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Collision-free Piecewise Quadratic Spline with Regular Quadratic Obstacles

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ABSTRACT

We classify mutual position of a quadratic Bézier curve and a regular quadric in three dimensional Euclidean space. For given first and last control point, we find the set of all quadratic Bézier curves having no common point with a regular quadric. This system of such quadratic Bézier curves is represented by the set of their admissible middle control points. The spatial problem is reduced to a planar problem where the regular quadric is represented by a conic section. Then, the set of all middle control points is found for each type of conic section separately. The key issue is to find the boundary of this set. It is formed from the middle control points of the Bézier curves touching the given conic section. Our results are applicable in collision-free paths computation for virtual agents where the obstacles are represented or bounded by regular quadrics. Another application can be found in searching for pointwise space-like curves in Minkowski space.

Key words: Bézier quadratic curve, regular quadric, intersection, collision-free paths

MSC2010: 65D17, 14P25, 51M04

Kvadratni splajnovi, po dijelovima bez kolizija, s regularnim kvadratnim barijerama

SAŽETAK

U trodimenzionalnom euklidskom prostoru klasificiramo međusobni odnos kvadratne Bézierove krivulje i regularne kvadrike. Za danu prvu i zadnju kontrolnu točku, nalazimo skup svih kvadratnih Bézierovih krivulja koje nemaju zajedničku točku s regularnom kvadrikom. Sustav ovakvih kvadratnih Bézierovih krivulja prikazuje se skupom njihovih dopustivih srednjih kontrolnih točaka. Prostorni problem svodi se na ravninski problem gdje konika predstavlja regularnu kvadriku. Tada se za svaku vrstu konike zasebno nalazi skup svih srednjih kontrolnih točaka. Glavna zadaća je naći granicu ovakvog skupa. Spomenutu granicu čine središnje kontrolne točke Bézierovih krivulja koje diraju koniku. Naši rezultati primjenjuju se u računanju putanja bez kolizija za virtualna sredstva gdje su barijere prikazane ili ograničene regularnim kvadrikama. Drugu primjenu nalazimo u istraživanju točkovnih prostornih krivulja u prostoru Minkowski.

Ključne riječi: Bézierove kvadratna krivulja, regularna kvadrika, presjek, dijelovi bez kolizija

1 Introduction

We consider a problem of finding conditions for collision-free piecewise quadratic path in $\mathbb{E}^d$ with respect to a regular quadric. If the quadric is a bounding domain of a complex object, we reduce the complexity of exact collision detection considerably.

For illustration, we work in three-dimensional space in which a regular quadric is situated. Taking two arbitrary fixed points out of the quadric in the same connected component, we find a set of all parabolic arcs connecting these two points such that this path and the regular quadric have no intersection. The spatial problem is reduced to a planar problem where the the regular quadric is represented by a conic section.

In most algorithms, the finding of collision-free path consists of two main steps. The first one is acquisition of linear spline path generated by sample-based planning algorithms. The second step is smoothing of the path because in mobile robotics a non smooth motions can cause slippage of wheels.

The finding of smooth collision-free path using Bézier curves was considered e.g. in [9], [11], [10]. But these algorithms are used as post processing, because they assume some linear collision-free path and they only smooth the path. Moreover, the algorithms provide only a numerical solution. Our method offers a direct analytical computation of all possible collision-free smooth paths without using sample-based planning algorithms. We assume an obstacle represented by a regular quadric and given start and end position of robot. We find all quadratic Bézier curves constituting the set of collision-free paths. One can use such a set for optimization of the sought path.

Sometimes the scene with obstacles is too much complicated and the smooth collision-free path cannot be found directly. Then, the use of sample-based planning algorithms is necessary. But the obtained linear path is jerky, because it contains many redundant nodes which was generated randomly. In order to remove these nodes the path pruning techniques as in [5] are used. Our results can also form a path pruning algorithm, where the node v_i can be removed if there is a quadratic Bézier collision-free path between nodes v_{i-1} and v_{i+1}. Such an algorithm is more flexible comparing with a piecewise linear approach.

There is another use for the affine three dimensional Minkowski space typically determined by a light cone. The points lying outside the cone are called space-like. If we take two space-like points as a first and a last control point, we can find all quadratic pointwise space-like Bézier curves. We analyze this problem in [3] for parabola. Some statements for other conic sections, we summarize in [13]

2 Theoretical background

In this section, we mention basic definitions of Bézier curves, regular quadrics and all concepts required later in the text.

Let $\mathbb{E}^3$ be three dimensional vector Euclidean space formed by vectors $x = (x_1, x_2, x_3)$ with scalar product $\langle\cdot,\cdot\rangle\colon \mathbb{E}^3 \times \mathbb{E}^3 \to \mathbb{R}$. By standard construction, we get an affine space with a Cartesian coordinate system $\langle O, e_1, e_2, e_3\rangle$. Let $\mathbb{E}^2$ be a Euclidean plane. We assign to each family of parallel lines a unique point at infinity, at which "all of such lines meet". All the points at infinity define the line at infinity l^∞. The *extended Euclidean plane*, denoted by $\overline{\mathbb{E}^2}$, is obtained as $\overline{\mathbb{E}^2} = \mathbb{E}^2 \cup l^\infty$. More can be found in [1], [2].

Let $M_{3,3}(\mathbb{R})$ be the set of 3×3 matrices with real coefficients. A quadratic form is the map $q\colon \mathbb{R}^3 \to \mathbb{R}$, where $q(x) = xQx^\top$ for the symmetric $Q \in M_{3,3}(\mathbb{R})$. We talk about *regular quadratic form* if the matrix Q is diagonal with entries $\lambda_{1,2,3} \in \{-1,1\}$ in a certain basis of $\mathbb{R}^3$. The unique symmetric bilinear form giving rise to q is denoted by P and called the *polar form* of q. We have $q(x) = P(x,x)$, $P(x,w) = xQw^\top$. In the associated affine space, we have $P(X,W) = XQW^\top$. Let Q be regular and the point $W \in \mathbb{R}^3$ be fixed. We call the set $W^\perp = \{X \in \mathbb{R}^3 \colon P(X,W) = 0\}$ the *polar (hyperplane)* of W. We say that X and W are conjugate with respect to the polar form P, and we denote this fact by $X \perp W$. For the *self-polar* point W holds $P(W,W) = 0$. In the affine space $\mathbb{R}^2$, the $W^\perp$ is the polar line determined by equation $WQX^\top = 0$, where the variable $X = (x,y,1)$. An image of a *regular quadric* κ is the set of all points $X \in \mathbb{E}^3$ such that for its position vector $x = X - O$ the equality $q(x) = 0$ holds. The properties of quadrics are in [8].

Let $Q_K \in M_{3,3}(\mathbb{R})$ be a symmetric matrix. The algebraic curve of degree 2 called *conic section* is the set $K = \{[x,y] \in \mathbb{R}^2 \colon f(x,y) = 0 \text{ for } f(x,y) = (x\ y\ 1)Q_K(x\ y\ 1)^\top\}$. In appropriate cases, we consider the equation of the conic section instead of K due to the fact that the field $\mathbb{R}$ is not algebraically closed. In the extended Euclidean plane, it is necessary to homogenize the equation of conic section by replacing $(x,y,1)$ with (x,y,z). We obtain the conic section $\overline{K} = \{[x,y,z] \in \overline{\mathbb{R}^2} \colon (x\ y\ z)Q_K(x\ y\ z)^\top = 0\}$ in homogeneous coordinates. More on plane algebraic curves we refer to the book [7]. The polar line $N^\perp$ of such a point N that $P(N,N) > 0$ splits the conic section K into some arcs. If $N^\perp \cap K = \{T_1, T_2\}$ then we denote the arc $\widehat{T_1T_2} = \{X \in K \colon P(N,X) > 0\}$.

From each point $X \in \mathbb{R}^2$ lying outside the regular conic section K, one can construct two tangent lines to $\overline{K} \subset \overline{\mathbb{R}^2}$. The corresponding points of contact may be either *affine* or *at infinity*. In the case of point of the contact at infinity, the conic section $K \subset \mathbb{R}^2$ is a hyperbola and the projective tangent line is called asymptote a in affine space. We denote its point of contact with K at infinity by a^∞, see fig. 7 for an example. We denote the set of all tangent lines to K by T_K. Hence, $T_K = \{\ell \subset \mathbb{R}^2 \colon \text{each point } X \in \ell \text{ satisfies } 0 = \langle \nabla f(X_0), X - X_0\rangle \mid X_0 \in K \cap \ell\}$. We denote by $\nabla f(x_0,y_0)$ the gradient $\left(\frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0)\right)$ of K at the point $[x_0,y_0] \in K$. Clearly, in a regular point, it is the normal vector of its tangent line.

For counting the number of real roots of real polynomial function in an interval, the theorem below is useful.

Theorem 1 (Budan-Fourier) *Let $f(x) = \sum_{i=0}^n a_i x^i \in P_n$, $n > 0, a_n \neq 0$. Let $\alpha < \beta$ and $f(\alpha)f(\beta) \neq 0$ and let $V(x)$ be the number of changes in sign in the sequence $\{f(x), f'(x), \ldots, f^{(n)}(x)\}$. Then, the number (including multiplicity) of real roots of the equation $f(x) = 0$ lying in the interval $\langle\alpha,\beta\rangle$ is equal to or is smaller, by an even number, than $V(\alpha) - V(\beta)$.*

We consider the collision-free path represented in Bézier form.

Definition 1 (Bézier curve) Bézier curve *of degree n in the space $\mathbb{E}^3$ is a polynomial map $b\colon [0,1] \to \mathbb{E}^3$ given by $b(t) = \sum_{i=0}^n B_i^n(t) b_i$. The points $b_i \in \mathbb{E}^3$ are called* control points, *the functions $B_i^n(t) = \binom{n}{i}(1-t)^{n-i}t^i$ for $i \in \{0,\ldots,n\}$ are Bernstein polynomials of degree n.*

More about the properties of Bézier curves can be found in [6], [4].

Lemma 1 (Determination of quadratic Bézier curve by its tangent) *Let the points $A,B,T \in \mathbb{R}^2$ be non-collinear and ℓ_T be a line such that segment $AB \cap \ell_T = \emptyset$, $T \in \ell_T$. Then, the quadratic Bézier curve $b(t)$ with the end points A,B, containing the point T and with the tangent line ℓ_T at the point T exists and is uniquely determined.*

Proof. Let the vector $\ell = (\ell_x, \ell_y) \neq (0,0)$ be the direction vector of the tangent line ℓ_T and $A = [a_x, a_y], B = [b_x, b_y], T = [t_x, t_y]$. Let the vector $n_\ell = (-\ell_y, \ell_x)$. We define the map τ such that $\tau(A,B,T,\ell) = C$ returns the middle control point of the Bézier curve $b(t)$.

$$\tau(A,B,T,\ell) = C = \frac{T - B_0^2(t_0)A - B_2^2(t_0)B}{B_1^2(t_0)} ,$$

where $t_0 \in (0,1)$ is a solution of the equation

$$0 = \alpha t^2 + 2\beta t + \gamma ,$$

and

$$\alpha = \ell_x(b_y - a_y) - \ell_y(b_x - a_x) ,$$
$$\beta = \ell_x(a_y - t_y) - \ell_y(a_x - t_x) ,$$
$$\gamma = -\beta .$$

The Bézier curve $b_{ACB}(t)$ with its control points A,C,B in this order satisfies the requirements of the theorem. For proving the existence of b_{ACB}, we use the affine transformation mapping the three independent points A,B,T to the points $A = [-1,0]$, $B = [1,0]$, $T = [0,1]$. We obtain the coefficients $\alpha = -2\ell_y$, $\beta = -\ell_x + \ell_y$, $\gamma = \ell_x - \ell_y$. After computing the discriminant, we do discussion with respect to direction of the vector ℓ. □

This statement is possible to extend in a natural way for degenerate situation in which collinearity of the points A,B,T or even $A = B$ hold.

3 Collision-free situation

Let us consider the Euclidean space $\mathbb{E}^3$, let $\langle O, e_1, e_2, e_3 \rangle$ be an affine coordinate system, with a regular quadric κ represented by the matrix Q. Let the points $A = [a_1, a_2, a_3]$ and $B = [b_1, b_2, b_3]$ be fixed and $a = A - O, b = B - O$ are their position vectors. Assuming the quadric is an enclosing volume of some obstacle and the points A,B are start and end position of a robot we require $q(a) > 0$ and $q(b) > 0$, i.e. A,B lie outside the quadric κ. We look for all collision-free (relative to quadric) paths supply by quadratic Bézier curves from the point A to the point B. In other words, we look for the set of all such points C that the Bézier curve $b_{ACB}(t)$ lie out of quadric. Thus, for all points $X \in b_{ACB}(t)$ and their position vectors $x = X - O$ the inequality $q(x) > 0$ holds. In applications, $A \neq B$ yields, however we solved also the case $A = B$ for the sake of completeness.

A generic quadratic Bézier curve is a part of a parabola, so it lies in the affine plane $\rho \subset \mathbb{E}^3$. Since the given points $A,B \in \rho$, the construction of the plane ρ may have several degrees of freedom depending on their positions. Using the equation $\rho = \{X \in \mathbb{E}^3; X = A + tv + sw, \text{for } t,s \in \mathbb{R}\}$, the degree of freedom is represented by the $\dim[v,w]$. If $A \neq B$, the degree of freedom is 1. As the position of the point C changes, the plane ρ accordingly contains the axis $\overleftrightarrow{AB}$, so we can choose as $v = B - A$ and the choice of the vector w is free. If $A = B$, the degree of freedom is 2 and the choice of both vectors v,w is free (up to linear dependency).

In any case, the intersection of the quadric κ and the plane ρ is a conic section K (see fig. 1(a)). The figure 1(b) shows all cases how the set S of all points X that $q(x) > 0$ in the possible types of plane ρ looks like. The collision-free Bézier curve $b_{ACB}(t) \subset S$. We present a solution in the plane ρ for each type of conic section and the planar results can be put together to form the spatial result.

As we shall see later, it is useful to consider ρ as an extended Euclidean plane. However, the control points $A,B,C \notin l^\infty$. Let $\langle O,x,y \rangle_\rho$ be any Cartesian coordinate system in the plane ρ. Let $A = [a_x, a_y]$, $B = [b_x, b_y]$ a $C = [c_x, c_y]$ be the local affine coordinates of the control points in $\langle O,x,y \rangle_\rho$.

Definition 2 (Set of admissible solutions) *Let $V_\rho(A,B)$ be a set of points $C \in \rho$ such that the curve b_{ACB} is collision-free. Then, we say that $V_\rho(A,B)$ is a* set of admissible solutions *in the plane ρ with respect to A,B. If no confusion arises, we say the set of admissible solutions.*

By $V_\rho^v(A,B)$, we denote the set of points $C \in \rho$ such that $b_{ACB} \cap K = M$, where M is the set of contact points of order 2 between $b_{ACB}(t)$ and K. We denote the set of points $C \in \rho$ such that b_{ACB} and K have transversal intersection (i.e. directions of the intersecting curves at the intersection point are linearly independent) by $V_\rho^t(A,B)$. The set M contains at most two such points, since two componentwise different quadratic curves may have at most two common points of contact of order 2 (see e.g. Bézout theorem, [7]).

For the given points A,B, the union of disjoint sets $V_\rho(A,B) \cup V_\rho^v(A,B) \cup V_\rho^t(A,B)$ gives the whole plane ρ.

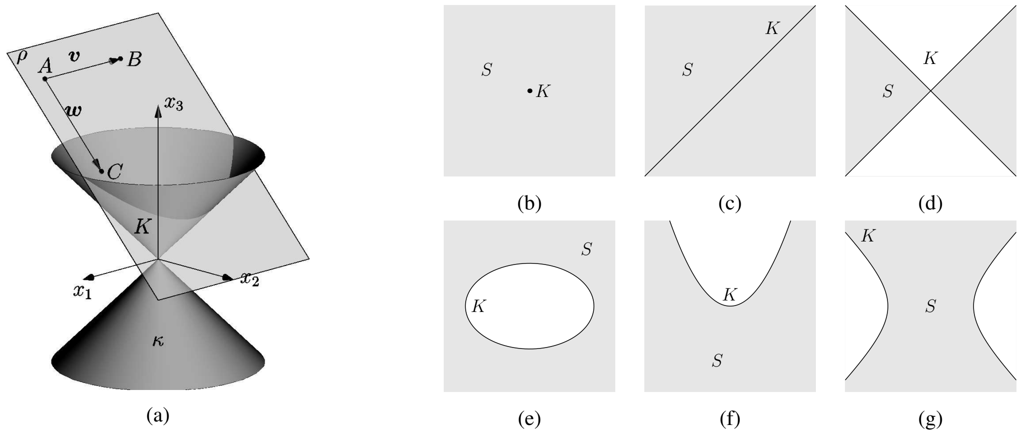


Figure 1: (a) *Plane* ρ *spans points* A,B,C. *In case of their non-collinearity, they generate* ρ *as affine hull. The conic section* K *is the intersection of the quadric* κ *and the plane* ρ.
(b–g) *Let* $K \subset \rho$ *be the conic section (point, double line, pair of lines, ellipse, parabola, hyperbola). The set* S *consists of all points lying out of quadric in the plane* ρ.

4 Exterior and interior points of contact

At first, we study the touching situation, that is the set $V^v_\rho(A,B)$.

Definition 3 (Set of points of contact) *We say that the set* $D \subset K$ *is the* set of points of contact *between* K *and the set of all* b_{ACB} *if for any point* $X_0 = [x_0, y_0] \in D$ *there is a point* C_0 *such that* $C_0 \in V^v_\rho(A,B)$ *and* $X_0 \in b_{AC_0B} \cap K$.

We say that b_{ACB} has *double contact*, if it has exactly two points of contact of order 2 with K (i.e. the set M contains exactly two points). We denote the middle control point of such Bézier curve by C_u. If K is a regular conic section, we denote the touching points by the letters $U_i, i = 1,2$ (see fig. 4). If K is a singular conic section, we denote the touching points by the symbol S_i, for $i = p,r$ (see fig. 5(b)).

When we obtain K as a connected component of a regular conic section, it may be an ellipse, a parabola or one component of a hyperbola. Since the connected component of regular conic section is convex, the tangent line is a supporting line of the component. In the case of singular conic section, the connected component may be a point (the top of the light-cone) or one isotropic line.

Let ℓ_T be a tangent line to a connected component K of conic section (regular or singular) at the point $T \in K$. It divides the plane ρ into two half-planes $\overline{H}^+_\ell, \overline{H}^-_\ell$ such that $K \subset \overline{H}^+_\ell$. We say, that ℓ_T *separates* the connected component K of the conic section and the arbitrary set of points O , if they lie in the different half-planes with respect to the tangent ℓ_T, i.e. $K \subset \overline{H}^+_\ell$ and $O \subset \overline{H}^-_\ell$. Additionally, in the case of hyperbola the asymptotes separate similarly K and O, but the corresponding touching point is infinite. We denote the set of all separating tangent lines and asymptotes by $T_{sep}(O,K)$. We denote by $S(O,K) \subset K$ the maximal open set of all *affine* points of contact of K and separating tangent lines.

Definition 4 (Exterior (interior) point of contact) *We say that the curve* b_{ACB} touches *a connected component of the regular conic section* K *from* outside (inside)*, if their common tangent is (is not) separating. Then, the point of contact is called* exterior (interior) point of contact*. The set of all exterior (interior) points of contact is denoted* D_{ext} (D_{in}).

The set of points of contact $D = D_{ext} \cup D_{in}$. Now, we describe the set of points of contact D for every type of conic section. First, we consider the regular conic sections, then we analyze the singular conic sections.

The polar line of the point A determined by the equation $AQ_KX^\top = 0$ splits the conic section K to some arcs. Let K be a regular conic section different from hyperbola. The condition that the point A is separated from K with respect to ℓ_T is that the point $T \in \{X \in K : AQ_KX^\top \geq 0\}$. So, the open set $S(A,K) = \{X \in K : AQ_KX^\top > 0\}$ (see fig. 2(a)). Now, let K_1, K_2 be the connected components of a hyperbola K. We consider each of them separately and we get $S(A,K_1) = \{X \in K_1 : AQ_KX^\top > 0\}$ and $S(A,K_2)$ (see also fig. 2(b)).

Let K be a connected component of a regular conic section. Let $\ell_T \in T_K$ and the point $T \in K$ be its point of contact. The tangent line $\ell_T \in T_{sep}(AB,K)$ if and only if $T \in \overline{S}(A,K) \cap \overline{S}(B,K) = \{X \in K : AQ_KX^\top \geq 0 \wedge BQ_KX^\top \geq 0\}$ (see fig. 2(a)).

We denote the set of all $X \in K$ such that the corresponding tangent line ℓ_X to K at X contains both control points A,B by D_{AB}. The set $D_{AB} = \{X \in K : AQ_KX^\top = 0 \wedge BQ_KX^\top = 0\}$.

(a) (b)

Figure 2: (a) *The set of all tangent lines of K, which separate the point A and K, is determined by arc* $\overset{\frown}{T_1T_2^+} = S(A,K)$. *Similarly, the arc* $\overset{\frown}{T_1^+T_2} = S(B,K)$. *The arc* $\overset{\frown}{T_1T_2}$ *determines all tangent lines* $T_{sep}(AB,K)$. *We obtain it as intersecion of the sets* $\overline{S}(A,K)\cap\overline{S}(B,K)$.

(b) *We have the set* $S(A,K_1) = \{X \in K_1 : AQ_KX^\top > 0\} = \overset{\frown}{Ta_2^\infty}$. *All tangent lines of connected component* K_2 *separate this component and the point A. Hence, we have the set* $S(A,K_2) = K_2$.

Theorem 2 (Set of exterior points of contact) *Let* K *be a regular connected component of conic section. The set of exterior points of contact* $D_{ext} \neq \emptyset$ *if and only if the segment* $AB\cap K = \emptyset$ *or* $AB\cap K = \{T\}$.
The set $D_{ext} = \{X \in K : AQ_KX^\top > 0 \wedge BQ_KX^\top > 0\} \cup D_{AB}$.

Proof. Sufficient condition. Let $AB \cap K = \{X_1, X_2\}$. Let ℓ be any tangent line to K. In order that the line ℓ separates AB and K (up to the point of contact if it exists), they must lie in the different half-planes with respect to ℓ (up to the point of contact). But the points $X_1, X_2 \in K$ lie in the same half-plane as AB. Hence, $\overline{S}(A,K)\cap\overline{S}(B,K) = \emptyset$ and there is no separating tangent line $\ell_T \in T_{sep}(AB,K)$. So, there exists no separating tangent line for any Bézier curve b_{ACB}. Consequently, the set $D_{ext} = \emptyset$.
Necessary condition. We discuss each case separately. Mainly, we use the fact that the quadratic Bézier curve is a convex curve, each of its tangent line defines a supporting half-plane to the curve. If $\ell_T \in T_{sep}(A,K)$ and $\ell_T \in T_{sep}(B,K)$, then $\ell_T \in T_{sep}(b_{ACB},K)$. So, for every $T \in \overline{S}(A,K)\cap\overline{S}(B,K)$ holds that $\ell_T \in T_{sep}(b_{ACB},K)$. At the end, we decide if endpoints T_1, T_2 of the intersection $\overline{S}(A,K)\cap\overline{S}(B,K)$ belong to the set D_{ext}. It is shown they do not in general. But if one of the triplets of points A,B,T_1 and A,B,T_2 is collinear, without loss of generality A,B,T_1, then the point $T_1 \in D_{AB}$. But $T_1 \in D_{ext}$ too, because the corresponding Bézier curve is a segment tangent to K. Hence, the set $D_{ext} = \{S(A,K)\cap S(B,K)\} \cup \{T_1\} \neq \emptyset$. □

Note, there might be two continuous arcs D^1_{ext}, D^2_{ext}, one on each connected component K_1, K_2 of the hyperbola K. Then, $D_{ext} = D^1_{ext} \cup D^2_{ext}$.
The line $\overleftrightarrow{AB}$ divides the plane ρ into two half-planes, the open half-plane ρ_-, and the closed half-plane $\overline{\rho_+}$. Let us sort the tangent lines from A and B to K, that are not in $T_{sep}(AB,K)$, into pairs. If there are only two tangent lines not in $T_{sep}(AB,K)$, we denote them ℓ_1^+, ℓ_2^+ (see fig. 3(a)). If there are four tangent lines not in $T_{sep}(AB,K)$, we determine two pairs ℓ_1^+, ℓ_2^+ and ℓ_1^-, ℓ_2^- such that the corresponding points of contact $T_i^\pm = \ell_i^\pm \cap K, i = 1,2$ lie in the same half-plane $T_i^+ \in \overline{\rho_+}$ and $T_i^- \in \rho_-, i = 1,2$. If $\ell_1^+ \cap \ell_2^+ = P^+ \in \overline{\rho_+}$. We say the tangents ℓ_1^+, ℓ_2^+ *converge*. If $P^+ \in \rho_-$ or P^+ is a point at infinity, we say they *diverge* (see fig. 3(b)).

(a) (b)

Figure 3: (a) *There are only two tangent lines not in* $T_{sep}(AB,K)$. *The corresponding points of contact* $T_1^+, T_2^+ \in \overline{\rho_+}$, *so we denote the tangent lines* ℓ_1^+, ℓ_2^+. *They diverge, because their intersection lies in the half-plane* ρ_-.
(b) *If there are four tangent lines not in* $T_{sep}(AB,K)$, *we determine two pairs* ℓ_1^+, ℓ_2^+ *and* ℓ_1^-, ℓ_2^- *according to corresponding points of contact. The pair* ℓ_1^+, ℓ_2^+ *converge, because their intersection* $P^+ \in \overline{\rho_+}$. *However, the pair* ℓ_1^-, ℓ_2^- *diverge.*

Theorem 3 (Set of interior points of contact) *Let K be a regular connected conic section. The set of interior points of contact $D_{in}^{+} \neq \emptyset$ ($D_{in}^{-} \neq \emptyset$) iff the pair of tangents ℓ_1^{+}, ℓ_2^{+} (ℓ_1^{-}, ℓ_2^{-}) converges.*
Moreover, let $\ell_1, \ell_2 \notin T_{sep}(AB,K)$ be a pair of converging tangent lines and $P^{+} = \ell_1 \cap \ell_2 \in \rho_{+}$. If there exists a point $C_u \in \rho_{+}$ such that curve b_{AC_uB} has double contact (at the points U_1, U_2), then the set $D_{in}^{+} = \{X \in K \cap \rho_{+} \colon AQ_KX^{\top} < 0 \wedge BQ_KX^{\top} < 0\} \setminus \{\widehat{U_1U_2}\}$ (see fig. 4). Else, the set $D_{in}^{+} = \{X \in K \cap \rho_{+} \colon AQ_KX^{\top} < 0 \wedge BQ_KX^{\top} < 0\}$. A similar statement holds for a pair of tangent lines converging in the half-plane ρ_{-} and D_{in}^{-}.

Proof. Necessary condition of the existence $D_{in}^{+} \neq \emptyset$. Let the pair of tangent lines ℓ_1^{+}, ℓ_2^{+} converges in the half-plane ρ_{+}. We consider the parallel line v_a, resp. v_b containing the point A, resp. B, and have no common points with K in the half-plane ρ_{+}. Let the lines v_a, v_b determine the point at infinity v^{∞}. Let $C_m = \frac{1}{2}A + \frac{1}{2}B$ be. Let the line v_m be parallel to the lines v_a, v_b and $C_m \in v_m$. Let us construct the sequence $\{C_i\}_{i\in\mathbb{N}} \subset \rho_{+}$ such that $C_i \in v_m$ and $\lim_{i\to\infty} C_i = v^{\infty}$. There exists $n \in \mathbb{N}$ such that for each C_i, where $i > n$, the Bézier curve $b_{AC_iB} \cap K = \emptyset$. On the other hand, there exists $k \in \mathbb{N}$ such that for C_k the Bézier curve b_{AC_kB} intersects the conic section K. So there must exists some real $l \in (k,n)$ that for the Bézier curve $b_{AC_lB} \cap K \in D_{in}^{+}$. The proof for the set D_{in}^{-} can be done in a similar way.
Now, we indicate how to get the expression of the set D_{in}^{+}. Let us construct the curve $\Gamma(X) = \tau(A,B,X,\ell_X)$, where $X \in \{X \in K \cap \rho_{+} \colon AQ_KX^{\top} < 0 \wedge BQ_KX^{\top} < 0\} = \widehat{T_1T_2}$ and ℓ_X is the direction vector of the tangent line ℓ_X to the conic section K at the point X. The curve Γ is connected, because the continuous map τ maps the connected set $\widehat{T_1T_2}$ onto one connected curve. Let $s\colon (0,1) \to \widehat{T_1T_2}$ be any regular parameterization of the arc $\widehat{T_1T_2}$ such that for $s \to 0$ be $X \to T_1$ and for $s \to 1$ be $X \to T_2$. Then, we study a self-intersections of the curve $\Gamma(X)$. It can be proved the curve Γ has at most one self-intersection.
If Γ has one self-intersection $C_u = \Gamma(U_1)$, it means, there exists another point $U_2 \in \widehat{T_1T_2}$ that $C_u = \Gamma(U_2)$ and the Bézier curve b_{AC_uB} has double contact with K. There exist two special Bézier curves lying in $\overline{\rho_{+}}$, the curve $b_{AC_{2T}B}$ having two transversal intersections with K and the curve $b_{AC_{4T}B}$ having four transversal intersections with K. We construct the set of Bézier curves $L(w) = (1-w)b_{AC_{2T}B} + wb_{AC_{4T}B}$, where $w \in [0,1]$. According to the Hurwitz theorem (Th.(1,5) in [12]), there exists the point $C_3 \in \overline{\rho_{+}}$ that Bézier curve b_{AC_3B} has two transversal intersections and one common point of contact T_3 with the conic section K. The point $T_3 \in \widehat{T_1T_2}$ and then $C_3 \in \Gamma$ for some parameter $s_3, s_1 < s_3 < s_2$. Hence, the points of the arc $\widehat{U_1U_2}$ generate the Bézier curves with transversal intersections with K, because there is not other self-intersection of Γ except C_u. The set has the form $D_{in}^{+} = \{X \in K \cap \rho_{+} \colon AQ_KX^{\top} < 0 \wedge BQ_KX^{\top} < 0\} \setminus \{\widehat{U_1U_2}\}$. Note, that $U_1, U_2 \in D_{in}^{+}$.
If Γ does not have a self-intersection, for all the points $X \in \widehat{T_1T_2}$, the corresponding Bézier curves $b_{A\Gamma(X)B}$ have only one common point with K, the point of contact X. Hence, the set $D_{in}^{+} = \{X \in K \cap \rho_{+} \colon AQ_KX^{\top} < 0 \wedge BQ_KX^{\top} < 0\}$. □

5 Set of admissible points of contact

As we said, the set of all points of order 2 contact $D = D_{ext} \cup D_{in}$. The following theorem describes the set D for various regular types of the conic section K.

Theorem 4 (Set of points of contact)

(a) *Let K be an ellipse. Then, the set of the points of contact D is either one arc of the exterior points of contact or one arc of exterior and one arc of interior points of contact or one or two arcs of interior points of contact.*

(b) *Let K be a parabola. Then, the set of the points of contact D is either one arc of the exterior points of contact or one arc of interior points of contact.*

(c) *Let K be a hyperbola. Then, the set of the points of contact D is either two arcs of the exterior points of contact or one arc of exterior and one arc of interior points of contact.*

The arcs of the interior points of contact may have two affiliated components. The set of exterior points of contact may contains only one point T, when segment $AB \cap K = \{T\}$.

Proof. a) (Ellipse) Let $AB \cap K = \emptyset$. Then among all tangents passing through A or B to K, there are two in the set $T_{sep}(AB,K)$. They determine one arc D_{ext}. The other two tangents are not in the set $T_{sep}(AB,K)$. If they converge, then there is also one arc D_{in}. If they diverge, $D_{in} = \emptyset$. If $AB \cap K = \{T\}$, the only difference is that the separating pair of tangent lines becomes one line $\overleftrightarrow{AB}$ and $D_{ext} = \{T\}$. Let $AB \cap K = \{X_1, X_2\}$. Then, we consider two pairs of tangent lines ℓ_1^{+}, ℓ_2^{+} and ℓ_1^{-}, ℓ_2^{-}. One pair, without loss of generality the pair ℓ_1^{+}, ℓ_2^{+}, always converge so $D_{in}^{+} \neq \emptyset$. The pair ℓ_1^{-}, ℓ_2^{-} may converge or diverge, so we can obtain D as one or two arcs of D_{in}.
b) c) The proof for parabola and hyperbola is omitted, because it can be derived in a similar way. □

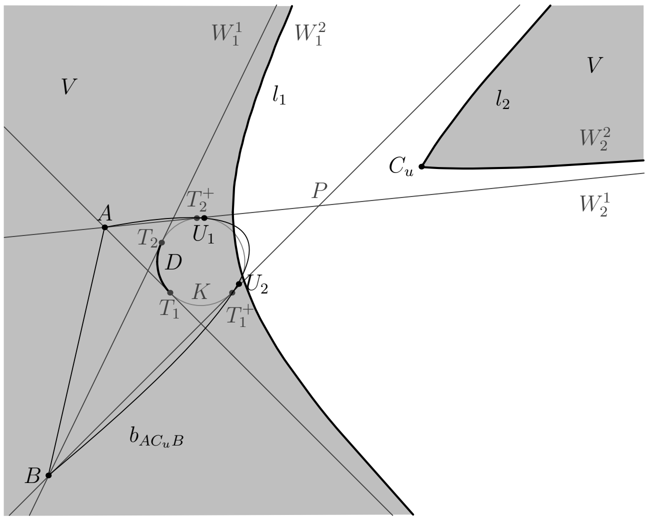


Figure 4: *The set* $D_{ext} = \{X \in K\colon AQ_KX^\top > 0 \wedge BQ_KX^\top > 0\} = \stackrel{\frown}{T_1T_2}$. *The set* $D_{in} \neq \emptyset$*, because the point* P *lies in the same half-plane generated by* $\overleftrightarrow{AB}$ *as the points* T_1^+, T_2^+. *The set* $D_{in} = \{X \in K\colon AQ_KX^\top < 0 \wedge BQ_KX^\top < 0\} \setminus \{\stackrel{\frown}{U_1U_2}\}$ *consists of two affiliated components* $\stackrel{\frown}{T_2^+U_1} \cup \stackrel{\frown}{U_2T_1^+} \cup \{U_1, U_2\}$. *The split of the arc is caused by the existence of the curve* b_{AC_uB}*, which has double contact with the conic section* K. *As one can see,* $b_{AC_uB} \cap K = \{U_1, U_2\}$. *Therefore, the set of points of contact* $D = D_{ext} \cup D_{in}$ *generates the curves* l_1, l_2 *such that* $l_1 \cup l_2 = \partial V_\rho(A,B)$. *Because the curve* l_1 *is generated by the exterior points of contact, the region* W_1^1 *containing the points* A, B *is subset* $W_1^1 \subset V_\rho(A,B)$. *The curve* l_2 *is generated by the interior points of contact. It bounds region* W_2^2 *not containing the points* A, B *and* $W_2^2 \subset V_\rho(A,B)$ *according to theorem 7 . The set of admissible solutions* $V_\rho(A,B) = W_1^1 \cup W_2^2$ *consists of two regions.*

Now, let K be a singular conic section. In the case of $K = \{V_Q\}$ (see fig. 1(b)), where V_Q is the top of the isotropic cone Q, we have $D = \{V_Q\}$ (see fig. 5(a)).
If $K = p$, where p is an isotropic double line (see fig. 1(c)), we must distinguish two cases. If A, B lie in the opposite half-planes generated by the line p, we have $D = \emptyset$. If A, B lie in the same half-plane, the set of points of contact $D = p$.
The last singular case is $K = p \cup r$, where p, r are a pair of distinct isotropic lines (see fig. 1(d)). Then, there are two regions of points lying out of K in the plane ρ. If A, B lie in the different regions, there is no collision-free Bézier curve b_{ACB}. Let A, B lie in the same region, which is determined by two half-lines $\overrightarrow{V_QP} \subset p$ and $\overrightarrow{V_QR} \subset r$ (see fig. 5(b)). Let $S_p \in p$ and $S_r \in r$ are the points of contact of the Bézier curve b_{AC_uB}, i.e. $b_{AC_uB} \cap K = \{S_p, S_r\}$ is double contact. Then $D = \overrightarrow{S_pP} \cup \overrightarrow{S_rR}$. The special case is $S_p = S_r = V_Q$.
In the tables 1 and 2, we see the structure of the set D for various types of conic sections. The numbers in the table 2 represent the number of connected arcs in the set D.

Table 1: *The set* D *for singular types of conic section* K.

	$\{V_Q\}$		p		$p \cup r$	
	D_{ext}	D_{in}	D_{ext}	D_{in}	D_{ext}	D_{in}
$AB \cap K = \emptyset \wedge \overleftrightarrow{AB} \cap K \neq \{T\}$	$\{V_Q\}$	$\emptyset$	p	$\emptyset$	$\overrightarrow{S_pP} \cup \overrightarrow{S_rR}$	$\emptyset$
$AB \cap K = \emptyset \wedge \overleftrightarrow{AB} \cap K = \{T\}$	$\{V_Q\}$	$\emptyset$	–	–	–	–
$AB \cap K = \{X_1, X_2\}$	–	–	–	–	–	–
$AB \cap K = \{T_0\}$	$\{V_Q\}$	$\emptyset$	–	–	–	–
$A = B$	$\{V_Q\}$	$\emptyset$	p	$\emptyset$	$\overrightarrow{V_QP} \cup \overrightarrow{V_QR}$	$\emptyset$

Table 2: *The set* D *for regular conic sections. The numbers indicate the number of connected arc components in the set* D.

	ellipse		parabola		hyperbola	
	D_{ext}	D_{in}	D_{ext}	D_{in}	D_{ext}	D_{in}
$AB \cap K = \emptyset \wedge \overleftrightarrow{AB} \cap K \neq \{T\}$	1	$\emptyset$,1	1	$\emptyset$	2	$\emptyset$
$AB \cap K = \emptyset \wedge \overleftrightarrow{AB} \cap K = \{T\}$	1	$\emptyset$	1	$\emptyset$	2	$\emptyset$
$AB \cap K = \{X_1, X_2\}$	$\emptyset$	1,2	$\emptyset$	1	1	1
$AB \cap K = \{T_0\}$	$\{T_0\}$	$\emptyset$,1	$\{T_0\}$	$\emptyset$	2	$\emptyset$
$A = B$	1	$\emptyset$	1	$\emptyset$	2	$\emptyset$

6 Boundary map

For the given points $A, B, X \in D \setminus D_{AB} \subseteq K$ and the tangent line ℓ_X at X to K, the Bézier curve b_{ACB} touching the conic section K is clearly identified. In order to find the middle control vertex C, we use the following map σ.

Definition 5 (Boundary map) *Let* D *be the set of points of contact for the given points* A, B *and* K. *The map* $\sigma\colon D \setminus D_{AB} \to \rho$ *is called* boundary map *if* $\sigma(X) = C$ *holds for some* C *from the definition 3 and* $X \in D \setminus D_{AB}$ *(see fig. 5(a)).*

It is not possible to define the map σ on the points in D_{AB}. If $T \in D_{AB} \subset K$ then the points A, B, T are collinear on the tangent line ℓ_T to K. Hence, there is an infinite number of points C such that the Bézier curve b_{ACB} touches K in T. All suitable C form a half-line, therefore we are interested in the end point of the half-line, the point C_S.

Theorem 5 *Let the conic section* $K \neq \{V_Q\}$ *and* $X = [x_0, y_0] \in D \setminus D_{AB}$. *Then, the corresponding boundary map* $\sigma\colon D \setminus D_{AB} \to \rho$ *has the form*

$$\sigma(X) = \frac{b(t_0) - B_0^2(t_0)A - B_2^2(t_0)B}{B_1^2(t_0)}, \qquad (1)$$

where $t_0 \in [0,1]$ *is a solution of the equation*

$$0 = \alpha t^2 + 2\beta t + \gamma \tag{2}$$

and for $A = [a_x, a_y, 1]$, $B = [b_x, b_y, 1]$, $X = [x_0, y_0, 1]$ *are*

$$\alpha = (A - B)Q_K X^\top,$$
$$\beta = -AQ_K X^\top,$$
$$\gamma = -\beta.$$

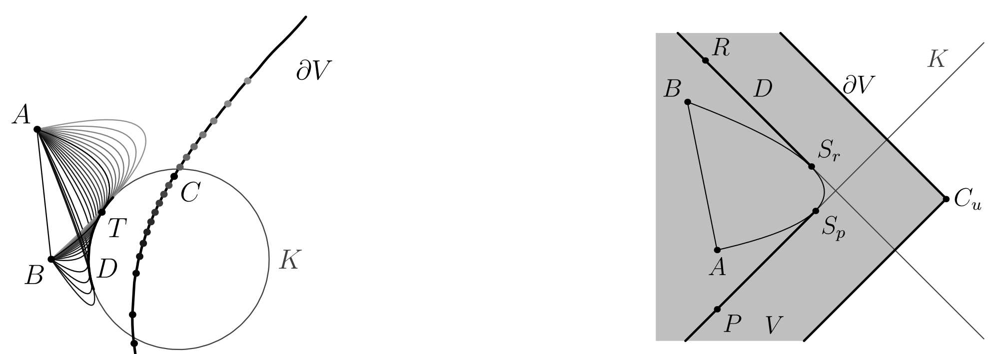
Proof. Since we consider only affine points $C \in \rho$, let K be a conic section with the matrix Q_K. Since the point of contact $X \in b_{ACB}(t)$, there exists $t_0 \in [0,1]$ such that $X = b_{ACB}(t_0) = B_0^2(t_0)A + B_1^2(t_0)C + B_2^2(t_0)B$. For the point $X \in K$ the equality $XQ_K X^\top = 0$ holds. So, $X \notin \{A, B\}$ and $t_0 \notin \{0,1\}$. Because the point of contact $X = [x_0, y_0] \in D \setminus D_{AB}$, the equality $\left\langle \nabla f(x_0, y_0), \frac{d}{dt} b_{ACB}(t_0) \right\rangle = 0$ holds. The equation (2) has a real solution if the discriminant $\Delta = (AQ_K X^\top)(BQ_K X^\top) \geq 0$. It holds, because if $X \in D_{ext}$ then both brackets are positive. On the other hand, if $X \in D_{in}$ then both brackets are negative and their product is positive. From this quadratic equation, we obtain two roots t_1, t_2. The question is, if they are both within $\langle 0,1 \rangle$. We use the theorem 1. The table 3 shows the values of the sequences $\{f(t), f'(t), f''(t)\}$ in the end points of the interval $\langle 0,1 \rangle$ for $f(t) = \alpha t^2 + 2\beta t + \gamma$. From the previous, the expressions $AQ_K X^\top$ and $BQ_K X^\top$ have the same sign. Hence, the table 4 shows the number of sign changes of the sequences $\{f(t), f'(t), f''(t)\}$ with respect to the signs of $AQ_K X^\top$ and $BQ_K X^\top$. According to theorem 1 applied to the function $f(t)$, only one root is in $\langle 0,1 \rangle$. If both $t_1, t_2 \in (0,1)$ and $t_1 \neq t_2$ then $X \in D_{AB}$. Let $t_1 \in (0,1)$. Then, we substitute $t_0 = t_1$ into the Bézier curve equation and obtain the relevant point C from the definition 3 for the point of contact X. Hence, $C = \sigma(X)$. □

Table 3: *The values of derivatives of the function* $f(t) = \alpha t^2 + 2\beta t + \gamma$ *at the end points of the interval* $\langle 0,1 \rangle$.

	$t = 0$	$t = 1$
$f(t)$	$AQ_K X^\top$	$-BQ_K X^\top$
$f'(t)$	$-2AQ_K X^\top$	$-BQ_K X^\top$
$f''(t)$	$2(A-B)Q_K X^\top$	

Table 4: *If we look on the numbers of sign changes, the differences* $2-1$ *and* $1-0$ *are both equal to* 1. *So, the function* $f(t)$ *have one real solution within the interval* $\langle 0,1 \rangle$.

	$AQ_K X^\top > 0 \wedge BQ_K X^\top > 0$		$AQ_K X^\top < 0 \wedge BQ_K X^\top < 0$	
	$t = 0$	$t = 1$	$t = 0$	$t = 1$
$f(t)$	+	−	−	+
$f'(t)$	−	−	+	+
$f''(t)$	±	±	±	±
# of sign changes	2 1	1 0	1 2	0 1



(a) (b)

Figure 5: (a) *The boundary map* σ *maps the points of the arc* D *to the points on* ∂V, *see that* $\sigma(T) = C$.
(b) *Let the conic section* $K = p \cup r$. *The points* A, B *have to lie in the same quadrant with respect to* K. *Let the quadrant be determined by two half-lines* $\overrightarrow{V_Q P} \subset p$ *and* $\overrightarrow{V_Q R} \subset r$. *If there exists a Bézier curve* b_{AC_uB} *with double contact* $M = \{S_p, S_r\}$ *with* K, *the set of exterior points of contact* $D_{ext} = \overrightarrow{S_p P} \cup \overrightarrow{S_r R}$. *Otherwise, we get* $S_p = S_r = V_Q$ *and* $D_{ext} = \overrightarrow{V_Q P} \cup \overrightarrow{V_Q R}$. *The set of interior points of contact is always* $D_{in} = \emptyset$. *The boundaries generated by the half-lines* $\overrightarrow{S_p P}, \overrightarrow{S_r R}$ *are connected due to the fact that* $\sigma(S_p) = \sigma(S_r) = C_u$. *Hence, the set of admissible solution* $V_\rho(A,B)$ *consist of one region. Because the boundary is generated by the exterior points of contact,* $A, B \in V_\rho(A,B)$.

For each $X \in D \setminus D_{AB}$, there exists exactly one point C such that the Bézier curve $b_{ACB}(t) \cap K = \{X\}$. How can we find the point C corresponding to a given point of contact $T \in D_{AB} \subset D$?

Lemma 2 *Let the points A, B and $T \in D_{AB} \subset K$ be collinear.*
(a) Let $AB \cap K = \{T\}$. The Bézier curve $b_{ACB} \cap K = \{T\}$ iff $C \in \overleftrightarrow{AB}$.
(b) Let $AB \cap K = \emptyset$. The Bézier curve $b_{ACB} \cap K = \{T\}$ if and only if $C \in \overrightarrow{C_S X} \subset \overleftrightarrow{AB}$, where $A, B \notin \overrightarrow{C_S X}$ and C_S is such that the derivative $\frac{d}{dt} b_{AC_SB}(t_0) = 0$ for $T = b_{AC_SB}(t_0)$. For the special case $A = B$, the point $C_S = A + 2(T - A)$.

In the equation (1) of the boundary map σ, the inequality $B_1^2(t_0) > 0$ holds for $t_0 \in (0,1)$, so the map σ is continuous. The σ maps the connected set $D_{ext} \setminus D_{AB}$ onto one connected curve l. If the point $T \in D_{AB} \neq \emptyset$, then $\lim_{X \to T} \sigma(X) = C_S$ and the union $l \cup \overrightarrow{C_S X}$ is a connected curve.

7 Set of admissible solutions

Theorem 6 (Boundary of the set $V_\rho(A,B)$) *The set of all such points C that for the Bézier curve $b_{ACB} \cap K \subset D$ yields, is the boundary of the set of admissible solutions $V_\rho(A,B)$. We denote it $\partial V_\rho(A,B)$.*

Proof. The point C belongs to the boundary of the set $V_\rho(A,B)$, if each neighborhood N of the point C contains both the point $C_1 \in V_\rho(A,B)$ and the point $C_2 \in \rho \setminus V_\rho(A,B)$. So, we prove the existence of points $C_1, C_2 \in N$ such that $b_{AC_1B} \cap K = \emptyset$ and $b_{AC_2B} \cap K = \{X_1, X_2\}$ with transversal intersection (see fig. 6).

Let A, B, u be given as in the lemma 1. Let C be such that $b_{ACB} \cap K \subset D$ and $T \in b_{ACB} \cap K$. Let the line ℓ_T with the direction vector u is the tangent line to K in T. Since $B_1^2(t_0) > 0$ for $t_0 \in (0,1)$, the map τ assigning to each point T its corresponding point C is continuous. It means, there exists a neighborhood M of the point T for each neighborhood N of the point C such that $\tau(M) \subset N$. For an arbitrary neighborhood N of the point C, the neighborhood M of the point T exists such that $\tau(M) \subset N$. Let $T_1 = T + k\nabla f(T)$ and $T_2 = T - k\nabla f(T)$, where $k > 0$ is such that $T_1, T_2 \in M$ and they satisfy the conditions of the lemma 1. Let the lines ℓ_1, ℓ_2 be parallel to the line ℓ_T (i.e. the vector u is their direction vector) and $T_1 \in \ell_1, T_2 \in \ell_2$. Then, the points $C_1 = \tau(A,B,T_1,u)$ and $C_2 = \tau(A,B,T_2,u)$ are $C_1, C_2 \in N$. We obtain the Bézier curves b_{AC_1B}, b_{AC_2B}. Since each tangent line ℓ_1, ℓ_2 defines the supporting half-plane to the convex quadratic Bézier curve and the points A, B lie out of K, it holds $b_{AC_1B} \cap K = \emptyset$ and $b_{AC_2B} \cap K = \{X_1, X_2\}$ with transversal intersection. □

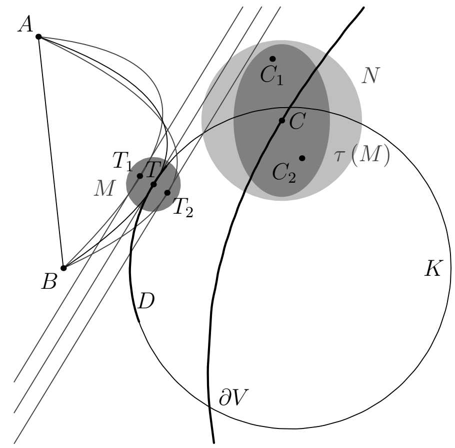


Figure 6: *Let C be such that $b_{ACB} \cap K \subset D$ and $T \in b_{ACB} \cap K$. For arbitrary neighborhood N of the point C, there exists a neighborhood M of the point T such that for $T_1, T_2 \in M$ the corresponding $C_1 = \tau(T_1), C_2 = \tau(T_2) \in N$. Moreover, $b_{AC_1B} \cap K = \emptyset$ and $b_{AC_2B} \cap K = \{X_1, X_2\}$ with transversal intersection. Hence, $C \in \partial V_\rho(A,B)$.*

We say that the boundary of the set of admissible solutions $\partial V_\rho(A,B)$ *is generated* by the set D mapped by the boundary map σ. The boundary of admissible solutions $\partial V_\rho(A,B)$ consists of one or two continuous unbounded curves. The degree of the curves is at most four in parameter t used in the map σ. It is clear from the expression for roots and the map σ in theorem 5.

If the conic section $K = p$, then the boundary of the set of admissible solutions ∂V is the parallel line with the line p. Let $K = p \cup r$. According to the table 1, the set of points of contact $D = \overrightarrow{S_p P} \cup \overrightarrow{S_r R}$ (in special case $S_p = S_r = V_Q$). The set ∂V consists from two half-lines parallel with p, resp. r, connected in the point C_u (see fig. 5(b)).

The region $V_\rho(A,B)$ is determined by its boundary. The boundary is parametrized piecewise using the map $\sigma\colon D \setminus D_{AB} \to \rho$. The next theorem determines, which parts of ρ with respect to the boundary belong to the region $V_\rho(A,B)$.

Theorem 7 (Set of admissible solutions)

(a) Let $K \neq \{V_Q\}$ be a connected component of the regular conic section. Let $l \subset \partial V_\rho(A,B)$ be a connected curve, which divides the plane into two regions W_1, W_2. If l is generated by D_{ext}, then $W_i \subseteq V_\rho(A,B)$ when $A, B \in W_i$. If l is generated by D_{ext} and $AB \subset l$, then $W_i \subseteq V_\rho(A,B)$ when $K \notin W_i$. If l is generated by D_{in}, then $W_i \subseteq V_\rho(A,B)$ when $A, B \notin W_i$.

(b) In the case of $K = \{V_Q\}$, both $W_1, W_2 \subset V_\rho(A,B)$. If $K = p$ or $K = p \cup r$, then $W_i \subseteq V_\rho(A,B)$ when $A, B \in W_i$.

Proof. (a) The intersection $AB \cap l \neq \emptyset$ iff $AB \cap K = \{T\}$. Then, the curve l is generated by exterior points of contact. The connected component K is a convex curve and the line $\overleftrightarrow{AB}$ determine the supporting half-plane to K. Hence, $W_i \subseteq V_\rho(A,B)$ if $K \notin W_i$.
If there exists only one curve $l = \partial V_\rho(A,B)$, we decide about W_i according to the point $C = \frac{1}{2}A + \frac{1}{2}B$.
Suppose that l_1 and l_2 exist such that $l_1 \cup l_2 = \partial V_\rho(A,B)$. According to the table 2, the conic section K is an ellipse. If both l_1, l_2 are generated by the sets of interior points of contact, in both cases we can decide about W_i according to the point $C = \frac{1}{2}A + \frac{1}{2}B$. The segment AB lies between the curves l_1, l_2 and the set $V_\rho(A,B)$ consists of two regions. Now, let the curve l_1 be generated by the set of exterior points of contact and the curve l_2 be generated by the set of interior points of contact (see fig. 4). The intersection $l_1 \cap l_2 = \emptyset$, because the Bézier curve with one exterior and one interior point of contact with K simultaneously does not exist. Let the curve l_1 divide the plane ρ into two components W_1, W_2 and $A, B \in W_1$. The curve $l_2 \subset W_2$, because for $C = \frac{1}{2}A + \frac{1}{2}B$ is $b_{ACB} \cap K = \emptyset$ and the set $W_1 \subset \partial V_\rho(A,B)$. Let the curve l_2 divide the region W_2 into two components W_3, W_4 and l_1 be the boundary between W_1, W_3. According to the theorem 6, the region W_3 consists of such points C that b_{ACB} and K have transversal intersections and the region $W_4 \subset \partial V_\rho(A,B)$. ☐

In the case of hyperbola, each component generates one set of admissible solutions. Hence, we obtain the regions $V_1(A,B), V_2(A,B)$ for the K_1, K_2. For every point $C \in V_1(A,B)$, the Bézier curve $b_{ACB} \cap K_1 = \emptyset$. We are looking for the set of points C, such that $b_{ACB} \cap (K_1 \cup K_2) = \emptyset$. It holds for every point $C \in V_1(A,B) \cap V_2(A,B)$, see fig. 7.

Finally, for the two given points A, B and the conic K, the set of acceptable solutions $V_\rho(A,B)$ consists of one or two regions, see the table 5. It depends on the number of arcs in the set D and on the type of the conic section K.

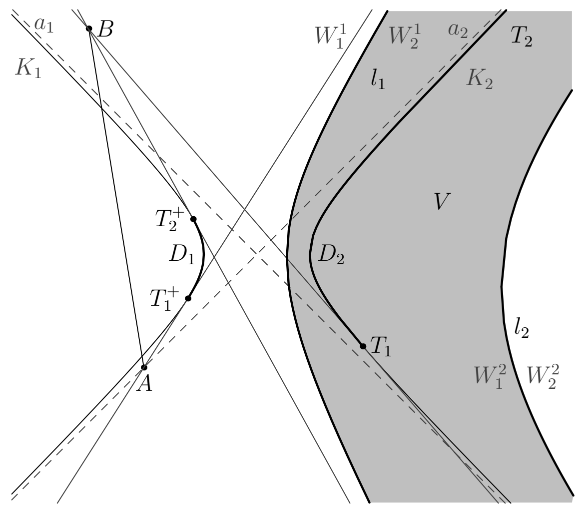


Figure 7: *For the component K_1, the set of exterior points of contact $D^1_{ext} = \emptyset$ and the set of interior points of contact $D^1_{in} = \overset{\frown}{T_1^+ T_2^+}$. The set of points of contact $D^1 = D^1_{in}$ generates the curve $l_1 \subset \partial V_\rho(A,B)$. The curve l_1 divides the plane ρ into two regions W^1_1, W^1_2. Let the points $A, B \in W^1_1$. According to the theorem 7, the set of admissible solutions for the component K_1 is $V^1(A,B) = W^1_2$.*
For the component K_2, since $T_2 = a_2^\infty$, the set of exterior points of contact $D^2_{ext} = \overset{\frown}{T_1 a_2^\infty}$ and the set of interior points of contact $D^2_{in} = \emptyset$. The set of points of contact $D^2 = D^2_{ext}$ generates the curve $l_2 \subset \partial V_\rho(A,B)$ and if the points $A, B \in W^2_1$, then $V^2(A,B) = W^2_1$.
Finally, the set of admissible solutions for the conic section K is $V_\rho(A,B) = V^1(A,B) \cap V^2(A,B)$.

Table 5: *The number of regions in the set of acceptable solutions $V_\rho(A,B)$.*

	$\{V_Q\}$	p	$p \cup r$	ellipse	parabola	hyperbola
$AB \cap K = \emptyset \wedge \overleftrightarrow{AB} \cap K \neq \{T\}$	2	1	1	1,2	1	1
$AB \cap K = \emptyset \wedge \overleftrightarrow{AB} \cap K = \{T\}$	1	–	–	1	1	1
$AB \cap K = \{X_1, X_2\}$	–	–	–	1,2	1	1
$AB \cap K = \{T_0\}$	2	–	–	1,2	1	1
$A = B$	1	1	1	1	1	1

8 Results and application

For the given regular quadric κ and the points A,B in three dimensional space, we described the set $V(A,B)$ of all such points C that the quadratic Bézier curve with control points A,C,B is collision-free path from the point A to the point B.
We solved the problem completely for each type of conic section K, which the plane ρ containing the points A,B cuts out in the regular quadric κ. The set $V_\rho(A,B)$ containing the admissible points $C \in \rho$ is described via the method of contact point. We proved, that the boundary of this set $\partial V_\rho(A,B)$ is determined by such points C that b_{ACB} touches the conic section K. Hence, we found the set of admissible points of contact D for the given points A,B as union of exterior and interior points of contact. Then, we showed how the set D determines the boundary $\partial V_\rho(A,B)$ using the boundary map σ.

We apply this study in the following example. Let the virtual agent starts at the point S_1 and needs to get to the point S_3 while passing through the point S_2. Moreover, he must avoid the house and two corn fields. First, we need to replace the obstacles with appropriate bounding objects. Conic sections as bounding objects well represent a wide range of obstacles. For example, ellipses are suitable for buildings, trees and things and hyperbolas and parabolas for coast or areas with special features. For our purpose, we choose an ellipse for house and a hyperbola for corn fields. The collision-free path is sought gradually. At first, we find the set V_e consisting of such points C that the Bézier curve $b_{S_1CS_2}$ is collision-free path between S_1,S_2 due to ellipse. Then, we find the set V_h consisting of such points C that the Bézier curve $b_{S_1CS_2}$ is collision-free path between S_1,S_2 due to hyperbola. Since we need to avoid both obstacles simultaneously, it is necessary to choose the final point $C_1 \in V_e \cap V_h$. The collision-free path between the points S_2,S_3 is found similarly as shown in fig. 8.

As can be seen in fig. 8, the points C_1,C_2 are chosen so that the spline is G^1-continuous. This choice may not be possible always. It depends on the existence of the segment C_1C_2 such that $S_2 \in C_1C_2$. All possible slopes of the segment C_1S_2 are determined by the tangent lines from the point S_2 to the set $V_e \cap V_h$. Possible slopes of the segment S_2C_2 are determined similarly. The necessary and sufficient condition for G^1-continuity of the spline is the existence of the same allowed slope for both segments.

9 Future work and conclusion

In the future, we look for the collision-free condition for Bézier curves of higher degree. They provide more natural-looking path and they are more flexible while avoiding obstacles including non-planar curves. We start to study conditions for cubic curves, using the knowledge and methods from the quadratic case. A possible application which is a matter of further work is the construction of collision-free splines with requirements for G^n continuity, for some $n \geq 2$.

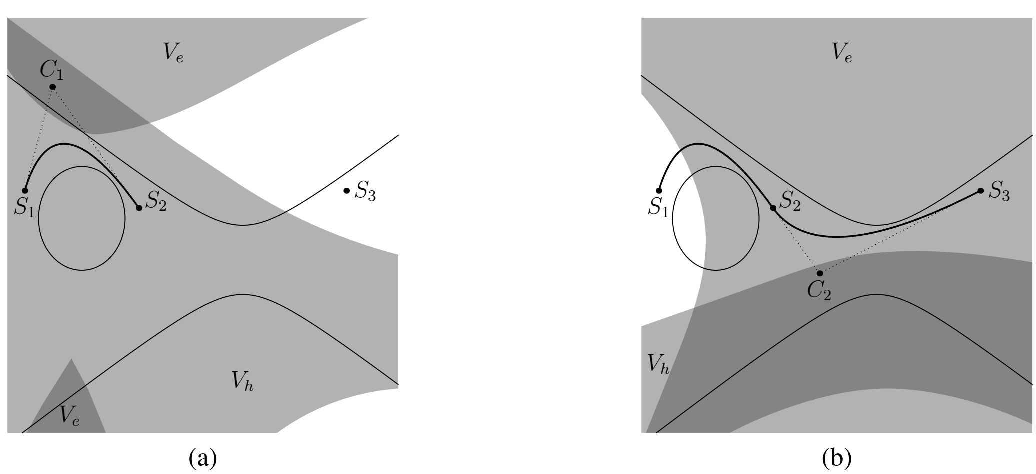


Figure 8: (a) *At first, we search the collision-free path between the points S_1,S_2. After finding the sets of admissible solutions V_e,V_h for each obstacle separately, we choose the point $C_1 \in V_e \cap V_h$.*
(b) *Then, we find the sets V_e,V_h for the points S_2,S_3 and choose the point $C_2 \in V_e \cap V_h$. We choose it so that the final spline is G^1-continuous. This approach might help in some cases of narrow passages when the obstacles are wrapped with appropriate quadrics.*

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