GLASNIK MATEMATIČKI
Vol. 39(59)(2004), 185 – 198

SOME QUESTIONS OF EQUIVARIANT MOVABILITY

P.S. Gevorgyan
Moscow State University, Russia

Abstract. In this article some questions of equivariant movability, connected with the substitution of the acting group G on closed subgroup H and with transitions to spaces of H-orbits and H-fixed points spaces, are investigated. In a special case, the characterization of equivariantly movable G-spaces is given.

1. Introduction

This paper is devoted to equivariant movability of G-spaces, i.e., topological spaces endowed with an action of a given compact group G.

More precisely, in § 3 we define the notion of equivariant movability or G-movability and we prove several theorems, including the following ones. If X is p-paracompact and $H \subseteq G$ is a closed subgroup, then G-movability of X implies its H-movability (§ 3, Theorem 3.3). G-movability of X also implies movability of the space $X[H]$ of H-fixed points in X (§ 4, Theorem 4.1). In particular, equivariant movability of a G-space X implies ordinary movability of the topological space X (§ 3, Corollary 3.5). We construct a non-trivial example which shows, that the converse, in general, is not true, even if we take for G the cyclic group Z_2 of order 2 (§ 5, Example 5.1). If X is a metrizable G-movable space and H is a closed normal subgroup of G, then the space $X|_H$ of its H-orbits is also G-movable (§ 6, Theorem 6.1). In the case $H = G$ we obtain that G-movability of a metrizable G-space implies ordinary movability of the orbit space $X|_G$ (§ 6, Corollary 6.2). The last assertion, in general, is not invertible (§ 6, Example 6.3). However, if X is metrizable, G is a compact Lie group and the action of G on X is free, then X is G-movable if and only if the orbit space $X|_G$ is movable (§ 7, Theorem 7.2). Examples 6.3 (§ 6) and

2000 *Mathematics Subject Classification.* 55P55, 54C56.
Key words and phrases. Equivariant shape theory, equivariant movability.

3 (§ 8) show that in the last theorem the assumption that the group G is a Lie group and the assumption that the action is free cannot be omitted.

Some of the above listed results with an outline of proof were given in [9].

Let us denote the category of all topological spaces and continuous maps by Top, the category of all metrizable spaces and continuous maps by M and the category of all p-paracompact spaces and continuous maps by P. Always in this article it is assumed that all topological spaces are p-paracompact spaces and the group G is compact.

The author is extremely grateful to the referee for his helpful remarks and comments.

The reader is referred to the books by K. Borsuk [4] and by S. Mardešić and J. Segal [15] for general information about shape theory and to the book by G. Bredon [5] for introduction to compact transformation groups.

2. Basic notions and conventions concerning equivariant topology

Let G be a topological group. A topological space X is called a G-space if there is a continuous map $\theta : G \times X \to X$ of the direct product $G \times X$ into X, $\theta(g, x) = gx$, such that

$$1) \quad g(hx) = (gh)x; \qquad 2) \quad ex = x,$$

for all $g, h \in G$, $x \in X$; here e is the unity of G. Such a (continuous) map $\theta : G \times X \to X$ is called an (continuous) action of the group G on the topological space X. An evident example is the so called trivial action of G on X: $gx = x$, for all $g \in G$, $x \in X$. Another example is the action of the group G on itself, defined by $(g, x) \to gx$ for all $g \in G$, $x \in G$.

If X and Y are G-spaces, then so is $X \times Y$, where $g(x, y) = (gx, gy)$, $g \in G$, $(x, y) \in X \times Y$.

A subset A of a G space X is called invariant provided $g \in G$, $a \in A$ implies $ga \in A$. It is evident, that an invariant subset of a G space is itself a G space. If A is an invariant subset of a G space X, then every neighborhood of A contains an open invariant neighborhood of A (see [17], Proposition 1.1.14).

Let X be any G-space and let H be a closed and normal subgroup of the group G. The set $Hx = \{hx; h \in H\}$ is called the H-orbit of the point $x \in X$. Clearly the H-orbits of any two points in X are either equal or disjoint, in other words X is partitioned by its H-orbits. We denote the set of all H-orbits of the G-space X by $X|_H$. The set $X|_H$ endowed with the quotient topology is called the H-orbit space of X. There is a continuous action of the group G on the space $X|_H$ defined by the formula $gHx = Hgx, g \in G, x \in X$. So, $X|_H$ is a G-space. In case $H = G$ the G-orbit of the point $x \in X$ is called the orbit of the point x and the G-orbit space is called the orbit space of the G-space X.

We denote by $X[H]$ the subspace of fixed points of H on X, or the H-fixed point subspace of the G-space X. Let us recall that $X[H] = \{x \in X; hx = x,$ for any $h \in H\}$.

The set $G_x = \{g \in G; g(x) = x\}$ is a closed subgroup of the group G, for every $x \in X$. G_x is called the stationary subgroup (or stabilizer) at the point x. The action of the group G on X (or the G-space X) is called free if the stationary subgroup G_x is trivial, for every $x \in X$. It is clear that $G_{gx} = gG_xg^{-1}$, i.e., the stationary subgroups at any two points of the same orbit are conjugate. The orbits Gx and Gy of points x and y, respectively, are said to have the same type if the stationary subgroups G_x and G_y are conjugate.

Let X, Y be G-spaces. A (continuous) map $f : X \to Y$ is called a G-map, or an equivariant map, if $f(gx) = gf(x)$ for every $g \in G$, $x \in X$. Note that the identity map $i : X \to X$ is equivariant and the composition of equivariant maps is equivariant. Therefore, all G-spaces and equivariant maps form a category. Let us denote the category of all topological G-spaces and equivariant maps by Top_G, the category of all metrizable G-spaces and equivariant maps by M_G and the category of all p-paracompact G-spaces and equivariant maps by P_G.

Let Z be a G-space and let $Y \subseteq Z$ be an invariant subset. A G-retraction of Z to Y is a G-map $r : Z \to Y$ such that $r|_Y = 1_Y$.

Let K_G be class of G-spaces. A G-space Y is called a G-absolute neighborhood retract for the class K_G or a $G - ANR(K_G)$ (G-absolute retract for the class K_G or a $G - AR(K_G)$), provided $Y \in K_G$ and whenever Y is a closed invariant subset of a G-space $Z \in K_G$, then there exist an invariant neighborhood U of Y and a G-retraction $r : U \to Y$ (there exists a G-retraction $r : Z \to Y$).

A G-space Y is called a G-absolute neighborhood extensor for the class K_G or a $G - ANE(K_G)$ (G-absolute extensor for the class K_G or a $G - AE(K_G)$), provided for any G-space $X \in K_G$ and any closed invariant subset $A \subseteq X$, every equivariant map $f : A \to Y$ admits an equivariant extension $\tilde{f} : U \to Y$, where U is an invariant neighborhood of A in X ($\tilde{f} : X \to Y$).

3. Movability and equivariant movability

The important shape invariant, called movability, was originally introduced by K. Borsuk [2] for metric compacta. Mardešić and Segal [14] generalized the notion of movability to compacta using the ANR-system approach. Kozlowski and Segal in [11] gave a categorical description of this property which applied to arbitrary topological spaces.

Following Mardešić and Segal [14], let us define the notion of equivariant movability or G-movability :

Definition 3.1. *An inverse G-system $\underline{X} = \{X_\alpha, p_{\alpha\alpha'}, A\}$ where each X_α, $\alpha \in A$, is a G-space and every $p_{\alpha\alpha'} : X_{\alpha'} \to X_\alpha$, $\alpha \leqslant \alpha'$, is a G-homotopy class, is called equivariantly movable or G-movable if for every $\alpha \in A$, there exists an $\alpha' \in A$, $\alpha' \geqslant \alpha$ such that for all $\alpha'' \in A$, $\alpha'' \geqslant \alpha$ there exists a G-homotopy class $r^{\alpha'\alpha''} : X_{\alpha'} \to X_{\alpha''}$ such that*

$$p_{\alpha\alpha''} \circ r^{\alpha'\alpha''} = p_{\alpha\alpha'}.$$

It is known (see [1], Theorem 2) that every G-space X admits a $G-ANR$-expansion in the sense of Mardešić (see [15], I, § 2.1), which is the same as saying that there is an inverse $G - ANR$-system (G-system consisting of $G-ANR$'s) $\underline{X} = \{X_\alpha, p_{\alpha\alpha'}, A\}$ associated with X in the sense of Morita [16].

Definition 3.2. *A G-space X is called equivariantly movable or G-movable if there is an equivariantly movable inverse $G - ANR$-system $\underline{X} = \{X_\alpha, p_{\alpha\alpha'}, A\}$ associated with X.*

Note that the last definition of equivariant movability coincides with the notion of ordinary movability if $G = \{e\}$ is the trivial group.

Let X be an equivariantly movable G-space. The evident question arises: does movability of the space X follows from its equivariant movability? The following, more general theorem gives a positive answer (Corollary 3.5) to the above question.

Theorem 3.3. *Let H be a closed subgroup of a group G. Every G-movable G-space is H-movable.*

To prove this theorem the next result is important.

Theorem 3.4. *Let H be a closed subgroup of a group G. Every $G - AR(P_G)$ $(G - ANR(P_G))$-space is an $H - AR(P_H)(H - ANR(P_H))$-space.*

Proof. According to a theorem of de Vries ([7], Theorm 4.4), it is sufficient to show that if X is a p-paracompact H-space, then the twisted product $G \times_H X$ is also p-paracompact. Indeed, since X is p-paracompact and G is compact, $G \times X$ is p-paracompact. Therefore, the twisted product $G \times_H X$ is p-paracompact. □

Proof of Theorem 3.3. Let X be any equivariantly movable G-space. With respect to the theorem of Smirnov ([18], Theorem 1.3), there is a closed and equivariant embedding of the G-space X to some $G - AR(P_G)$-space Y. Let us consider all open G-invariant neighborhoods of type F_σ of the G-space X in Y. By a result of R. Palais ([17], Proposition 1.1.14), these neighborhoods form a cofinal family in the set of all open neighborhoods of X in Y, in particular, in the set of all open and H-invariant neighborhoods of the H-space X in the H-space Y, which, by Theorem 3.3 is an $H - AR(P_H)$-space. Hence, from the G-movability of the above mentioned family follows its H-movability, i.e. from the G-movability of the G-space X follows the H-movability of the H-space X. ∎

From Theorem 3.3 we obtain the following corollary if we consider the trivial subgroup $H = \{e\}$ of the group G.

Corollary 3.5. *Every equivariantly movable G-space X is movable.*

The converse, in general, is not true, even if one takes for G the cyclic group Z_2 of order 2 (see Example 5.1).

4. Movability of the H-fixed point space

Theorem 4.1. *Let H be a closed subgroup of a group G. If a G-space X is equivariantly movable, then the H-fixed point space $X[H]$ is movable.*

The proof requires the use of the following theorem.

Theorem 4.2. *Let H be a closed subgroup of a group G. Let X be a $G - AR(P_G)(G - ANR(P_G))$- space. Then the H-fixed point space $X[H]$ is an $AR(P)(ANR(P))$-space.*

Proof. Let X be a $G - AR(P_G)(G - ANR(P_G))$-space. By Theorem 3.4, it is sufficient to prove the theorem in the case $H = G$. I.e., we must prove that $X[G]$ is $AR(P)$-space. By a theorem of Smirnov ([18], Theorem 1.3), we can consider X as a closed G-subspace of a $G - AR(P_G)$-space $C(G, V) \times \prod D_\lambda$ where V is a normed vector space and thus an $AE(M)$-space, $C(G, V)$ is the space of continuous maps from G to V with the compact-open topology and with the action $(g'f)(g) = f(gg'), g, g' \in G, f \in C(G, V)$ of the group G and D_λ is a closed ball of a finite-dimensional Euclidean space E_λ with the orthogonal action of the group G.

First, let us prove that the set $(C(G, V) \times \prod D_\lambda)[G]$ of all fixed points of the G-space $C(G, V) \times \prod D_\lambda$ is an $AR(P)$-space. The spaces $C(G, V)$ and E_λ are normed spaces. Since the actions of the group G on $C(G, V)$ and E_λ are linear, the sets $C(G, V)[G]$ and $E_\lambda[G]$ will be closed convex sets of locally convex spaces $C(G, V)$ and E_λ, respectively. Therefore, by a well-known theorem of Kuratowski and Dugundji [3], $C(G, V)$ and E_λ are absolute retracts for metrizable spaces. By a theorem of Lisica [12], they are also absolute retracts for p-paracompact spaces. For a closed ball $D_\lambda \subset E_\lambda$ the last conclusion is true since the set $D_\lambda[G] = D_\lambda \bigcap E_\lambda[G]$ is closed and convex in E_λ.

Since the group G acts on the product $C(G, V) \times \prod D_\lambda$ coordinate-wise,

$$(C(G, V) \times \prod D_\lambda)[G] = C(G, V)[G] \times \left(\prod D_\lambda\right)[G].$$

Hence, $(C(G, V) \times \prod D_\lambda)[G]$ is an $AR(P)$-space, because it is a product of two $AR(P)$-spaces.

Now let us prove that $X[G]$ is an $AR(P)$-space. Since X is a $G-AR(P_G)$-space, it is a G-retract of the product $C(G,V)\times\prod D_\lambda$. Therefore, $X[G]$ is a retract of the $AR(P)$-space $(C(G,V)\times\prod D_\lambda)[G]$, hence, it is an $AR(P)$-space.

The absolute neighborhood retract case is proved similarly. □

Proof of Theorem 4.1. Let X be a G-movable space. By Theorem 3.3, it is sufficient to prove the theorem in the case $H=G$. So, we must prove movability of the space $X[G]$ of all G-fixed points. We consider the G-space X as a closed and G-invariant space of some $G-AR(P_G)$-space Y ([18], Theorem 1.3). The family of all open, G-invariant F_σ-type neighborhoods U_α of the G-space X in Y, is cofinal in the set of all open neighborhoods of X in Y ([17], Proposition 1.1.14). It consists of $G-ANR(P_G)$-spaces. The intersections $U_\alpha\cap Y[G]=U_\alpha[G]$ are $ANR(P)$-spaces (Theorem 4.2). They form a cofinal family of neighborhoods of the space $X[G]$ in $Y[G]$. Indeed, for any neighborhood U of the set $X[G]$ in $Y[G]$ there is a neighborhood V of the set $X[G]$ in Y such that $V\cap Y[G]=U$. Then the set $W=(Y\setminus Y[G])\cup V$ is a neighborhood of the set X in Y, moreover, $W\cap Y[G]=U$. There is an α such that $U_\alpha\subset W$ and therefore $U_\alpha[G]\subset U$. So the family of neighborhoods $U_\alpha[G]$ is cofinal.

Since X is G-movable, for every U_α there is a neighborhood $U_{\alpha'}\subset U_\alpha$ such that, for any other neighborhood $U_{\alpha''}\subset U_{\alpha'}$, there exists a G-equivariant homotopy $F:U_{\alpha'}\times I\to U_\alpha$ such that $F(y,0)=y$ and $F(y,1)\in U_{\alpha''}$, for any $y\in U_{\alpha'}$. It is not difficult to verify that the homotopy $F[G]:U_{\alpha'}[G]\times I\to U_\alpha[G]$, induced by F, satisfies the condition of movability of $X[G]$. □

5. Example of a movable, but not equivariantly movable space

Example 5.1. We will use the idea of S. Mardešić [13]. Let us consider the unit circle $S=\{z\in C; |z|=1\}$. Let us denote $B=[S\times\{1\}]\cup[\{1\}\times S]$. B is the wedge of two copies of the unit circle S with base point $\{1\}$. Let us define a continuous map $f:B\to B$ by the formulas:

$$f(z,1)=\begin{cases}(z^4,1), & 0\leqslant arg(z)\leqslant\frac{\pi}{2}\\ (1,z^4), & \frac{\pi}{2}\leqslant arg(z)\leqslant\pi\\ (z^{-4},1), & \pi\leqslant arg(z)\leqslant\frac{3\pi}{2}\\ (1,z^{-4}), & \frac{3\pi}{2}\leqslant arg(z)\leqslant 2\pi\end{cases}$$

$$f(1,t)=\begin{cases}(t^{-4},1), & 0\leqslant arg(t)\leqslant\frac{\pi}{2}\\ (1,t^{-4}), & \frac{\pi}{2}\leqslant arg(t)\leqslant\pi\\ (t^4,1), & \pi\leqslant arg(t)\leqslant\frac{3\pi}{2}\\ (1,t^4), & \frac{3\pi}{2}\leqslant arg(t)\leqslant 2\pi\end{cases}$$

for every z and t from S. Let us consider the ANR-sequences

$$B \xleftarrow{f} B \xleftarrow{f} B \xleftarrow{f} \cdots$$

and

$$\Sigma B \xleftarrow{\Sigma f} \Sigma B \xleftarrow{\Sigma f} \Sigma B \xleftarrow{\Sigma f} \cdots$$

where Σ is the operation of suspension. Let us denote

$$P = \varprojlim \{B, f\}.$$

Then

$$\Sigma P = \varprojlim \{\Sigma B, \Sigma f\}.$$

Let us define an action of the group $Z_2 = \{e, g\}$ on ΣB by the formulas

$$e[x,t] = [x,t]; \qquad g[x,t] = [x,-t].$$

for every $[x,t] \in \Sigma B, -1 \leqslant t \leqslant 1$. It induces an action on ΣP.

Proposition 5.2. *The space ΣP has trivial shape, but it is not Z_2-movable.*

Proof. The triviality of shape of the space ΣP is proved by the method of Mardešić [13]. Let us prove that the space ΣP is not Z_2-movable. Consider the set $\Sigma P[Z_2]$ of all fixed-points of Z_2-space ΣP. It is obvious that $\Sigma P[Z_2] = P$. Hence, by Theorem 4.1, it is sufficient to prove the following proposition. ∎

Proposition 5.3. *The space P is not movable.*

Proof. Since the movability of an inverse system remains unchanged under the action of a functor, it is sufficient to prove non-movability of the inverse sequence of groups

$$\pi_1(B) \xleftarrow{f_*} \pi_1(B) \xleftarrow{f_*} \pi_1(B) \xleftarrow{f_*} \cdots , \tag{1}$$

where $\pi_1(B)$ is the fundamental group of the space B and f_* is the homomorphism induced by the mapping $f : B \to B$.

It is known that for sequences of groups movability implies the following condition of Mittag-Leffler, abbreviated as ML ([15], p. 166, Corollary 4):

The inverse system $\{G_\alpha, p_{\alpha\alpha'}, A\}$ of the $pro-GROUP$ category is said to be ML provided for every $\alpha \in A$, there exist $\alpha' \in A, \alpha' \geqslant \alpha$, such that $p_{\alpha\alpha'}(G_{\alpha'}) = p_{\alpha\alpha''}(G_{\alpha''})$, for any $\alpha'' \in A, \alpha'' \geqslant \alpha$.

Thus, it sufficient to prove that the sequence (1) does not satisfy condition ML. Let us observe that $\pi_1(B)$ is a free group with two generators a and b, and f_* is the homomorphism defined by the formulas

$$f_*(a) = aba^{-1}b^{-1}, \qquad f_*(b) = a^{-1}b^{-1}ab.$$

f_* is a monomorphism, because $f_*(a) \neq f_*(b)$, but not an epimorphism, because, for example, $f_*(x) \neq a$, for all $x \in \pi_1(B)$. Hence, for any natural m and $n, Imf_*^m \supsetneqq Imf_*^n$ only if $m > n$. It means that the inverse sequence (1) does not satisfy condition ML. $\square$

6. Movability of the orbit space

THEOREM 6.1. *Let X be a metrizable G-space. If X is G-movable then for any closed and normal subgroup H of the group G, the H-orbit space $X|_H$ is also G-movable.*

PROOF. Without losing generality one may suppose that X is a closed G-invariant subset of some $G - AR(M_G)$-space Y ([18], Theorem 1.1). $X|_H$ is a closed G-invariant subset of $Y|_H$ ([5], Theorem 3.1).

Let $\{X_\alpha, \alpha \in A\}$ be the family of all G-invariant neighborhoods of X in Y. Let us consider the family $\{X_\alpha|_H, \alpha \in A\}$, where each $X_\alpha|_H \in G-ANR(M_G)$ and is a G-invariant neighborhood of $X|_H$ in $Y|_H$. Let us prove that the family $\{X_\alpha|_H, \alpha \in A\}$ is cofinal in the family of all neighborhoods of $X|_H$ in $Y|_H$. Let U be an arbitrary neighborhood of $X|_H$ in $Y|_H$. By a theorem of Palais ([17], Proposition 1.1.14), there exists a G-invariant neighborhood $V \supset X|_H$ laying in U. Let us denote $\tilde{V} = (pr)^{-1}(V)$, where $pr : Y \to Y|_H$ is the H-orbit projection. It is evident that $\tilde{V}$ is a G-invariant neighborhood of the space X in Y and $V = \tilde{V}|_H$. So in any neighborhood of the space $X|_H$ in $Y|_H$, there is a neighborhood of type $X_\alpha|_H$, where X_α is a G-invariant neighborhood of X in Y.

Now let us prove the G-movability of the space $X|_H$. Let X be G-movable. It means that the inverse system $\{X_\alpha, i_{\alpha\alpha'}, A\}$ is G-movable. We must prove that the induced inverse system $\{X_\alpha|_H, i_{\alpha\alpha'}|_H, A\}$ is G-movable. Let $\alpha \in A$ be any index. By the G-movability of the inverse system $\{X_\alpha, i_{\alpha\alpha'}, A\}$, there is $\alpha' \in A, \alpha' > \alpha$, such that for any other index $\alpha'' \in A, \alpha'' > \alpha$, there exists a G-mapping $r^{\alpha'\alpha''} : X_{\alpha'} \to X_{\alpha''}$, which makes the following diagram G-homotopy commutative

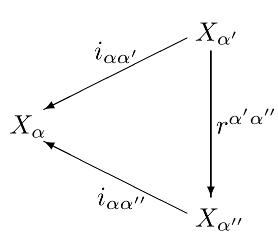

DIAGRAM 1.

It turns out that, for given $\alpha \in A$, the obtained index $\alpha' \in A, \alpha' > \alpha$, also satisfies the condition of G-movability of the inverse system

$$\{X_\alpha|_H, i_{\alpha\alpha'}|_H, A\}.$$

This is obvious, because the G-homotopy commutativity of Diagram 1 implies the G-homotopy commutativity of the following diagram

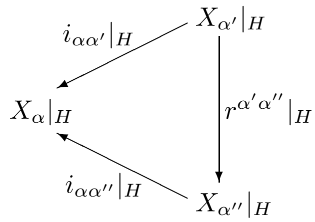

Diagram 2.

where $r^{\alpha'\alpha''}|_H : X_{\alpha'}|_H \to X_{\alpha''}|_H$ is induced by the mapping $r^{\alpha'\alpha''}$. So, the G-movability of the space $X|_H$ is proved. □

Corollary 6.2. *Let X be a metrizable G-space. If X is G-movable, then the orbit space $X|_G$ is movable.*

Proof. In the case $H = G$ from the last theorem we obtain that the orbit space $X|_G$ with the trivial action of the group G is G-movable. Therefore, it will be movable by Corollary 3.5. □

Corollary 6.2 in general is not invertible:

Example 6.3. Let Σ be a solenoid. It is known ([4], Theorem 13.5) that Σ is a non-movable compact metrizable Abelian group. By Corollary 3.5, the solenoid Σ with the natural group action is not Σ-movable although the orbit space $\Sigma|_\Sigma$ as a one-point set is movable.

The converse of Corollary 6.2 is true if the group G is a Lie group and the action is free (see Theorem 7.2).

7. Equivariant movability of a free G-space

Theorem 7.1. *Let G be a compact Lie group and let Y be a metrizable $G - AR(M_G)$-space. Suppose that a closed invariant subset X of Y has an invariant neighborhood whose orbits have the same type. If the orbit space $X|_G$ is movable, then X is equivariantly movable.*

Proof. The orbit space $X|_G$ is closed in $Y|_G$, which is a $G - AR(M)$-space. Let U be an arbitrary invariant neighborhood of X in Y. By the assumption of the theorem, it follows that there exists a cofinal family of neighborhoods of X in Y, whose orbits have the same type. Therefore, one may suppose that all orbits of the neighborhood U have the same type. The orbit set $U|_G$ will be a neighborhood of $X|_G$ in $Y|_G$. From the movability of $X|_G$ it follows that, for the neighborhood $U|_G$, there is a neighborhood $\tilde{V}$ of the space $X|_G$ in $Y|_G$, which lies in the neighborhood $U|_G$ and contracts to any preassigned neighborhood of the space $X|_G$.

Let us denote $V = (pr)^{-1}(\tilde{V})$, where $pr : Y \to Y|_G$ is the orbit projection. It is evident that V is an invariant neighborhood of the space X lying in U. Let us prove that V contracts in U to any preassigned invariant neighborhood of X. Let W be any invariant neighborhood of X in Y. We must prove the existence of an equivariant homotopy $F : V \times I \to U$, which satisfies the condition

$$F(x, 0) = x, \quad F(x, 1) \in W,$$

for any $x \in V$. Since $W|_G$ is a neighborhood of the space $X|_G$ in $Y|_G$, there is a homotopy $\tilde{F} : V|_G \times I \to U|_G$ such that

$$F(\tilde{x}, 0) = \tilde{x}, \quad \tilde{F}(\tilde{x}, 1) \in W|_G, \tag{2}$$

for any $\tilde{x} \in V|_G$. The homotopy $\tilde{F} : V|_G \times I \to U|_G$ preserves the G-orbit structure, because $V \subset U$ and all orbits of U have the same types (see Diagram 3).

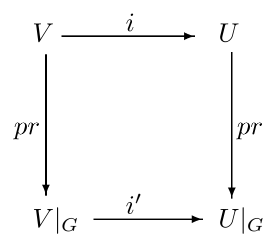

Diagram 3.

By the covering homotopy theorem of Palais ([17], Theorem 2.4.1), there is an equivariant homotopy $F : V \times I \to U$, which covers the homotopy $\tilde{F}$ and satisfies $F(x, 0) = i(x) = x$. That is, the following diagram is commutative (Diagram 4).

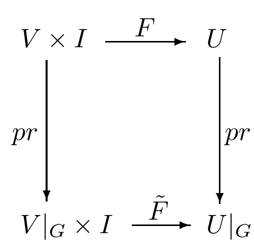

Diagram 4.

$F : V \times I \to U$ is the designed equivariant homotopy. It only remains to verify that $F(x, 1) \in W$. But this immediately follows from (2) and the commutativity of Diagram 4. □

THEOREM 7.2. *Let G be a compact Lie group. A metrizable free G-space X is equivariantly movable if and only if the orbit space $X|_G$ is movable.*

PROOF. The necessity in a more general case was proved in Corollary 6.2. Let us prove the sufficiency. Let the orbit space $X|_G$ be movable. One can consider the G-space X as a closed and invariant subset of some $G-AR(M_G)$-space Y. Let $P \subset X$ be any orbit. From the existence of slices it follows that around P there is such an invariant neighborhood $U(P)$ in Y that $typeQ \geqslant typeP$, for any orbit Q from $U(P)$ ([5], Corollary 5.5). Since the action of the group G on X is free, $typeQ = typeP = typeG$, for any orbit Q lying in $U(P)$. Let us denote $V = \cup\{U(P); P \in X|_G\}$. It is evident that V is an invariant neighborhood of the space X in Y and that all of its orbits have the same type. Then, by Theorem 7.1, X is equivariantly movable. □

Example 6.3 shows that the assumption that G is a Lie group is essential in the above theorem. The Example 8.1 which follows shows that the condition of freeness of the action of the group G is also essential in the above theorem.

8. EXAMPLE OF A NON-FREE NOT Z_2-MOVABLE SPACE WITH A MOVABLE ORBIT SPACE

EXAMPLE 8.1. Let us consider the space $P = \varprojlim\{B, f\}$ constructed in Example 5.1. Let us define an action of the group $Z_2 = \{e, g\}$ on the space B by the formulas

$$\begin{aligned} e(z,1) &= (z,1) \\ e(1,t) &= (1,t) \\ g(z,1) &= (1,z^{-1}) \\ g(1,t) &= (t^{-1},1), \end{aligned} \tag{3}$$

for any z and t from S. B is a $Z_2 - ANR(M_{Z_2})$ space with the fixed-point $b_0 = (1, 1)$.

PROPOSITION 8.2. *The mapping $f : B \to B$, defined by formulas (3), is equivariant.*

PROOF. It is necessary to prove the following two equalities:

$$\begin{aligned} f(g(z,1)) &= g(f(z,1)) \\ f(g(1,t)) &= g(f(1,t)), \end{aligned} \tag{4}$$

for any z and t from S. Let us prove the first one. Consider the following cases:

Case 1. $0 \leqslant arg z \leqslant \frac{\pi}{2} \quad \Leftrightarrow \quad \frac{3\pi}{2} \leqslant arg z^{-1} \leqslant 2\pi$.
Then $f(g(z,1)) = f(1,z^{-1}) = (1,z^{-4}) = g(z^4,1) = gf(z,1)$.
Case 2. $\frac{\pi}{2} \leqslant arg z \leqslant \pi \quad \Leftrightarrow \quad \pi \leqslant arg z^{-1} \leqslant \frac{3\pi}{2}$.
Then $f(g(z,1)) = f(1,z^{-1}) = (z^{-4},1) = g(1,z^4) = gf(z,1)$.
Case 3. $\pi \leqslant arg z \leqslant \frac{3\pi}{2} \quad \Leftrightarrow \quad \frac{\pi}{2} \leqslant arg z^{-1} \leqslant \pi$.
Then $f(g(z,1)) = f(1,z^{-1}) = (1,z^4) = g(z^{-4},1) = gf(z,1)$.
Case 4. $\frac{3\pi}{2} \leqslant arg z \leqslant 2\pi \quad \Leftrightarrow \quad 0 \leqslant arg z^{-1} \leqslant \frac{\pi}{2}$.
Then $f(g(z,1)) = f(1,z^{-1}) = (z^4,1) = g(1,z^{-4}) = gf(z,1)$.

The second equality of (4) is proved in a similar way. □

PROPOSITION 8.3. *P is a connected, compact, metrizable and equivariantly non-movable Z_2-space which is free at all points except at the only fixed point $(b_0, b_0, ...)$ and $sh(P|_{Z_2})$=0.*

PROOF. P is a Z_2-space because it is an inverse limit of $Z_2 - ANR(M_{Z_2})$-spaces B and f is an equivariant mapping. The uniqueness of the fixed point is evident. The connectedness, compactness and metrizability follows from the properties of inverse systems ([8], Theorem 6.1.20, Corollary 4.2.5). The non Z_2-movability follows from Proposition 5.3 and Corollary 3.5.

Let us prove that $sh(P|_{Z_2}) = 0$ and thus the orbit space $P|_{Z_2}$ is movable.

Let $X = \varprojlim\{B|_{Z_2}, f|_{Z_2}\}$. X is equimorphic to the orbit space $P|_{Z_2}$. Indeed, let us define a mapping $h : X \to P|_{Z_2}$ in the following way:

$$h(([x_1],[x_2],...)) = [(x_1,x_2,...)]$$

where $([x_1],[x_2],...) \in X$, and $x_1, x_2, ...$ are selected from the classes $[x_1],[x_2],...$ in such way that $(x_1,x_2,...) \in P$ or what is the same $f(x_{n+1}) = x_n$, for any $n = 1,2,....$ Let us prove that the mapping h is defined correctly. Let $\tilde{x}_1, \tilde{x}_2, ...$ be some other representatives of the classes $[x_1],[x_2],...$, respectively, satisfying the conditions $f(\tilde{x}_{n+1}) = \tilde{x}_n$ for any $n \in N$. Since each class $[x_n]$ has two representatives: x_n and gx_n, where $g \in Z_2 = \{e, g\}$, either $\tilde{x}_n = gx_n$ or $\tilde{x}_n = x_n$. But it is obvious that, if for some $n_0 \in N, \tilde{x}_{n_0} = gx_{n_0}$, then, for any $n \in N, \tilde{x}_n = gx_n$, because f is equivariant. Thus, in the case of another choice of the representatives of the classes $[x_1],[x_2],...$, we have

$$h(([x_1],[x_2],...)) = [(\tilde{x}_1,\tilde{x}_2,...)] = [(gx_1,gx_2,...)] = \\ = [g(x_1,x_2,...)] = [(x_1,x_2,...)].$$

However, h is a continuous bijection and thus, it is a homeomorphism ([8], Theorem 3.1.13).

Consequently,

$$P|_{Z_2} = \varprojlim\{B|_{Z_2}, f|_{Z_2}\},$$

where $B|_{Z_2} \cong S$ and the mapping $\bar{f} = f|_{Z_2} : S \to S$ is defined by the formulas:

$$\bar{f}(z) = \begin{cases} z^4, & 0 \leqslant arg(z) \leqslant \frac{\pi}{2} \\ z^{-4}, & \frac{\pi}{2} \leqslant arg(z) \leqslant \frac{3\pi}{2} \\ z^4, & \frac{3\pi}{2} \leqslant arg(z) \leqslant 2\pi \end{cases} \tag{5}$$

for any $z \in S$. Thus, we conclude that the orbit space $P|_{Z_2}$ is a limit of the inverse sequence

$$S \xleftarrow{\bar{f}} S \xleftarrow{\bar{f}} S \xleftarrow{\bar{f}} \cdots$$

By formula (5), the mapping $\bar{f}$ induces a homomorphism $\bar{f}_* : \pi_1(S) \to \pi_1(S)$, which acts as follows:

$$\bar{f}_*(a) = aa^{-1}a^{-1}a,$$

where $a \in \pi_1(S) \cong Z$ is the generator of the group Z. From the above formula, it follows that $\bar{f}_*$ is the null-homomorphism and thus, $deg\bar{f} = 0$. For any $k = 1, 2, \cdots, \bar{f}_*^k$ is also a null-homomorphism and thus, $deg\bar{f}^k = 0$. Therefore, by the classical Hopf theorem ([10], Section 2.8, Theorem H^n) all $\bar{f}^k : S \to S$ are null-homotopic and $sh(P|_{Z_2}) = 0$. □

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Department of Higher Geometry and Topology
Moscow State University
Russia
E-mail: gevorgyan@mech.math.msu.su, pgev@angelfire.com

Received: 31.08.2001
Revised: 22.01.2003. & 26.05.2003