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On Some Regular Polygons in the Taxicab 3−Space

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ABSTRACT

In this study, it has been researched which Euclidean regular polygons are also taxicab regular and which are not. The existence of non-Euclidean taxicab regular polygons in the taxicab 3-space has also been investigated.

Key words: taxicab geometry, Euclidean geometry, regular polygons

MSC2010: 51K05, 51K99, 51N25

O nekim pravilnim mnogokutima u taxicab trodimenzionalnom prostoru

SAŽETAK

U ovom se radu proučava koji su euklidski pravilni mnogokuti ujedno i taxicab pravilni, a koji to nisu. Također se istražuje postojanje taxicab pravinih mnogokuta koji nisu pravilni u euklidskom smislu.

Ključne riječi: taxicab geometrija, euklidska geometrija, pravilni mnogokuti

1 Introduction

The taxicab 3-dimensional space $\mathbb{R}^3_T$ is almost the same as the Euclidean analytical 3-dimensional space $\mathbb{R}^3$. The points, lines and planes are the same in Euclidean and taxicab geometry and the angles are measured in the same way, but the distance function is different. The taxicab metric is defined using the distance function as in [4], [6]

$$d_T(A,B) = |b_1 - a_1| + |b_2 - a_2| + |b_3 - a_3|. \tag{1}$$

Also, for a vector $\overrightarrow{V} = (v_1, v_2, v_3) \in \mathbb{R}^3$, taxicab norm of $\overrightarrow{V}$ is noted as $\left\|\overrightarrow{V}\right\|_T$ as in [5] and

$$\left\|\overrightarrow{V}\right\|_T = |v_1| + |v_2| + |v_3|. \tag{2}$$

Since taxicab plane and 3-dimensional taxicab space have different distance function from that in the Euclidean plane and 3-dimensional space, it is interesting to study the taxicab analogues of topics that include the distance concept in the Euclidean plane and 3-dimensional space. During the recent years, many such topics have been studied in the taxicab plane and 3-dimensional space (see [1], [2], [3], [5], [6], [7], [8], [9], [10]).

In $\mathbb{R}^2$, the equation of taxicab circle is

$$|x - x_0| + |y - y_0| = r \tag{3}$$

which is centered at $M = (x_0, y_0)$ point with radius r.

In $\mathbb{R}^3$, the equation of taxicab sphere is

$$|x - x_0| + |y - y_0| + |z - z_0| = r \tag{4}$$

which is centered at $M = (x_0, y_0, z_0)$ point with radius r (see [7], [8], [9]). Then, a taxicab circle can be defined by a plane and a taxicab sphere.

In $\mathbb{R}^3$, the taxicab circle is the intersection set of taxicab sphere and $ax + by + cz + d = 0$, $(a,b,c,d \in \mathbb{R})$, plane which passes through the center of taxicab sphere.

Recently regular polygons have been studied in the taxicab plane (see [3]). Although there do not exist Euclidean and taxicab regular triangles in the taxicab plane (see [3]), there exist Euclidean and taxicab regular triangles in the taxicab 3-dimensional space $\mathbb{R}^3_T$ (see Example 1, 2). Therefore, it can be interesting to study regular polygons in the taxicab 3-dimensional space. On the $x = k_1, y = k_2, z = k_3$ $(k_1, k_2, k_3 \in \mathbb{R})$ planes in the $\mathbb{R}^3_T$, Euclidean and taxicab regular polygons can be investigated in the same way as in the taxicab plane $\mathbb{R}^2_T$. Therefore, in this study, regular polygons which are not on $x = k_1, y = k_2, z = k_3$ $(k_1, k_2, k_3 \in \mathbb{R})$ planes are researched and we answer the following question: Which Euclidean regular polygons are also the taxicab regular, and which are not? In addition, we investigate the existence and nonexistence of taxicab regular polygons in the taxicab 3-dimensional space $\mathbb{R}^3_T$.

2 Taxicab Regular Polygons

As in the Euclidean 3-space, a polygon in the taxicab 3-space consists of three or more coplanar line segments, the line segments (sides) intersect only at endpoints, each endpoint (vertex) belongs to exactly two line segments and no two line segments with a common endpoint are collinear. If the number of sides of a polygon is n for $n \geq 3$ and $n \in \mathbb{N}$, then the polygon is called an $n-$gon. The following definitions for polygons in the taxicab 3-space are given by means of the taxicab lengths instead of the Euclidean lengths:

Definition 1 *A polygon in the plane is said to be taxicab equilateral if the taxicab lengths of its sides are equal.*

Definition 2 *A polygon in the plane is said to be taxicab equiangular if the measures of its interior angles are equal.*

Definition 3 *A polygon in the plane is said to be taxicab regular if it is both taxicab equilateral and equiangular.*

Definition 2 does not give a new equiangular concept because the taxicab and the Euclidean measure of an angle are the same. That is, every Euclidean equiangular polygon is also the taxicab equiangular, and vice versa. However, since the taxicab 3-space has a different distance function, Definition 1 and Definition 3 are new concepts [3].
An n-gon can be formed by n vectors with a total of zero on a plane. If the lengths of n vectors are the same and all the angles between two consecutive vectors are equal, the n-gon is regular (see Figure 1). Therefore, we need to find n vectors that allow these requirements to form a regular n-gon.

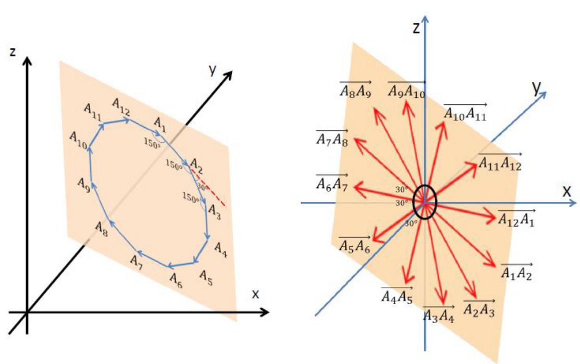

Figure 1.

Let W be a vector set and the members of W are sides of taxicab regular polygon as in Figure 1. It is clear that all the vectors are on the same plane and all the angles between two consecutive vectors are equal.
Let w be a vector in $\mathbb{R}^3$. We would like to find geometric locations of the w_i vectors where $\|w\|_E = \|w_i\|_E$ and $\|w\|_T = \|w_i\|_T$ (since there exist an infinite number of points on a circle as in Case 2, it can be said that $i \in \mathbb{N}$). Let $k \in \mathbb{R}$.
If the direction vector of w is $(\pm k, \pm k, \pm k)$, then $\|w\|_E = \frac{\|w\|_T}{\sqrt{3}}$.
If the direction vector of w is a member of $\{(\pm k, 0, 0), (0, \pm k, 0), (0, 0, \pm k)\}$, then $\|w\|_E = \|w\|_T$.
If the direction vector of w is not a member of $\{(\pm k, 0, 0), (0, \pm k, 0), (0, 0, \pm k), (\pm k, \pm k, \pm k)\}$, then $\frac{\|w\|_T}{\sqrt{3}} < \|w\|_E < \|w\|_T$.
To determine w_i vectors, there exist three different cases depending on the Euclidean and taxicab lengths of the w vector.
Case 1. For $\|w\|_E = \frac{\|w\|_T}{\sqrt{3}}$, as it is shown in Figure 2, the members of the intersection set of the Euclidean and taxicab sphere are

$$w_i = (\pm\frac{\|w\|_T}{3}, \pm\frac{\|w\|_T}{3}, \pm\frac{\|w\|_T}{3}). \tag{5}$$

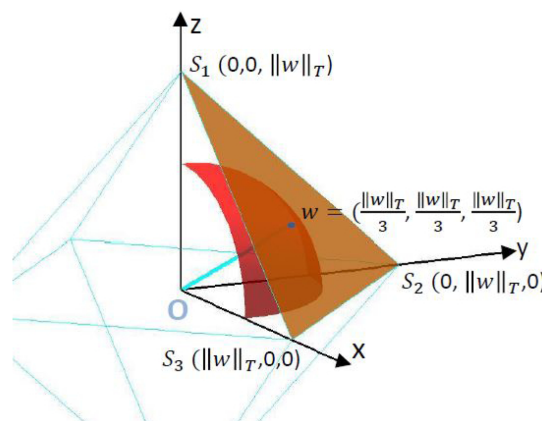

Figure 2.

Case 2. For $\frac{\|w\|_T}{\sqrt{3}} < \|w\|_E < \|w\|_T$, as it is shown in Figure 3, the intersection sets are 8 circles on the taxicab sphere. One of the circles, C, is on $x+y+z = \|w\|_T$ plane and points of C are intersection of

$$x+y+z = \|w\|_T \tag{6}$$

plane and

$$\left(x-\frac{\|w\|_T}{3}\right)^2+\left(y-\frac{\|w\|_T}{3}\right)^2+\left(z-\frac{\|w\|_T}{3}\right)^2=\frac{3\|w\|_E^2-\|w\|_T^2}{3} \tag{7}$$

Euclidean sphere. All the circles, for $1 \leq j \leq 8$ C_j, are intersection of

$$\left(x\pm\frac{\|w\|_T}{3}\right)^2+\left(y\pm\frac{\|w\|_T}{3}\right)^2+\left(z\pm\frac{\|w\|_T}{3}\right)^2=\frac{3\|w\|_E^2-\|w\|_T^2}{3} \tag{8}$$

Euclidean spheres and

$$\pm x \pm y \pm z = \|w\|_T \tag{9}$$

planes. Thus, w_i vectors where $\|w\|_E = \|w_i\|_E$ and $\|w\|_T = \|w_i\|_T$ are members of $\bigcup_{j=1}^{8} C_j$.

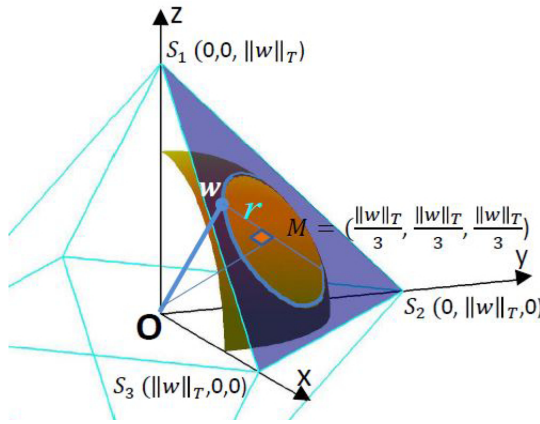

Figure 3.

Case 3. For $\|w\|_E = \|w\|_T$, as it is shown in Figure 20, the members of the intersection set of Euclidean and taxicab sphere are

$$w_i = (\pm\|w\|_T, 0, 0), (0, \pm\|w\|_T, 0), (0, 0, \pm\|w\|_T). \tag{10}$$

Let us introduce the following abbreviations for the following Proposition 1.
$\pm x \pm y \pm z = \|\overrightarrow{w}\|_T$ plane equation is shown as $P_{\pm,\pm,\pm}$ and $\left(x \pm \frac{\|w\|_T}{3}\right)^2 + \left(y \pm \frac{\|w\|_T}{3}\right)^2 + \left(z \pm \frac{\|w\|_T}{3}\right)^2 = \frac{3\|w\|_E^2 - \|w\|_T^2}{3}$ sphere equation is shown as $S_{\pm,\pm,\pm}$.

Proposition 1 *Let w be a vector. The geometric locations of the w_i $(i \in \mathbb{N})$ vectors which fulfill the following conditions $\|w\|_E = \|w_i\|_E$ and $\|w\|_T = \|w_i\|_T$ are $\bigcup_{j=1}^{9} C_j$ and C_j, for $0 \le j \le 9$, can be defined as:*

Assume $\|w\|_E \neq \|w\|_T$
If $0 \le x, y, z \le \|w\|_T$ *then* $C_1 = S_{-,-,-} \cap P_{+,+,+}$.
If $0 \le x, z \le \|w\|_T$ *and* $-\|w\|_T \le y \le 0$ *then* $C_2 = S_{-,+,-} \cap P_{+,-,+}$.
If $0 \le z \le \|w\|_T$ *and* $-\|w\|_T \le x, y \le 0$ *then* $C_3 = S_{+,+,-} \cap P_{-,-,+}$.
If $0 \le y, z \le \|w\|_T$ *and* $-\|w\|_T \le x \le 0$ *then* $C_4 = S_{+,-,-} \cap P_{-,+,+}$.
If $0 \le x, y \le \|w\|_T$ *and* $-\|w\|_T \le z \le 0$ *then* $C_5 = S_{-,-,+} \cap P_{+,+,-}$.
If $0 \le x \le \|w\|_T$ *and* $-\|w\|_T \le y, z \le 0$ *then* $C_6 = S_{-,+,+} \cap P_{+,-,-}$.
If $-\|w\|_T \le x, y, z \le 0$ *then* $C_7 = S_{+,+,+} \cap P_{-,-,-}$.
If $0 \le y \le \|w\|_T$ *and* $-\|w\|_T \le x, z \le 0$ *then* $C_8 = S_{+,-,+} \cap P_{-,+,-}$.
Assume $\|w\|_E = \|w\|_T$.
$C_9 = \{(\pm\|w\|_T, 0, 0), (0, \pm\|w\|_T, 0), (0, 0, \pm\|w\|_T)\}$.

Corollary 1 *Let W be a vector set of the edges of Euclidean regular polygon and $w \in W$. A Euclidean regular polygon is also taxicab regular if only if $W \subseteq \bigcup_{j=1}^{9} C_j$ where $\bigcup_{j=1}^{9} C_j$ is the same as in Proposition 1.*

Proof. For $n \ge 3$, if an $n-$gon is Euclidean and taxicab regular, then all the w side vectors of $n-$gon are members of the intersection set of an origin-centered taxicab sphere with a radius of $\|w\|_T$ and an origin-centered Euclidean sphere with a radius of $\|w\|_E$. It is clear that the intersection set is $\bigcup_{j=1}^{9} C_j$ as in Proposition 1. □

3 Euclidean Regular Polygons in Taxicab 3-Space

Euclidean regular polygons are Euclidean equiangular. Since the taxicab angles are measured in the same way as Euclidean (see [3], [6]), the polygons are also taxicab equiangular. Thus, we are just interested in which Euclidean regular polygons are taxicab equilateral and which are not (see Definitions 1, 2, 3).

Theorem 1 *None of Euclidean regular $n-$gon $(n > 12)$ is taxicab regular.*

Proof. All the regular polygons are planar and the intersection set of Euclidean and taxicab spheres forms eight circles (see Figure 4). A plane can intersect only 6 out of 8 circles on the taxicab sphere. Thus, the number of intersection points of the plane and 6 circles are max 12 (see Figures 4, 5). Therefore, maximum 12 parts of lines which have equal taxicab length and equal Euclidean length can be obtained. As a result, no Euclidean regular $n-$gon $(n > 12)$ is taxicab regular. □

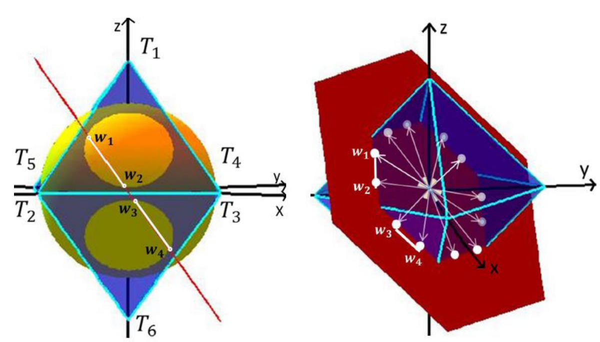

Figure 4.

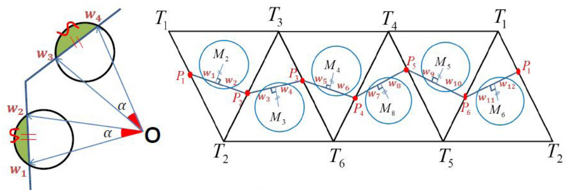

Figure 5.

Theorem 2 *There exist* 12*-gons that are Euclidean and taxicab regular and they are lying on a plane parallel to one of the* $\pm x \pm y \pm z = 0$ *planes.*

Proof. According to the proof presented in Theorem 1, it is possible that there exist 12 points on the intersection set of Euclidean and taxicab spheres and a plane. Thus, we need to prove that the angle between two consecutive vectors can be 30^o and that all the points can lay on a plane parallel to one of the $\pm x \pm y \pm z = 0$ planes.
As it was stated before, it is easy to see that the plane can intersect max 6 out of 8 circles (see Figure 4) with $r = \frac{1}{\sqrt{3}}\sqrt{3\|w\|_E^2 - \|w\|_T^2}$ radius and $M = \left(\pm\frac{\|w\|_T}{3}, \pm\frac{\|w\|_T}{3}, \pm\frac{\|w\|_T}{3}\right)$ center. When $a,b,c > 0$, $ax + by + cz = 0$ plane intersect octahedron (taxicab sphere) on its $T_1T_2T_3$, $T_6T_2T_3$, $T_6T_3T_4$, $T_6T_4T_5$, $T_1T_5T_4$, $T_1T_2T_5$ faces. For $k,m,n \in \mathbb{R}$, the intersection set of plane and edges of octahedron is

$$\left\{\begin{array}{c} P_1 = (-k, 0, k - \|w\|_T), P_2 = (0, -m, \|w\|_T - m), \\ P_3 = (n, n - \|w\|_T, 0), P_4 = (k, 0, k - \|w\|_T), \\ P_5 = (0, m, m - \|w\|_T), P_6 = (-n, \|w\|_T - n, 0) \end{array}\right\}.$$

For $1 \le i \le 6$, the plane P_i are on

$$ax + by + cz = 0. \tag{11}$$

To obtain

$$\begin{aligned} \measuredangle w_1Ow_2 = \measuredangle w_3Ow_4 = \measuredangle w_5Ow_6 = \measuredangle w_7Ow_8 \\ = \measuredangle w_9Ow_{10} = \measuredangle w_{11}Ow_{12} = \alpha, \end{aligned}$$

it should be satisfied that

$$\|\overrightarrow{w_1w_2}\| = \|\overrightarrow{w_3w_4}\| = \|\overrightarrow{w_5w_6}\| = \|\overrightarrow{w_7w_8}\| = \|\overrightarrow{w_9w_{10}}\| = \|\overrightarrow{w_{11}w_{12}}\| \tag{12}$$

(see Figures 4, 5). Also, to obtain (12) it should be satisfied that

$$\begin{aligned} d_E(M_2, [P_1P_2]) &= d_E(M_3, [P_2P_3]) = d_E(M_4, [P_3P_4]) \\ &= d_E(M_8, [P_4P_5]) = d_E(M_5, [P_5P_6]) \\ &= d_E(M_6, [P_6P_1]). \end{aligned} \tag{13}$$

For $\|w\|_T = 3$, each of equations (11) and (13) are solved on the Maple Computer Math Program. It is seen that $|a| = |b| = |c|$ and $m = k = n = \frac{\|w\|_T}{2}$. Therefore, $ax + by + cz = 0$ planes are $\pm x \pm y \pm z = 0$.
If $w_1 = P_1$ and $w_2 = P_2$, then $\alpha = 60^o$ (see Figure 6).
If $w_1 = w_2$, then $\alpha = 0^o$ (see Figure 7). So $0^o \le \alpha \le 60^o$. Then, it is possible that $\alpha = 30^o$. For $\alpha = 30^o$ and $1 \le i \le 12$, Euclidean and taxicab regular 12−gon can be created by w_i vectors. □

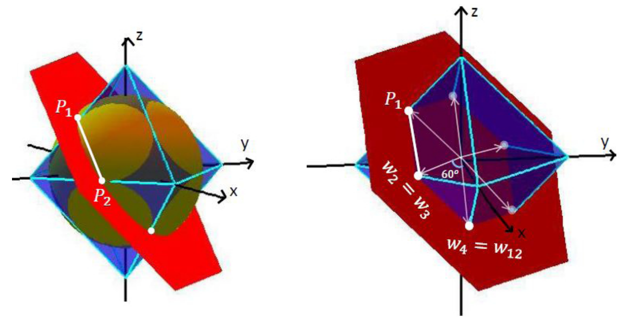

Figure 6.

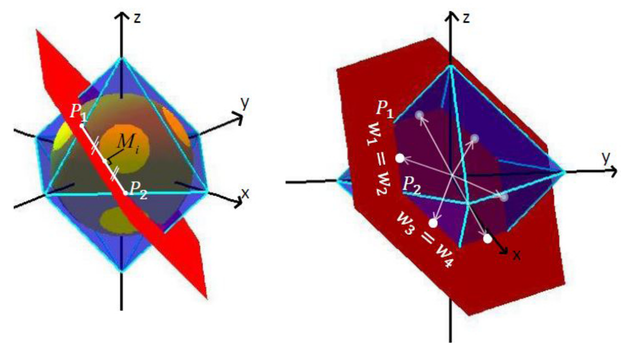

Figure 7.

Now we are going to explain the connection between taxicab and Euclidean lengths of the sides of a regular 12−gon: Because of $P_2 = (0, -\frac{\|w\|_T}{2}, \frac{\|w\|_T}{2})$ and $P_3 = (\frac{\|w\|_T}{2}, -\frac{\|w\|_T}{2}, 0)$, P_2 and P_3 points have an equal distance from the origin. Therefore, the midpoint of the line segment $[P_2P_3]$ is $S_{23} = (\frac{\|w\|_T}{4}, -\frac{\|w\|_T}{2}, \frac{\|w\|_T}{4})$ and this point, S_{23}, is on the taxicab sphere.
The angles of the right triangle $S_{23}V_4O$ are $15^o, 75^o, 90^o$ (see Figure 8). As a result,

$$d_E(S_{23}, O) = \cos 15^o d_E(w_4, O) \tag{14}$$

and

$$\begin{aligned} d_E(S_{23}, O) &= \sqrt{\left(\frac{\|w\|_T}{4}\right)^2 + \left(-\frac{\|w\|_T}{2}\right)^2 + \left(\frac{\|w\|_T}{4}\right)^2} \\ &= \frac{\sqrt{6}}{4}\|w\|_T \end{aligned} \tag{15}$$

$$d_E(w_4, O) = \|w\|_E \tag{16}$$

$$cos 15^o = \frac{2+\sqrt{3}}{\sqrt{6}+\sqrt{2}}. \tag{17}$$

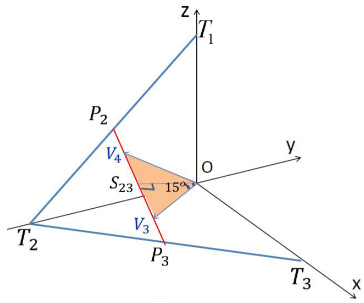

Figure 8.

When equations (15), (16) and (17) are written in 14, it is found that

$$\|w\|_T = \frac{3+\sqrt{3}}{3}\,\|w\|_T\,. \tag{18}$$

Thus we can get the following corollary.

Corollary 2 *Let $A_1A_2...A_{12}$ be a Euclidean regular polygon on one of the $\pm x \pm y \pm z = k$ planes, $k \in \mathbb{R}$. If the taxicab length of one of the sides is $\frac{3+\sqrt{3}}{3}$ times of the Euclidean length, $A_1A_2...A_{12}$ is also taxicab regular.*

Theorem 3 *There do not exist Euclidean regular* 10*-gons that are taxicab regular at the same time.*

Proof. Let $A_1A_2A_3A_4A_5A_6A_7A_8A_9A_{10}$ be a taxicab and Euclidean regular 10-gon. In this case, taxicab lengths of vectors $\overrightarrow{A_1A_2}$, $\overrightarrow{A_2A_3}$, $\overrightarrow{A_3A_4}$, $\overrightarrow{A_4A_5}$, $\overrightarrow{A_5A_6}$, $\overrightarrow{A_6A_7}$, $\overrightarrow{A_7A_8}$, $\overrightarrow{A_8A_9}$, $\overrightarrow{A_9A_{10}}$, $\overrightarrow{A_{10}A_1}$ are the same. Thus, these vectors are on the intersection set of an origin centered taxicab sphere with a radius of $d_T(A_1,A_2)$ and an origin-centered Euclidean sphere with a radius of $d_E(A_1,A_2)$ and the plane $ax+by+cz=0$. Also, the angle between each consecutive vectors is 36^o. For the number of intersection points between 6 circles and the plane to be 10, the plane must intersect four or five circles with two points. Since all the circles are symmetric with respect to the origin, the plane must intersect 4 circles with two points and 2 circles with 1 point as shown in Figures 9, 10.

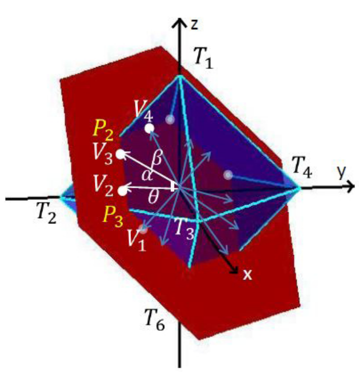

Figure 9.

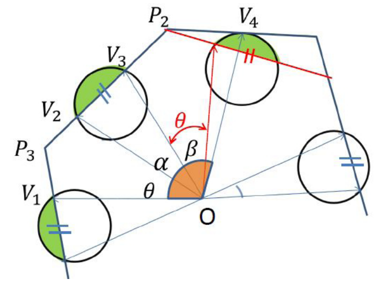

Figure 10.

Let $\measuredangle V_1OV_2 = \theta$, $\measuredangle V_2OV_3 = \alpha$ and $\measuredangle V_3OV_4 = \beta$ (see Figures 9, 10). Since $\beta > \theta$, β and θ cannot be 36^o at the same time. As a result, no Euclidean regular $10-$gon is taxicab regular. □

Theorem 4 *None of the Euclidean regular* $5,7,9,11-$*gons is taxicab regular at the same time.*

Proof. Let $A_1A_2A_3A_4A_5$ Euclidean regular $5-$gon be also taxicab regular. Then, the taxicab lengths of $\overrightarrow{A_1A_2}$, $\overrightarrow{A_2A_3}$, $\overrightarrow{A_3A_4}$, $\overrightarrow{A_4A_5}$, $\overrightarrow{A_5A_1}$ vectors of the $5-$gon are the same. If $\overrightarrow{A_2A_1}$, $\overrightarrow{A_3A_2}$, $\overrightarrow{A_4A_3}$, $\overrightarrow{A_5A_4}$, $\overrightarrow{A_1A_5}$ vectors are added into $\overrightarrow{A_1A_2}$, $\overrightarrow{A_2A_3}$, $\overrightarrow{A_3A_4}$, $\overrightarrow{A_4A_5}$, $\overrightarrow{A_5A_1}$ vectors, ten vectors on the same plane are obtained. The taxicab lengths of 10 vectors are the same and the angle between each consecutive vectors is 36^o (see Figure 11). This is a contradiction, because it is shown in Theorem 3 that none of the Euclidean regular $10-$gons is taxicab regular. Also, in Theorem 1, no Euclidean regular $n-$gon ($n > 12$) is taxicab regular, so there is not any Euclidean and taxicab regular $n-$gon for $n = 14, 18, 22$. Similarly, it can be understood that there is not Euclidean and taxicab regular $n-$gon for $n = 7, 9, 11$. □

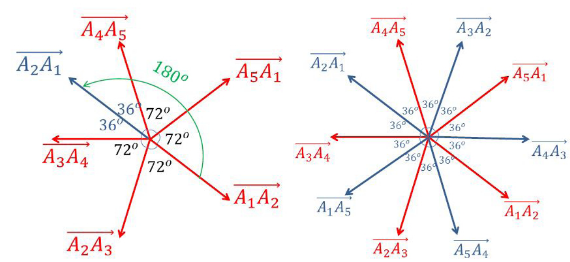

Figure 11.

Theorem 5 *If two symmetry planes of Euclidean regular* $8,6,4,3-$*gons satisfy the following conditions, these polygons are taxicab regular at the same time.*

i) *Two symmetry planes are not orthogonal.*

ii) *At least one of them is on the corners of polygons.*

iii) *Two symmetry planes are members of*

$\{x \mp y = k_1,\ x \mp z = k_2, y \mp z = k_3, x = k_4, y = k_5, z = k_6\}$ *set* (*for* $1 \le i \le 6,\ k_i \in \mathbb{R}$)

Proof. A reflection about the plane set is a taxicab isometry [6]. Let $A_1A_2A_3A_4A_5A_6A_7A_8$ be a Euclidean regular polygon, A_1A_5 and A_2A_6 are symmetry planes. According to the A_1A_5 symmetry plane;

$$d_T(A_1,A_2)=d_T(A_1,A_8),\ d_T(A_2,A_3)=d_T(A_8,A_7)$$
$$d_T(A_3,A_4)=d_T(A_7,A_6),\ d_T(A_4,A_5)=d_T(A_6,A_5) \quad (19)$$

According to the A_2A_6 symmetry plane;

$$d_T(A_2,A_3)=d_T(A_2,A_1),\ d_T(A_3,A_4)=d_T(A_1,A_8)$$
$$d_T(A_4,A_5)=d_T(A_8,A_7),\ d_T(A_5,A_6)=d_T(A_7,A_6) \quad (20)$$

From (19) and (20) equations, it is obtained that taxicab lengths of polygon sides are the same. It is clear that the other symmetry planes can give the same result.
As it is known that the angles between two non-orthogonal symmetry planes of a Euclidean regular 8−gon are 22.5^o, 45^o and 67.5^o (see Figure 12). The angle between $x+z=k_2$ and $x=k_4$ planes is 45^o. Thus, if the following three features are present in the Euclidean regular 8−gon, then it is said to be taxicab regular.
i) Polygon center of gravity is on the intersection line of $x+z=k_2$ and $x=k_4$,
ii) Polygon is orthogonal to $x+z=k_2$ and $x=k_4$,
iii) One of the corner of the polygon on $x+z=k_2$, the other corner on $x=k_4$.
Similarly, other pairs of planes can be found. □

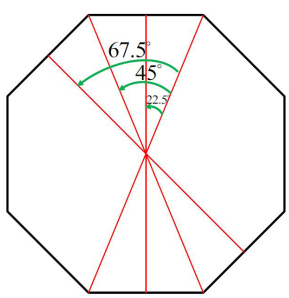

Figure 12.

In addition to Theorem 5, while a regular 8−gon is on one of the $x=k_4$, $y=k_5$, $z=k_6$ planes, the connection between taxicab length and Euclidean length of the sides of the regular 8−gon are researched. As it is shown in Figure 13 the result is researched for $z=0$.

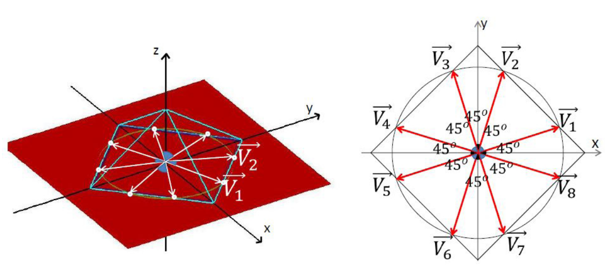

Figure 13.

Let $S_1=(d_T(A_1,A_2),0,0)$ and $S_2=(0,d_T(A_1,A_2),0)$. Midpoint of S_1 and S_2 is $S_{12}=(\frac{d_T(A_1,A_2)}{2},\frac{d_T(A_1,A_2)}{2},0)$ on the taxicab sphere. The angles of $S_{12}V_1O$ right triangle are $22.5^o, 67.5^o, 90^o$ (see Figure 14). Hence,

$$d_E(S_{12},O)=\cos(22,5^o).d_E(V_1,O) \quad (21)$$

and

$$d_E(S_{12},O)=\sqrt{\left(\frac{d_T(A_1,A_2)}{2}\right)^2+\left(\frac{d_T(A_1,A_2)}{2}\right)^2+0^2}$$
$$=\frac{\sqrt{2}}{2}d_T(A_1,A_2) \quad (22)$$

$$d_E(V_1,O)=d_E(A_1,A_2) \quad (23)$$

$$\cos 22,5^o=\frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}}. \quad (24)$$

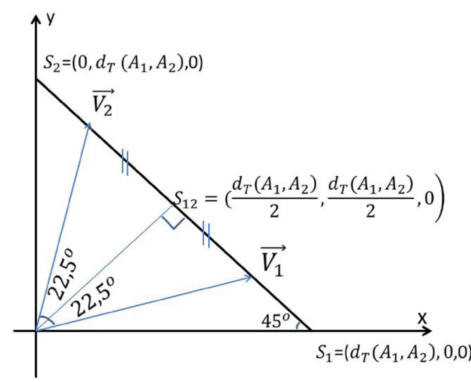

Figure 14.

When equations (22), (23) and (24) are written in (21), the following equation can be found

$$d_T(A_1,A_2)=\frac{\sqrt{2\sqrt{2}+2}}{2}\cdot d_E(A_1,A_2). \quad (25)$$

Since every Euclidean translation of $\mathbb{R}^3$ is an isometry of $\mathbb{R}^3_T$ [6], this result is also the same on $z=k$ $(k\in\mathbb{R})$ planes. Thus, we can give the following corollary.

Corollary 3 *Let $A_1A_2A_3A_4A_5A_6A_7A_8$ be a Euclidean regular polygon on one of the $x=k_1, y=k_2, z=k_3$ planes. If the taxicab length of one of the sides is $\frac{\sqrt{2\sqrt{2}+2}}{2}$ times of the Euclidean length, $A_1A_2A_3A_4A_5A_6A_7A_8$ is also taxicab regular.*

Example 1 *For $A(0,1,0)$, $B(1,0,0)$, $C(0,0,1)$ points, ABC is a triangle. Since*

$$d_E(A,B)=d_E(A,B)=d_E(A,B)=\sqrt{2},$$
$$d_T(A,B)=d_T(A,B)=d_T(A,B)=2$$

ABC is also Euclidean and taxicab regular. Symmetry axes of the triangle are $x-y=0$, $y-z=0$, $x-z=0$.

Symmetry planes are not enough to determine taxicab regular polygons. Because, all the symmetry planes of taxicab regular polygons are not members of

$$\{x \mp y = k_1, x \mp z = k_2, y \mp z = k_3, x = k_4, y = k_5, z = k_6\}$$

set (for $1 \leq i \leq 6$, $k_i \in \mathbb{R}$).

Example 2 *For* $A(0,0,1)$, $B(-\frac{1}{3},-\frac{1}{3},-\frac{1}{3})$, $C(1,0,0)$ *points, ABC is a taxicab regular triangle. But only one of the symmetry planes is a member of*

$$\{x \mp y = k_1, x \mp z = k_2, y \mp z = k_3, x = k_4, y = k_5, z = k_6\}$$

set (for $1 \leq i \leq 6$, $k_i \in \mathbb{R}$*). Other two symmetry planes are not members.*

Corollary 4 *Let* $A_1A_2A_3A_4A_5A_6$ *be a Euclidean regular polygon on one of the* $\pm x \pm y \pm z = k$ *planes* ($k \in \mathbb{R}$)*. If the taxicab length of one of the sides is* $\sqrt{2}$ *times of the Euclidean length,* $A_1A_2A_3A_4A_5A_6$ *is also taxicab regular.*

Proof. Let $d_T(A_1,A_2) = \sqrt{2}d_E(A_1,A_2)$. The intersection points between $x+y+z=0$ plane and an origin-centered taxicab sphere with a radius of $d_T(A_1,A_2)$ are the midpoints of the 6 edges.

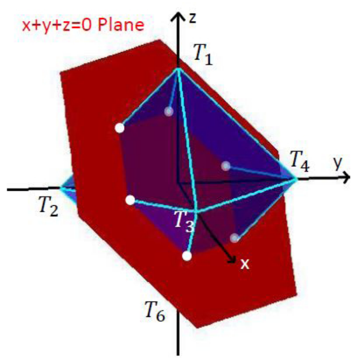

Figure 15.

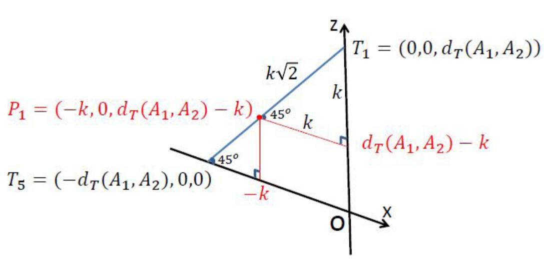

Figure 16.

Also $P_1 = (-\frac{d_T(A_1,A_2)}{2}, 0, \frac{d_T(A_1,A_2)}{2})$. Since

$$d_E(P_1,O) = \sqrt{(-\frac{d_T(A_1,A_2)}{2})^2 + 0^2 + \left(\frac{d_T(A_1,A_2)}{2}\right)^2}$$
$$= \frac{\sqrt{2}}{2}d_T(A_1,A_2),$$

then $d_E(P_1,O) = d_E(A_1,A_2)$. Thus, the intersection points between $x+y+z=0$ plane and an origin-centered Euclidean sphere with a radius of $d_E(A_1,A_2)$ and an origin-centered taxicab sphere with a radius of $d_T(A_1,A_2)$ are the midpoints of the 6 edges. The angle between each consecutive vectors is 60^o (see Figure 17). Thus, the polygon $A_1A_2A_3A_4A_5A_6$ is taxicab regular at the same time. □

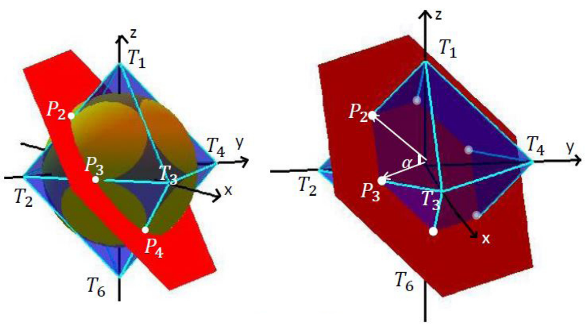

Figure 17.

Example 3 *Let us consider* $A_1 = (0,0,0)$, $A_2 = (0,-1,1)$, $A_3 = (1,-2,1)$, $A_4 = (2,-2,0)$, $A_5 = (2,-1,-1)$, $A_6 = (1,0,-1)$. *It is clear that* $A_1A_2...A_6$ *is a Euclidean and taxicab regular* $6-$*gon. Besides,* $d_T(A_1,A_2) = \sqrt{2}d_E(A_1,A_2)$ *and* A_1,A_2,A_3,A_4,A_5,A_6 *points are on the* $x+y+z=0$ *plane.*

Corollary 5 *For* $n = 3,4$ *and* 6, $n-$*gons on one of the* $\pm x \pm y \pm z = k$ *planes are Euclidean regular. If the taxicab length of one of the sides is* $\frac{3+\sqrt{3}}{3}$ *times of Euclidean length,* $n-$*gon is also taxicab regular.*

Proof. As it is seen in Theorem 2, there are 12 vectors on the same plane with the same taxicab length and the angles between two consecutive vectors are 30^o (see Figure 18).

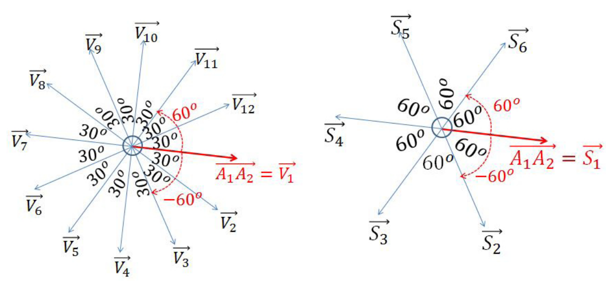

Figure 18.

If we choose 6 vectors whose angles between two consecutive vectors are 60^o, these vectors can create a taxicab regular $6-$gon. □

Corollary 6 *Let us consider, $A_1A_2A_3A_4A_5A_6$ is a Euclidean regular polygon on one of the $\pm x \pm y \pm z = k$ planes. If the taxicab length of one of the sides is $\frac{2\sqrt{6}}{3}$ times of the Euclidean length, $A_1A_2A_3A_4A_5A_6$ is also taxicab regular.*

Proof. Let $A_1A_2A_3A_4A_5A_6$ be a Euclidean regular 6−gon on $x+y+z=k$ plane and the taxicab length of the $\overrightarrow{A_1A_2}$ vector is $\frac{2\sqrt{6}}{3}$ times of the Euclidean length. Then $d_T(A_1,A_2) = \frac{2\sqrt{6}}{3} d_E(A_1,A_2)$ and let the intersection points between $x+y+z=0$ plane and a origin-centered Euclidean sphere with a radius of $d_E(A_1,A_2)$ and a origin-centered taxicab sphere with a radius of $d_T(A_1,A_2)$ be $w_i = (x,y,z)$ for $i \in \mathbb{N}$. Then

$$\|\overrightarrow{w_i}\|_T = \frac{2\sqrt{6}}{3} \|\overrightarrow{w_i}\|_E \tag{26}$$

$$|x|+|y|+|z| = \frac{2\sqrt{6}}{3}\sqrt{x^2+y^2+z^2} \tag{27}$$

and

$$x+y+z=0 \tag{28}$$

and

$$\|\overrightarrow{w_i}\|_T = d_T(A_1,A_2) \tag{29}$$

$$|x|+|y|+|z| = d_T(A_1,A_2). \tag{30}$$

Equations (27), (28) and (30) are solved by the Maple math program. As a result, it is found that all points are

$$\begin{aligned}
w_1 &= \left(-\frac{d_T(A_1,A_2)}{2}, \frac{d_T(A_1,A_2)}{4}, \frac{d_T(A_1,A_2)}{4}\right),\\
w_2 &= \left(-\frac{d_T(A_1,A_2)}{4}, -\frac{d_T(A_1,A_2)}{4}, \frac{d_T(A_1,A_2)}{2}\right),\\
w_3 &= \left(\frac{d_T(A_1,A_2)}{4}, -\frac{d_T(A_1,A_2)}{2}, \frac{d_T(A_1,A_2)}{4}\right),\\
w_4 &= \left(\frac{d_T(A_1,A_2)}{2}, -\frac{d_T(A_1,A_2)}{4}, -\frac{d_T(A_1,A_2)}{4}\right),\\
w_5 &= \left(\frac{d_T(A_1,A_2)}{4}, \frac{d_T(A_1,A_2)}{4}, -\frac{d_T(A_1,A_2)}{2}\right),\\
w_6 &= \left(-\frac{d_T(A_1,A_2)}{4}, \frac{d_T(A_1,A_2)}{2}, -\frac{d_T(A_1,A_2)}{4}\right).
\end{aligned}$$

It is clear that $\|\overrightarrow{w_i}\| = \|\overrightarrow{w_j}\|$ and $\|\overrightarrow{w_i}\|_T = \|\overrightarrow{w_i}\|_T$ for $1 \le i,j \le 6$. Also the angle between $\overrightarrow{w_1}$ and $\overrightarrow{w_2}$ is α and $\alpha = 60^o$. Similarly, it can be shown that the angles between $\overrightarrow{w_i}$ and $\overrightarrow{w_j}$ $(1 \le i,j \le 6)$ are also 60^o. Therefore, it can be concluded that the Euclidean polygon is a taxicab regular at the same time. □

Example 4 *By using $A_1 = (0,0,0)$, $A_2 = (3,-6,3)$, $A_3 = (9,-9,0)$, $A_4 = (12,-6,-6)$, $A_5 = (9,0,-9)$, $A_6 = (3,3,-6)$ points, we can create $A_2A_3A_4A_5A_6$ Euclidean and taxicab regular 6−gon. Also $d_T(A_1,A_2) = \frac{2\sqrt{6}}{3} d_E(A_1,A_2)$ and all the A_1, A_2, A_3, A_4, A_5, A_6 points are on the $x+y+z=0$ plane.*

Corollary 7 *Let $\overrightarrow{V} \in \{(1,0,0),(0,1,0),(0,0,1)\}$. If there is a rotation whose axis is $\overrightarrow{V}$ between two consecutive sides of a quadrilateral, it is Euclidean and taxicab regular.*

Proof. Let $A_1A_2A_3A_4$ be a Euclidean regular quadrilateral and there is a rotation between $[A_1A_2]$ and $[A_1A_4]$. Since the angle between $[A_1A_2]$ and $[A_1A_4]$ is $\frac{\pi}{2}$, the rotation angle is $\frac{\pi}{2}$. Therefore, the rotation is a taxicab isometry [6]. Thus $d_T(A_1,A_2) = d_T(A_1,A_4)$ and $d_E(A_1,A_2) = d_E(A_1,A_4)$. As a result, the quadrilateral is Euclidean and taxicab regular. □

Corollary 8 *Let $A_1A_2A_3$ be a Euclidean regular triangle on one of the $\pm x \pm y \pm z = k$ planes. If the taxicab length of one of the sides is $\sqrt{2}$ or $\frac{2\sqrt{6}}{3}$ times of the Euclidean length, $A_1A_2A_3$ is also taxicab regular.*

Proof. As it is seen in Corollary 6, there are 6 vectors on the same plane with the same taxicab length and the angles between two consecutive vectors are 60^o. If we choose 3 vectors whose angles between two consecutive vectors are 120^o (see Figure 19), these vectors can create a taxicab regular 3−gon. □

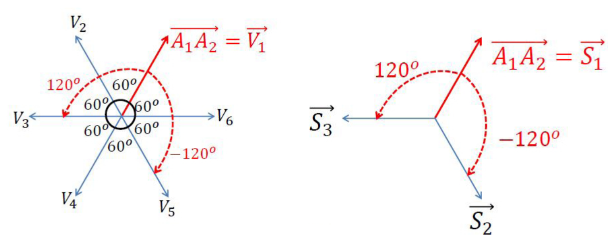

Figure 19.

Example 5 *For $A_1 = (1,2,-1)$, $A_2 = (4,-4,2)$, $A_3 = (7,-1,-4)$ points, ABC is a Euclidean and taxicab regular triangle.*

In addition, $d_T(A_1,A_2) = \frac{2\sqrt{6}}{3} d_E(A_1,A_2)$ and A_1, A_2, A_3, A_4, A_5, A_6 points are on $x+y+z=2$ plane.

Example 6 *For $A_1 = (0,0,0)$, $A_2 = (0,-1,1)$, $A_3 = (1,-1,0)$ points, ABC is a Euclidean and taxicab regular triangle.*

Furthermore, $d_T(A_1,A_2) = \sqrt{2} d_E(A_1,A_2)$ and A_1, A_2, A_3, A_4, A_5, A_6 points are on $x+y+z=0$ plane.

4 Taxicab Regular Polygons in Taxicab 3-Space

Theorem 6 *There do not exist any taxicab regular triangles which are not Euclidean.*

Proof. Let ABC be non-Euclidean taxicab regular triangle. Then $s(\hat{A}) = s(\hat{B}) = s(\hat{C}) = 60^o$. Thus, ABC is a Euclidean regular triangle and this is a contradiction. □

Theorem 7 *There exist taxicab regular* $4-$*gons which are not Euclidean.*

Proof. Draw an origin-centered Euclidean and taxicab sphere with a radius 1 (see Figure 20). Let $P \in]AE[$ line segment.

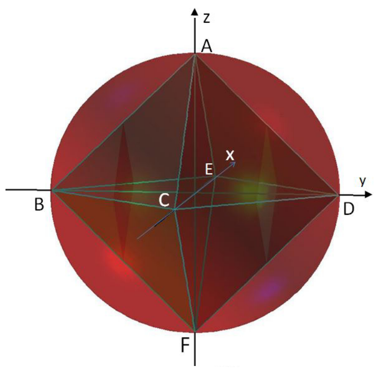

Figure 20.

Since part of line $[OD]$ is perpendicular to the $x = 0$ plane and the AEO triangle is on the $x = 0$ plane, $[OP] \perp [OD]$. Besides, $d_E(O,D) = d_T(O,D) = d_T(O,P) = 1$ but $d_E(O,P) < 1$. Because OPH is a right triangle and $d_E(O,P) < d_E(O,H) + d_E(H,P) = d_T(O,P) = 1$ as it is shown in Figure 21. According to [6] Proposition 3.1, every Euclidean translation is a taxicab isometry. Thus, we can create a non-Euclidean taxicab regular $4-$gon by translating $[OP], [OD]$ parts of the lines as in Figure 22. Since $P \in]AE[$ line segment, there are many P points. Therefore, there exist many non-Euclidean taxicab regular $4-$gons. □

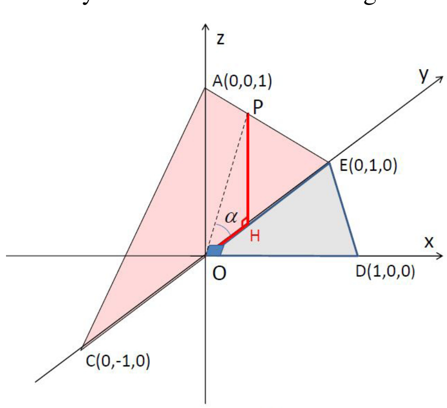

Figure 21.

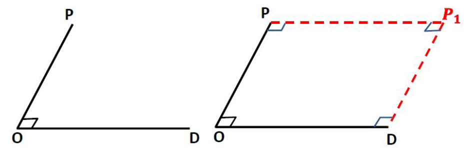

Figure 22.

Example 7 *Let us consider* $A(0,0,0)$, $B(2,0,0)$, $C(2,1,1)$, $D(0,1,1)$ *points. ABCD is a* $4-$*gon. It is clear that;*

$$[AB] \perp [BC], [BC] \perp [CD], [CD] \perp [AD], [AD] \perp [AB]$$

and

$$d_T(A,D) = d_T(B,C) = d_T(A,B) = d_T(C,D) = 2$$

but $d_E(A,D) \neq d_E(A,B)$*. Thus, ABCD is a non-Euclidean taxicab regular* $4-$*gon.*

Theorem 8 *When a line segment is given as a side, it is possible to create a taxicab regular* $2n-$*gons* $(n \geq 2)$ *including the given side.*

Proof. In $\mathbb{R}^3$, let us consider now any given line segment $[A_1A_2]$ in the taxicab plane. Since all the corners of polygons are planar, a plane E must be chosen which includes the given line segment to draw a $2n$-sided polygon (see Figure 23). After then, let us draw an A_1-centered taxicab sphere with a radius of $d_T(A_1,A_2)$ (see Figure 24).

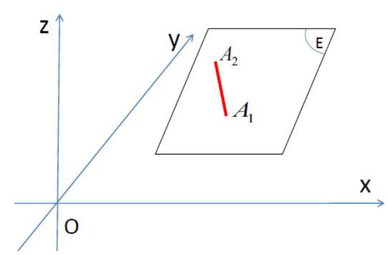

Figure 23.

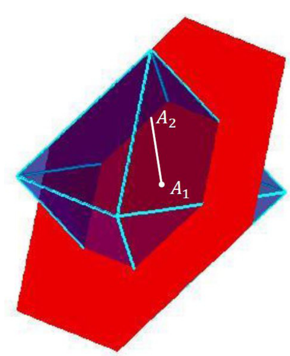

Figure 24.

It is clear that the plane and taxicab sphere intersect. The intersection of a plane and a taxicab sphere is a taxicab circle on plane E as the intersection of a plane and a Euclidean sphere is circle in $\mathbb{R}^3$. It is known that the measure of each interior angle of a regular $2n-$gon is $\frac{\pi(n-1)}{n}$ radians. It is obvious that $(n-1)$ line segments A_iA_{i+1}, $(2 \leq i \leq n)$, having the same taxicab length $d_T(A_1,A_2)$, can be drawn such that the measure of the angle between every two consecutive segments is $\frac{\pi(n-1)}{n}$ radians, by using the taxicab circles with center A_i and a radius of $d_T(A_1,A_2)$, as in Figure 25.

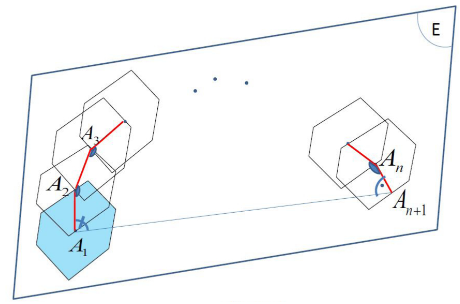

Figure 25.

$\measuredangle A_2A_1A_{n+1} + \measuredangle A_nA_{n+1}A_1 = \frac{\pi(n-1)}{n}$. If we continue to draw line segments $A'_iA'_{i+1}$ which are symmetric to A_iA_{i+1}, $1 \leq i \leq n$, about the midpoint of A_1A_{n+1}, respectively, we get a $2n-$gon (see Figure 26). Since the symmetry to a point is taxicab isometry, both taxicab lengths and angle measures are preserved.
Thus, for $1 \leq i \leq n$,

$$d_T(A_i,A_{i+1}) = d_T(A'_i,A'_{i+1}) = d_T(A_1,A_2) \tag{31}$$

and

$$\begin{aligned}\measuredangle A_1A_2A_3 &= \measuredangle A_{n+1}A'_2A_1 = \ldots \\ &= \measuredangle A_{n-1}A_nA_{n+1} + \measuredangle A'_{n-1}A'_nA_1 = \frac{\pi(n-1)}{n}.\end{aligned} \tag{32}$$

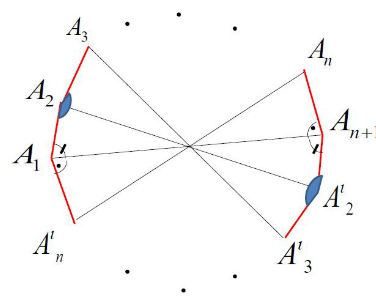

Figure 26.

Due to the equalities (31) and (32), this is a taxicab regular $2n-$gon. Since there are many planes which include the line segment, it is possible that many taxicab regular $2n-$gons can be drawn by using each plane. □

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