x_1, x_2 \in M_t$ then $d(x_1, x_2) \le a' + 2\varepsilon$.

PROOF. First we observed that the funcion $d^* : [0,1] \to R^+$ given by $d^*(t) = d(m_1, r(t))$ is increasing. To see this compare the two triangles $\triangle\, Om_1r(0)$ and $\triangle\, Om_1r(t)$ where $0 < t \le 1$. Let α_t be the smaller angle between $\overline{Om_1}$ and $\overline{Or(t)}$ for $0 \le t \le 1$. Note that $\overline{Om_1}$ is of fixed length, r is the length of $\overline{Or(t)}$ for each $0 \le t \le 1$, and $\alpha_{t''} > \alpha_{t'}$ for $0 \le t' < t'' \le 1$. So, $d^*(m_1, r(t))$ increases as t increases. Hence, $d(m_1, m_2) < d(m_1, x) < a' + 2\varepsilon$. □

In this case as in case 1, we see that for $y' \in p_\varepsilon(R)$,

$$\operatorname{diam}(p_\varepsilon^{-1}\{y'\}) \le a' + 2\varepsilon.$$

Now we define q_ε in four different cases.

q_ε CASE 1: $b \neq w$

We define q_ε for ε where $0 < \varepsilon < \frac{1}{4}\min\{d(w,b), d(w,v)\}$. Pick $p \in \overline{sw}$ such that $0 < d(w,p) < \varepsilon$. Let $u \in \overline{yv}$ such that $\overline{pu}$ is perpendicular to $\overline{sw}$. For $0 \le t \le 1$, let

$$p_t = tp + (1-t)w,$$

$$u_t = tu + (1-t)v, \text{ and}$$

$$L_t = \overline{p_t u_t}.$$

Let P_2 be the portion of the plane which is bound by

$$B_2 = \overline{sp} \cup \overline{pu} \cup uy \cup \{re^{i\theta} | \theta_y \le \theta \le \theta_s\}$$

together with its boundary B_2. We define $q_\varepsilon : L \to \overrightarrow{ws}$ as follows

a) q_ε/L_t is the constant function which sends each point of L_t to P_t,
b) q_ε/P_2 is the perpendicular projection of P_2 into $\overrightarrow{ws}$.

From previous observations we can see that for $y' \in q_\varepsilon(L)$,

$$\operatorname{diam}(q_\varepsilon^{-1}\{y'\}) \le b' + 2\varepsilon.$$

q_ε CASE 2: $b = w$

We define q_ε for ε where $0 < \varepsilon < \frac{1}{4}\min\{d(v,w), d(w,s)\}$. Pick $p \in \overline{ws}$ such that $0 < d(w,p) < \varepsilon$. Pick $p_1 \in \overline{ps}$ such that $0 < d(p_1, p) < \varepsilon$. Let $p_2 \in X$ such that $\overline{p_1p_2}$ is perpendicular to $\overline{ws.}$ Let $p_2 = re^{i\theta_{p_2}}$. For $0 \le t \le 1$ let

$$u_t = ty + (1-t)v,$$

$$p_t = tp + (1-t)w, \text{ and}$$

$$L_t = \overline{u_t p_t}.$$

Let $X_{p_2y} = \{re^{i\theta} | \theta \in [\theta_y, \theta_{p_2]}\}$. Let $l : [0,1] \to X_{p_2y}$ be a continuous surjective function where $l(0) = p_2$, $l(1) = y$. Let $p_{1t} = (1-t)p_1 + tp$ and $U_t = \overline{p_{1t}l(t)}$. Let P_2 be the portion of the plane which is bound by $B_2 = \overline{sp_1} \cup \overline{p_1p_2} \cup X_{p_2s}$ where $X_{p_2s} = \{re^{i\theta} | \theta_{p_2} \le \theta \le \theta_s\}$ together with its boundary B_2. We define $q_\varepsilon : L \to \overrightarrow{ws}$ as follows

a) $q_{\varepsilon/_{L_t}}$ is the constant function which takes each point of L_t to the point p_t,
b) $q_{\varepsilon/_{U_t}}$ is the constant function which sends each point of U_t to the point p_{1t},
c) $q_{\varepsilon/_{P_2}}$ is the perpendicular projection of P_2 into $\overrightarrow{ws}$.

From previous observations we can see that for $y' \in q_\varepsilon(L)$,

$$\operatorname{diam}(q_\varepsilon^{-1}\{y'\}) \le b' + 2\varepsilon.$$

q_ε Case 3: $c \ne v$

We define q_ε for ε where $0 < \varepsilon < \frac{1}{4}\min\{d(v,w), d(v,c)\}$. Pick $u \in \overline{cv}$ such that $0 < d(u,v) < \varepsilon$. Let $p \in \overline{sw}$ such that $\overline{pu}$ is perpendicular to $\overline{vy}$. Let

$$u_t = tu + (1-t)v,$$

$$p_t = tp + (1-t)w, \text{ and}$$

$$L_t = \overline{u_tp_t}.$$

Let P_2 be the portion of the plane bound by

$$B_2 = \overline{yu} \cup \overline{up} \cup \overline{sp} \cup X_{ys}$$

where $X_{ys} = \{e^{i\theta} | \theta_y \le \theta \le \theta_s\}$ together with its boundary B_2. Let $q : L \to \overrightarrow{vy}$ be defined as follows

a) q/L_t is the constant function that sends each point of L_t to u_t,
b) q/P_2 is the perpendicular projection of P_2 into $\overrightarrow{vy}$.

Let $q(L) = \overline{vy'}$. Let $q^* : \overline{vy'} \to \overline{sw}$ be a surjective continuous map such that $q^*(v) = w, q^*(u) = p, q^*(y') = s$. Let $q_\varepsilon = q^* \circ q$. From previous observations it is clear that if $y' \in \overline{sw}$, $\operatorname{diam} q_\varepsilon^{-1}\{y'\} \le c' + 2\varepsilon$.

q_ε Case 4: $v = c$

We define q_ε for ε where $0 < \varepsilon < \frac{1}{4}\min\{d(v,w), d(v,y)\}$. Pick $u \in \overline{vy}$ such that $0 < d(v,u) < \varepsilon$. Pick $u_1 \in \overline{uy}$ such that $0 < d(u,u_1) < \varepsilon$. Let $u_2 \in X$ such that $\overline{u_1u_2}$ is perpendicular to $\overline{vy}$. Let $u_2 = re^{i\theta u_2}$. Let

$$u_t = tu + (1-t)v,$$

$$p_t = ts + (1-t)w.$$

Let $L_t = \overline{U_tp_t}$. Let $X_{u_2s} = \{re^{i\theta} | \theta_{u_2} \le \theta \le \theta_s\}$. Let $l : [0,1] \to X_{u_2s}$ be a continuous surjective function where $l(0) = u_2$ and $l(1) = s$. Let $u_{1t} = (1-t)u_1 + tu$ and $U_t = \overline{u_{1t}l(t)}$. Let $B_2 = \overline{u_1y} \cup \overline{u_1u_2} \cup X_{yu_2}$ where $X_{yu_2} =$

$\{re^{i\theta}|\theta_y \leq \theta \leq \theta_{u_2}\}$. Let P_2 be the portion of the plane bound by B_2 together with its boundary B_2. We define a function $q : L \to \overrightarrow{vy}$ as follows

a) q/L_t is the constant function that sends each point of L_t to U_t,
b) q/U_t sends each point of u_t to u_{1t},
c) q/P_2 is the perpendicular projection of P_2 into $\overrightarrow{u_1y}$.

Let $q(L) = \overline{vy'}$. Let $q^* : \overline{vy'} \to \overline{sw}$ be a surjective continuous map such that $q^*(v) = w, q^*(y') = s$. Let $q_\varepsilon = q^* \circ q$. From previous observations it is clear that if $x' \in \overline{sw}$ then $\text{diam} q_\varepsilon^{-1}\{x'\} \leq c' + 2\varepsilon$.

Let Y be the continuum as given above. We consider 7 cases as given below:

Case A1: $s_X = \max\{a', b'\}$ $a \neq w \neq b$
Case B1: $s_X = \max\{a', c'\}$ $a \neq w, c \neq v$
Case C1: $s_X = \max\{a', b'\}$ $a = w = b$
Case D1: $s_X = \max\{a', c'\}$ $a = w, c = v$
Case E1: $s_X = \max\{a', b'\}$ $a = w, b \neq w$
Case F1: $s_X = \max\{a', c'\}$ $a = w, c \neq v$
Case G1: $v = re^{i\frac{\pi}{2}}$ $s_x = d(v, w)$

Let $C \subseteq Y \times Y$ be a continuum such that $p_1[C] \subseteq p_2[C] \subseteq Y$.

Case A1: $s_X = \max\{a', b'\}$ $a \neq w \neq b$

Let $p : L \cup R \to \overrightarrow{ws} \cup \overrightarrow{wt}$ be given by $p/R = p_\varepsilon$ as defined in case 1 for p_ε. $p/L = q_\varepsilon$ as defined in case 1 for q_ε.

Consider $p \circ p_1, p \circ p_2 : C \to \overrightarrow{ws} \cup \overrightarrow{wt}$. The functions $p \circ p_1$ and $p \circ p_2$ are continuous, $p \circ p_1[C] \subseteq p \circ p_2[C] = J \subset \overrightarrow{ws} \cup \overrightarrow{wt}$. Clearly J is an interval and there is a $c \in C$ such that $p \circ p_1(c) = p \circ p_2(c)$. From previous observations we see that diam $(p^{-1}\{p \circ p_1(c)\}) \leq \max\{a' + 2\varepsilon, b' + 2\varepsilon\}$. So, $d(p_1(c), p_2(c)) \leq \max\{a' + 2\varepsilon, b' + 2\varepsilon\}$. Since this is true for all $\varepsilon > 0$, we conclude that

$$\tau(Y) \leq \max\{a', b'\} = s_X = \tau(X) \text{ where } \tau \text{ is any of the spans.}$$

Case B1: $s_X = \max\{a', c'\}$, $a \neq w, c \neq v$

In this case we define $p : L \cup R \to \overrightarrow{ws} \cup \overrightarrow{wt}$ by $p/R = p_\varepsilon$ as in case 1 for p_ε and $p/L = q_\varepsilon$ as in case 3 for q_ε.

The rest of this case is handled as in case A1. Our conclusion now is that

$$\tau(Y) \leq \max\{a', c'\} = s_X = \tau(X) \text{ where } \tau \text{ is any of the spans.}$$

Case C1: $s_X = \max\{a', b'\}$, $a = w = b$

In this case we define $p : L \cup R \to \overrightarrow{ws} \cup \overrightarrow{wt}$ by $p/R = p_\varepsilon$ as defined in case 2 for p_ε and $p/L = q_\varepsilon$ as defined in case 2 for q_ε.

In a manner similar to the previous cases we can conclude that

$$\tau(Y) \leq \max\{a', b'\} = s_X = \tau(X) \text{ where } \tau \text{ is any of the spans.}$$

Case D1: $s_X = \max\{a', c'\}$, $a = w, c = v$

In this case we define $p : L \cup R \to \overrightarrow{sw} \cup \overrightarrow{wt}$ by $p/R = p_\varepsilon$ as in case 2 for p_ε and $p/L = q_\varepsilon$ as in case 4 for q_ε.

As in the previous cases we can conclude that

$$\tau(Y) \le \max\{a', c'\} = s_X = \tau(X) \text{ where } \tau \text{ is any of the spans.}$$

Case E1: $s_X = \max\{a', b'\}$, $a = w, b \ne w$

In this case we define $p : L \cup R \to \overrightarrow{sw} \cup \overrightarrow{wt}$ by $p/R = p_\varepsilon$ as defined in case 2 for p_ε and $p/L = q_\varepsilon$ as defined in case 2 for q_ε.

In this case we can conclude that

$$\tau(Y) \le \max\{a', b'\} = s_X = \tau(X) \text{ where } \tau \text{ is any of the spans.}$$

Case F1: $s_X = \max\{a', c'\}$, $a = w, c \ne v$

In this case we define $p : R \cup L \to \overrightarrow{sw} \cup \overrightarrow{wt}$ by $p/R = p_\varepsilon$ as in case 2 for p_ε and $p/L = q_\varepsilon$ as in case 3 for q_ε.

Our conclusion in this case is that

$$\tau(Y) \le \max\{a', c'\} = s_x = \tau(X) \text{ where } \tau \text{ is any of the spans.}$$

Case G1:

We define $p : R \cup L \to \overrightarrow{wt} \cup \overrightarrow{ws}$ when $v = re^{i\theta}$. In this case $s_X = d(v, w)$. Pick ε where $0 < \varepsilon < \frac{1}{4}\min\{d(w,t), d(w,s)\}$. Pick $m \in \overline{wt}$ such that $0 < d(w,m) < \varepsilon$. Let $n \in X$ such that $\overline{mn}$ is perpendicular to $\overline{wt}$. Pick $u \in \overline{ws}$ such that $0 < d(w,u) < \varepsilon$. Let $p = re^{i\theta\rho} \in X$ such that $\overline{pu}$ is perpendicular to $\overline{xs}$. Let $B_1 = \overline{nm} \cup \overline{mt} \cup X_{tn}$ where $X_{tn} = \{re^{i\theta} | \theta \in [0, \theta_n] \cup [\theta_t, 2\pi)$ if $0 \le \theta_n < \frac{\pi}{2}$, $\theta \in [\theta_t, \theta_n]$ if $\frac{3\pi}{2} \le \theta_n < 2\pi\}$. Let P_1 be the portion of the plane bound by B_1 together with its boundary B_1. Let $r : [0,1] \to X_{nv}$ where $X_{nv} = \{re^{i\theta} | \theta_n \le \theta \le \frac{\pi}{2}$ if $0 \le \theta_n < \frac{\pi}{2}$, $\theta \in [\theta_n, 2\pi) \cup [0, \frac{\pi}{2}]$ if $\frac{3\pi}{2} < \theta_n < 2\pi\}$ be a continuous, surjective function such that $r(0) = v$ and $r(1) = n$. Let $m_t = (1-t)w + tm$. Let $R_t = \overline{m_t r(t)}$. Let $l : [0,1] \to X_{vp}$ where $X_{vp} = \{re^{i\theta} | \frac{\pi}{2} \le \theta \le \theta_p\}$ be a continuous surjective function such that $l(0) = v, l(1) = p$. Let $u_t = (1-t)w + tu$. Let $L_t = \overline{u_t l(t)}$. Let $B_2 = \overline{su} \cup \overline{up} \cup X_{ps}$ where $X_{ps} = \{re^{i\theta} | \theta_p \le \theta \le \theta_s\}$. Let P_2 be the portion of the plane bound by B_2 together with its boundary B_2. We define $p : R \cup L \to \overrightarrow{wt} \cup \overrightarrow{ws}$ as follows:

p/P_1 is the perpendicular projection of P_1 into $\overrightarrow{mt}$

p/R_t is the constant function which sends each point in R_t to m_t.

p/L_t is the constant function which sends each point in L_t to u_t.

p/P_2 is the perpendicular projection of P_2 into $\overrightarrow{vs}$.

Observation 4:

Note that the continuous function $d^* : [0,1] \to R^+$ given by $d(w, r(t))$ is decreasing. So, for each $t \in [0,1]$, $d(m_t, r(t)) \le d(v,w) + \varepsilon$. Similarly, it follows that $d(u_t, d(t)) \le d(v,w) + \varepsilon$ for $t \in [0,1]$. Using this observation together with observation 1, we see that for $y' \in p(R \cup L)$, diam $p^{-1}\{y'\} \le d(v,w) + \varepsilon$. Since this is true for all $\varepsilon > 0$, we can conclude that

$$\tau(Y) \leq d(v, w) = s_X = \tau(X) \text{ where } \tau \text{ is any of the spans.}$$

□

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