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MIXED-INTEGER NONLINEAR PROGRAMMING BASED OPTIMAL TIME SCHEDULING OF CONSTRUCTION PROJECTS UNDER NONCONVEX COSTS

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Original scientific paper

Optimal project scheduling under nonconvex time-cost relations represents a challenging problem in construction management. The nonconvex time-cost relations may appear in a construction project when several different duration options are available for its activities due to alternative technological processes enabled for their realization or wide accessibility of production resources. The source of nonconvexity of the project scheduling optimization problem can also be the project penalty- or bonus-duration relations arranged within the construction contract. The aim of this paper is to present the mixed-integer nonlinear programming (MINLP) based optimal time scheduling of construction projects under nonconvex costs. For this purpose, the MINLP model was developed and applied. A numerical example from literature and an example of construction project time-cost trade-off analysis under practical nonconvex penalty function are given in the paper to demonstrate advantages of MINLP optimization. The example from literature first presented the capability of the MINLP approach to obtain the optimal solution for difficult, highly combinatorial nonconvex discrete project scheduling problem. Thereupon, the following example revealed that the optimal project time-cost curve may take very nonuniform shape on account of discrete nature of activity direct cost options and nonconvex relation between project duration and total cost. In this way, the presented study intends to provide practitioners with new information from the field of optimization techniques for project scheduling as well as an alternative view on performance of total cost when project duration is changed.

Keywords: *construction management; discrete optimization; mixed-integer nonlinear programming; nonconvex costs; time scheduling*

Optimalno terminsko planiranje građevinskih projekata s nekonveksnim troškovima pomoću mješovitog cjelobrojnog nelinearnog programiranja

Izvorni znanstveni članak

Optimalno terminsko planiranje projekata s nekonveksnim troškovima predstavlja zahtjevan problem u organizaciji građenja. Nekonveksni odnosi između vremena i troškova mogu nastupiti u građevinskom projektu kada su na raspolaganju različite varijante za trajanje njegovih aktivnosti zbog mogućnosti izbora različitih tehnoloških procesa za izvođenje radova ili široke pristupačnosti proizvodnih resursa. Izvor nekonveksnosti problema troškovne optimizacije terminskog plana je moguće naći i u dogovorenom odnosu između trajanja projekta te penala ili premije iz ugovora o građenju. Cilj ovoga rada je predstaviti optimalno terminsko planiranje projekata s nekonveksnim troškovima pomoću mješovitog cjelobrojnog nelinearnog programiranja. U tu svrhu je razvijen i primijenjen optimizacijski model. Za prikaz prednosti optimizacije s mješovitim cjelobrojnim nelinearnim programiranjem su u radu predstavljeni uporaba razvijenog modela na primjeru iz literature i primjer analize ovisnosti ukupnih troškova građevinskog projekta o dužini njegovoga trajanja uzimajući u obzir praktičnu nekonveksnu funkciju penala. Primjer iz literature je najprije prikazao sposobnost pristupa mješovitog cjelobrojnog nelinearnog programiranja da pronađe optimalno rješenje za zahtjevan, vrlo kombinatoričan, nekonveksan i diskretan problem planiranja projekta. Sljedeći primjer je zatim u nastavku pokazao, da optimalna krivulja ovisnosti ukupnih troškova projekta od njegova trajanja može imati veoma neujednačen oblik zbog utjecaja diskretno definiranih direktnih troškova za varijante izvođenja aktivnosti te nekonveksnog odnosa između trajanja projekta i ukupnih troškova. Predstavljeni rad na ovaj način namjerava praktičarima ponuditi nove informacije s područja optimizacijskih tehnika za planiranje projekata kao i jedan drugačiji pogled na ponašanje ukupnih troškova projekta kada se njegovo trajanje promijeni.

Ključne riječi: *diskretna optimizacija; mješovito cjelobrojno nelinearno programiranje; nekonveksni troškovi; organizacija građenja; terminsko planiranje*

1 Introduction

The problem of cost optimal project time scheduling, i.e. the time-cost trade-off problem (TCTP), was originally initiated by Kelley and Walker in 1959 [1]. Since the invention of critical path method, the TCTP has attracted considerable interest in research community and has generated a large number of papers due to its practical relevance. The objective of the TCTP is to minimize the total cost of the project by crashing and scheduling its activities in a given project network. When the total cost becomes nonlinearly time-dependent and the project schedule is given in discrete duration units (e.g. working days), the TCTP turns into the nonlinear discrete problem.

The construction industry is traditionally affected by inefficiency and ineffectiveness [2]. Since construction organizations are required to adapt on specific market demands to achieve competitive performance [3], the project scheduling and effective management are very important for success in construction business [4, 5]. In order to state whether it is acceptable to the contractor to take on the potential project, it is necessary to analyse two basic parameters, i.e. the cost and the duration [6].

The optimal solution of the nonlinear discrete TCTP has received a substantial attention among the researchers and various techniques have been proposed in literature, e.g. genetic algorithms [7÷12], simulated annealing [13, 14], tabu search [14, 15], neural networks [16], ant colony optimization [17÷20], particle swarm optimization [21], differential evolution [22], harmony search [23] mixed-integer linear programming [24÷28] and hybrid methods, such as genetic algorithm and simulated annealing [28], genetic algorithm and dynamic programming [29], cutting plane method and Monte Carlo simulation [30], etc.

The existing solution approaches are mostly focused on TCTPs with linear, concave, convex and hybrid concave-convex time-cost relations. However, nonconvex time-cost relations may appear in a project when several duration options are available for its activities due to alternative technological processes enabled for their realization or wide accessibility of resources. The source of nonconvexity can also be the project penalty- or bonus-duration relations arranged in the construction contract.

Klanšek [31] has found, on the basis of obtained optimization results for different nonlinear discrete transportation problems, that a specific nonconvex and discrete network problem, similar to the TCTP addressed here, can be appropriately solved to optimality by MINLP approach. The aim of this paper is to present the MINLP based optimal solution of the nonconvex discrete TCTP. For this purpose, the MINLP model was developed and applied. A numerical example from literature and an example of project time-cost trade-off analysis under practical nonconvex penalty function are given in the paper to demonstrate advantages of MINLP optimization.

2 Time-cost relations in project scheduling problem

The minimization of the total project cost is often set as the objective of the TCTP. In early studies, the time-cost relations were assumed to be linear [32], see Fig. 1a. Still, even the early works revealed that the linear function rarely reflects the actual dependence between duration and cost. Afterwards, the time-cost relations were often set with convex functions [33÷37], see Fig. 1b.

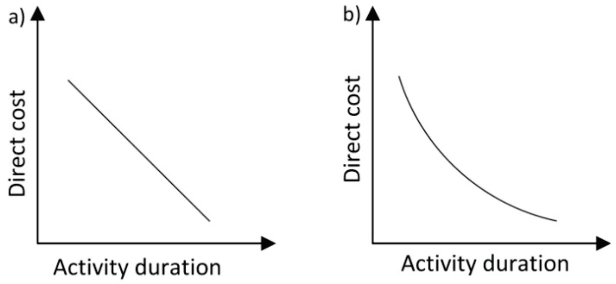

Figure 1 Linear and convex time-cost relations

In some cases, the convexity assumption was also relaxed by enabling the time-cost function to take a different shape, e.g. concave [38] or hybrid concave-convex [39], see Figs. 2a and 2b, respectively.

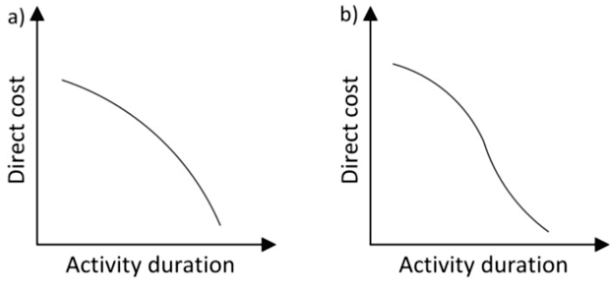

Figure 2 Concave and hybrid convex-concave time-cost relations

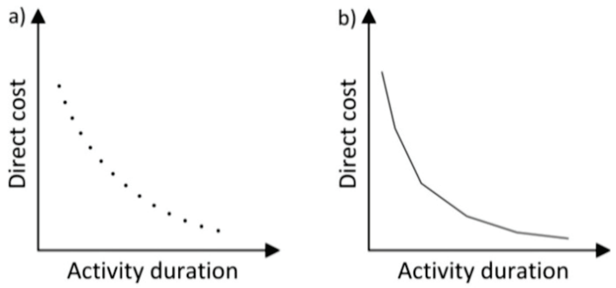

Figure 3 Discrete and piece-wise linear time-cost relations

Meyer and Shaffer [40] presented one of the first mixed-integer linear programming (MILP) models for TCTP with discrete time-cost relations. Thereupon, the MILP has been mainly used to solve exactly the discrete TCTP [24÷28]. Since the MILP can only handle linear dependences between variables, the nonlinear time-cost relations were usually set with discrete or piece-wise linear terms, see Figs. 3a and 3b, respectively.

Time-cost relations can in many cases be suitably formulated with (piece-wise) linear, concave, convex or hybrid concave-convex functions. However, when several different time-cost options are available for activities, the abovementioned approximations may be less favourable. Namely, the differences between the values of parameters for actual time-cost dependences, gained from the project analysis, and the approximated ones, included into the TCTP model, may lead the search algorithm to attain the sub-optimal solution. Although the nonconvex time-cost relations can also be discretized, the advantages of gradient information will remain unexploited during the search process. In such cases, it can be suitable to use the nonconvex time-cost relations in the TCTP model.

The indirect cost in construction project often includes initial costs, equipment operating and overhead costs, and business operating costs. Generally, the project indirect cost can be defined in the TCTP model by different terms. However, the linear dependence between project duration and its indirect cost, usually defined as daily cost, is commonly used in construction projects.

The penalties and bonuses may influence project performance when they are included in the construction contract. Generally, penalties and bonuses can be single or they can be set with linear or nonlinear time-dependences. Still, they are usually arranged within the contract as the unconstrained- or as the constrained linear penalties/bonuses, see Figs. 4a and 4b, respectively.

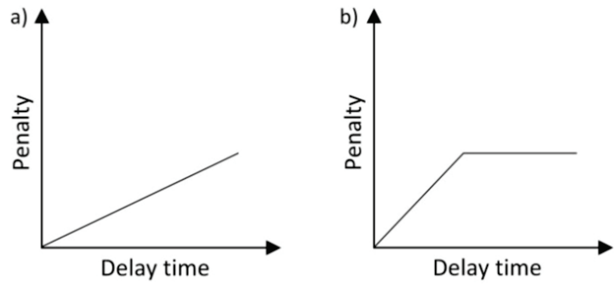

Figure 4 Unconstrained and constrained linear penalties

It is common situation in construction business that constrained linear penalties are set in the contract. For instance, the current Special Construction Practice Code [41] in Slovenia regulates that the daily penalty for project delay is 0,1 % of the contract amount while the highest penalty is limited by 5 % of the contract amount. Similar practice may also be found in other countries.

Thus, the single- and the unconstrained linear penalty/bonus-duration relations can be defined as linear parts of the TCTP model while the constrained linear penalty/bonus is the discontinuous relation for which the continuous formulation may be achieved with nonconvex function. So, the project penalty- or bonus-duration relations can also be the source of nonconvexity of the TCTP.

3 MINLP optimization

MINLP is an exact mathematical programming technique which executes discrete optimization of discrete variables simultaneously with the continuous optimization of continuous variables. Since the MINLP can manage nonlinear relations between decision variables, it was selected to find the optimal solution of the nonconvex discrete TCTP. Generally, the nonlinear continuous / discrete MINLP optimization problem can be given as follows:

$$\begin{aligned}
&\text{Minimize } z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{y} + f(\boldsymbol{x}) \\
&\text{subject to:} \\
&\boldsymbol{h}(\boldsymbol{x}) = 0 \\
&\boldsymbol{g}(\boldsymbol{x}) \le 0 \\
&\boldsymbol{By} + \boldsymbol{Cx} \le \boldsymbol{b} \\
&\boldsymbol{x} \in X = \left\{ x \in R^n : \boldsymbol{x}^{\mathrm{LO}} \le \boldsymbol{x} \le \boldsymbol{x}^{\mathrm{UP}} \right\} \\
&\boldsymbol{y} \in Y = \{0,1\}^m
\end{aligned} \qquad \text{(MINLP-G)}$$

where $\boldsymbol{x}$ denotes the vector of continuous variables determined from the compact set X and $\boldsymbol{y}$ stands for vector of binary 0–1 variables. The continuous variables $\boldsymbol{x}$ may be applied linearly or nonlinearly in the objective function z and constraints while the binary 0–1 variables y can be used only in linear terms. Functions $f(\boldsymbol{x})$, $\boldsymbol{h}(\boldsymbol{x})$ and $\boldsymbol{g}(\boldsymbol{x})$ are nonlinear functions with continuous variables contained in the objective function, equality and inequality constraints, respectively. All functions $f(\boldsymbol{x})$, $\boldsymbol{h}(\boldsymbol{x})$ and $\boldsymbol{g}(\boldsymbol{x})$ must be continuous and differentiable. Finally, the expression $\boldsymbol{By} + \boldsymbol{Cx} \le \boldsymbol{b}$ stands for subset of mixed linear (in)equality constraints.

The MINLP solution of a nonlinear discrete optimization problem may be achieved in three steps. The first step consists of generation of a superstructure with different alternatives for discrete solutions of continuous variables, the second one involves formulation of a MINLP model and the last one comprises a solution for the MINLP problem. Generating an appropriate superstructure of discrete solution options is an important phase in MINLP optimization [42]. That is because the quality of the obtained solution directly depends on the quality and quantity of the discrete solution alternatives included into MINLP problem superstructure. Since the optimal discrete solution has to be found from the developed superstructure of alternatives, considering all feasible variations, the selection procedure requires a discrete decision optimization. After the superstructure of discrete alternatives is successfully generated, the MINLP optimization model can be formulated applying suitable modelling software and solved by using an appropriate algorithm [31].

4 MINLP model formulation for nonconvex discrete TCTP

Taking MINLP-G formulation into consideration, the MINLP model for the nonconvex discrete TCTP is more specific, particularly in view of variables and constraints. Thus, it consists of the cost objective function, the precedence relationship constraints, the project duration constraints and the logical constraints. Therefore, the objective of optimization is to minimize total project cost while satisfying project network precedence relationship constraints, project duration limitations and logical constraints for selection of optimal discrete start times and durations of activities. MINLP optimization model of nonconvex discrete TCTP may be given as follows:

$$\begin{aligned}
&\text{Minimize } COST = \sum_{i \in I} C_i(D_i) + CI(DP) + P(DL) - \\
&- B(DE) + \varepsilon \sum_{i \in A} S_i - \varepsilon \sum_{i \in Z} S_i
\end{aligned} \tag{1}$$

subject to:

$$S_i + D_i + L_{i,j} \le S_j,\ i \in I,\ j \in J(i),\ (i,j) \in FS \tag{2}$$

$$S_i + L_{i,j} \le S_j,\ i \in I,\ j \in J(i),\ (i,j) \in SS \tag{3}$$

$$S_i + D_i + L_{i,j} \le S_j + D_j,\ i \in I,\ j \in J(i),\ (i,j) \in FF \tag{4}$$

$$S_i + L_{i,j} \le S_j + D_j,\ i \in I,\ j \in J(i),\ (i,j) \in SF \tag{5}$$

$$S_{i\omega} + D_{i\omega} - S_{i\alpha} \le DP,\ i\alpha, i\omega \in I \tag{6}$$

$$DP - DL + DE = DT \tag{7}$$

$$DLDE = 0 \tag{8}$$

$$D_i = \sum_{e \in E} y_{i,e} DT_{i,e},\ i \in I \tag{9}$$

$$\sum_{e \in E} y_{i,e} = 1,\ i \in I \tag{10}$$

$$S_i^{\mathrm{LO}} \le S_i \le S_i^{\mathrm{UP}},\ i \in I,\ S_i \in R^+ \tag{11}$$

$$D_i^{\mathrm{LO}} \le D_i \le D_i^{\mathrm{UP}},\ i \in I,\ D_i \in R^+ \tag{12}$$

$$DE^{\mathrm{LO}} \le DE \le DE^{\mathrm{UP}},\ DE \in R^+ \tag{13}$$

$$DL^{\mathrm{LO}} \le DL \le DL^{\mathrm{UP}},\ DL \in R^+ \tag{14}$$

$$DP^{\mathrm{LO}} \le DP \le DP^{\mathrm{UP}},\ DP \in R^+ \tag{15}$$

$$y_{i,e} \in \{0,1\},\ i \in I,\ e \in E \tag{16}$$

Eq. (1) represents the objective function. The variable *COST* denotes total project cost, the set I contains project activities i, $i \in I$, $C_i(D_i)$ stands for direct cost-duration functions of activities, $CI(DP)$ is the project indirect cost-duration function, $P(DL)$ is the penalty-duration function and $B(DE)$ is the bonus-duration function. The dependence between direct cost C_i of an activity and its duration D_i in construction project is often nonlinear. When various activity duration options must be included into the MINLP model of the TCTP, the nonconvex approximation can be used for $C_i(D_i)$ relations. The amount of indirect cost CI often linearly depends on project duration DP. However, the nonlinear functions $CI(DP)$ may also be used if they are required.

The penalty cost P depends on the amount of delay time DL, while the award bonus B depends on the amount of time the project is early DE. In published literature, the $P(DL)$ and $B(DE)$ terms were usually set into the TCTP model as constants (i.e. single penalty/bonus) or as unconstrained linear functions. When discontinuous constrained linear project penalty is defined in the construction contract, the next nonconvex continuous approximation can be used for $P(DL)$ relation:

$$P(DL) = PM\left\{\frac{DL}{DG}\left[\frac{\arctan(CF\,DL)}{\pi}+\frac{1}{2}\right] + \left(1-\frac{DL}{DG}\right)\left[\frac{\arctan(CF(DL-DG))}{\pi}+\frac{1}{2}\right]\right\} \tag{17}$$

where the PM is the maximum amount of penalty allowed in the construction contract and DG is the delay time period during which the penalty is time-dependent and rises linearly with gradient DL/DG (i.e. the amount of time when the per period penalty is incurred), see Fig. 5.

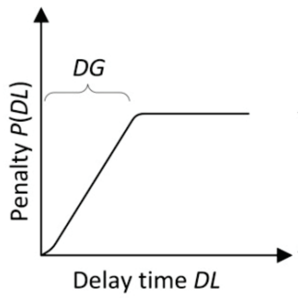

Figure 5 Continuous approximation of constrained linear penalties

At the end of the mentioned period, penalty $P(DL)$ achieves maximum value PM. If the delay time DL exceeds the time period DG, the penalty function $P(DL)$ will give a constant value PM for any larger value of DL. Tightness of the approximation function is controlled by a large value of the curve-fitting parameter CF. Note that Eq. (17) can also be suitably used in a similar manner for continuous approximation of constrained linear bonuses, if they are arranged in the construction contract.

Project network includes critical activities (i.e. activities on the critical path) and also several activities with some float times (i.e. noncritical activities). The noncritical project activities may start with some delay without influencing project duration and total cost. The expressions $+\varepsilon S_i$ and $-\varepsilon S_i$ are added into the objective function for those project activities with floats, $i \in A$ and $i \in Z$, that are desired to start as soon as possible and as late as possible, respectively. The terms εS_i include the constant ε with a very small value that has no practical effect on the objective function value [26]. However, these expressions, defined in the objective function that should be minimized, force start times of activities S_i to take discrete values since their duration times D_i are also forced, by the logical constraints, to take discrete values.

Eqs. (2) ÷ (5) denote the Finish-to-Start (*FS*), Start-to-Start (*SS*), Finish-to-Finish (*FF*) and Start-to-Finish (*SF*) precedence relationships between project activities i, $i \in I$, and their succeeding activities j, $j \in J(i)$, respectively. While start times S_i and durations D_i of activities are defined with continuous variables, the lag/lead times $L_{i,j}$ between the activity i, $i \in I$, and its succeeding activities j, $j \in J(i)$, are defined with discrete constant parameters.

Since the activities must be performed between the project start and finishing times, the project duration constraints were set by Eq. (6) to limit their completion times. The variables $S_{i\omega}$ and $D_{i\omega}$ denote start times and durations of final activities $i\omega$, $i\omega \in I$, while $S_{i\alpha}$ represents start times of initial activities $i\alpha$, $i\alpha \in I$. The relationship between project duration DP, the delay time DL, the amount of time the project is early DE and the target project duration DT is given by Eq. (7). Since only one of the variables DL and DE may take a nonzero value in any scheduling solution, they are extra constrained by Eq. (8).

Logical constraints in Eqs. (9) ÷ (10) must be fulfilled for the selection of discrete solutions for continuous variables. The set E is defined to include discrete solution options e, $e \in E$, into the MINLP model. The discrete constants $DD_{i,e}$ represent the superstructure of discrete duration alternatives for activities. Since each discrete constant $DD_{i,e}$ is a potential discrete solution of its corresponding continuous variable D_i, the selection of the optimal discrete solution from the superstructure of alternatives is performed applying binary decision variables $y_{i,e}$.

Each discrete constant option is selected to be the discrete solution of the corresponding continuous variable only if the found value of the allocated binary variable is equal to 1. If the obtained value of the assigned binary variable is equal to 0, the discrete alternative is rejected. Eq. (10) assures that only one discrete value $DD_{i,e}$ can be selected as the discrete solution for each variable D_i. Eqs. (11) ÷ (15) define that optimal values for variables S_i, D_i, DE, DL and DP should be found between their lower and upper bounds (i.e. between crashed and normal values), while Eq. (16) defines the domain of binary decision variables $y_{i,e}$.

5 Numerical examples

A numerical example from literature and an example of the construction project time-cost trade-off analysis under practical nonconvex penalty function are presented in this section to show advantages of MINLP optimization. The TCTPs were solved on a 64-bit operating system by PC Intel Core i7, 2.93 GHz, 8 GB RAM, 1 TB hard disc.

5.1 Example from literature

The considered discrete TCTP was originally introduced in research work of Sonmez and Bettemir [28] and solved to optimality with application of a hybrid strategy developed using genetic algorithms, simulated annealing, and quantum simulated annealing techniques. The applied project contains 63 activities. The precedence relations between activities and the direct cost-duration options can be found in the reference [28].

The direct costs of activities were set in the MINLP model using polynomial approximations. Thus, the direct cost-duration functions were gained for activities using the polynomial curve-fitting calculations on the proposed cost-duration options. First, the quadratic polynomial approximation was applied for activities that contained three different cost-duration options. Thereupon, the cubic and the quartic polynomial approximations were used for activities with four and five different cost-duration options, respectively. The project indirect cost is \$2300/day as suggested in the reference [28].

Following the example project data, the objective function (1), the precedence relationship constraints (2), the project duration constraints (6), the logical constraints (9÷10), the bounds on continuous variables (11, 12 and

15), and the bounds on binary variables (16) of the MINLP model, presented in section 4, were used to find the optimal solution of the example TCTP. The MINLP optimization model for the adopted TCTP contained an objective variable (e.g. variable *COST*), 127 continuous variables (e.g. 63 variables S_i; 63 variables D_i; variable *DP*), 296 discrete variables (e.g. binary 0–1 variables $y_{i,e}$) and 217 constraints (e.g. 71 precedence relationship constraints; 20 project duration constraints; 126 logical constraints).

Taking the example data into account, one can find that the MINLP model for the 63-activity TCTP consists of 2 activities with 3 crashing options, 15 activities with 4 crashing options and 46 activities with 5 crashing options. As an outcome, the total number of feasible discrete time schedule combinations for the 63-activity TCTP can be calculated as $3^2 \times 4^{15} \times 5^{46}$ = 1,373×10^{42}. Therefore, the MINLP model for the considered TCTP represents a highly combinatorial nonconvex discrete optimization problem. A high-level language GAMS (General Algebraic Modelling System) [43] was used for modelling and for data inputs/outputs.

The main objective of the MINLP optimization was to determine the optimal project time schedule in which all defined activities are desired to start as soon as possible with respect to minimum total project cost, subjected to project precedence relationship constraints, project duration limitations and logical constraints for the selection of optimal discrete start times and durations of activities.

The project activities were initially set to their normal direct cost option. It should be noted here, that the normal project scheduling solution, in a view of required total solver time consumption, does not represent the most efficient initial point for optimization. Namely, the convergence of the search algorithm can be achieved much faster, if the initial point is determined closer to the optimal solution. However, the normal project schedule generated as initial point for optimization is often used in practice since it usually represents the first feasible solution known to a planner.

The simple branch and bound method (SBB) [44] was employed to solve the nonconvex discrete TCTP of the 63-activity example project to optimality. The SBB method incorporates the branch and bound optimization method (BB) known from MILP and NLP local search techniques. Initially the relaxed MINLP problem was solved from the provided starting point. The space of feasible solutions for discrete decision variables was subdivided in the BB process. Then, the constraints on the discrete variables were tightened to new integer values to eliminate current non-integer solutions.

After the bounds were tightened, a new, tighter NLP sub-problem was solved beginning the search from the precedent NLP solution. CONOPT (generalized reduced-gradient algorithm) [45] was applied to solve the NLP sub-problems. The objective function value gained from the result of the NLP sub-problem was determined as lower bound on the objective in the reduced feasible space. The SBB search terminated when all sub-problems were solved or fathomed.

Table 1 Optimal solution for nonconvex discrete TCTP of the 63-activity example project

Project activity	Optimal solution		
	Start time (day)	Duration (days)	Direct cost ($)
1	1	12	4 250
2	1	18	14 800
3	1	24	22 450
4	1	19	17 800
5	1	28	31 180
6	13	44	54260
7	13	39	47600
8	19	52	62 140
9	25	63	72 750
10	20	57	66 500
11	29	63	83 100
12	57	68	75 500
13	52	40	34 250
14	71	33	52 750
15	88	47	38 140
16	88	75	94 600
17	77	60	78 450
18	92	81	127 150
19	92	36	82 500
20	125	41	48 350
21	92	64	85 250
22	104	53	79 100
23	135	43	66 450
24	163	66	72 500
25	137	50	70 100
26	173	84	93 500
27	166	67	78 500
28	156	66	85 000
29	157	76	92 700
30	178	34	27 500
31	187	96	145 000
32	257	43	43 150
33	257	52	61 250
34	222	74	89 250
35	233	138	183 000
36	229	54	47 500
37	283	29	26 750
38	300	51	61 250
39	309	67	81 150
40	296	41	45 250
41	371	23	32 300
42	283	44	36 400
43	283	75	66 800
44	312	82	102 750
45	376	59	84 750
46	376	66	94 250
47	337	54	73 500
48	327	41	36 750
49	394	147	312 000
50	435	101	47 800
51	442	83	84 600
52	391	31	23 150
53	368	39	31 500
54	541	18	24 300
55	422	29	23 400
56	536	38	41 250
57	559	30	53 400
58	422	24	12 500
59	451	27	34 600
60	574	31	28 500
61	589	20	33 500
62	605	25	38 750
63	609	22	12 700
		Indirect project cost ($):	1 449 000
		Total project cost ($):	5 421 120

The total solver time required by the SBB method to find the optimal solution for the example project TCTP was about two minutes. The gained minimum total project cost value was identical to that reported by Sonmez and Bettemir [28]. However, a more detailed comparison between the solution found in their study and the one obtained in this work was not performed due to lack of project schedule data.

Nevertheless, the optimal solution found by the SBB search method contains a minimum total cost of $ 5 421 120 and the optimal project duration of 630 working days. Tab. 1 demonstrates the optimal solution found for nonconvex discrete TCTP of the 63-activity example project.

The time-cost curve was also obtained for the example project after running repetitively the developed MINLP optimization model by SBB method for all feasible discrete project durations between normal and crashed project duration, see Fig. 6. The project time-cost curve includes minimum total project cost values achieved by repetitive MINLP optimization. In this way, Fig. 6 shows that the found cost optimal solution for the project time schedule was situated in a shallow minimum.

Fig. 7 presents critical paths of normal project scheduling solution and the optimal one as well as the accelerated activities which are presented in the project network diagram in white boxes. Fig. 7 shows that the normal project scheduling solution contains only one (initial) critical path while the optimal one includes three critical paths, i.e. initial path and two additional critical paths gained on the account of project compression from normal to optimal duration.

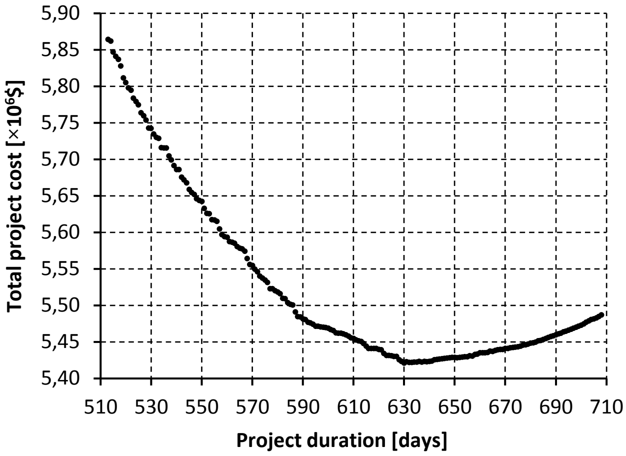

Figure 6 Time-cost curve of the 63-activity example project

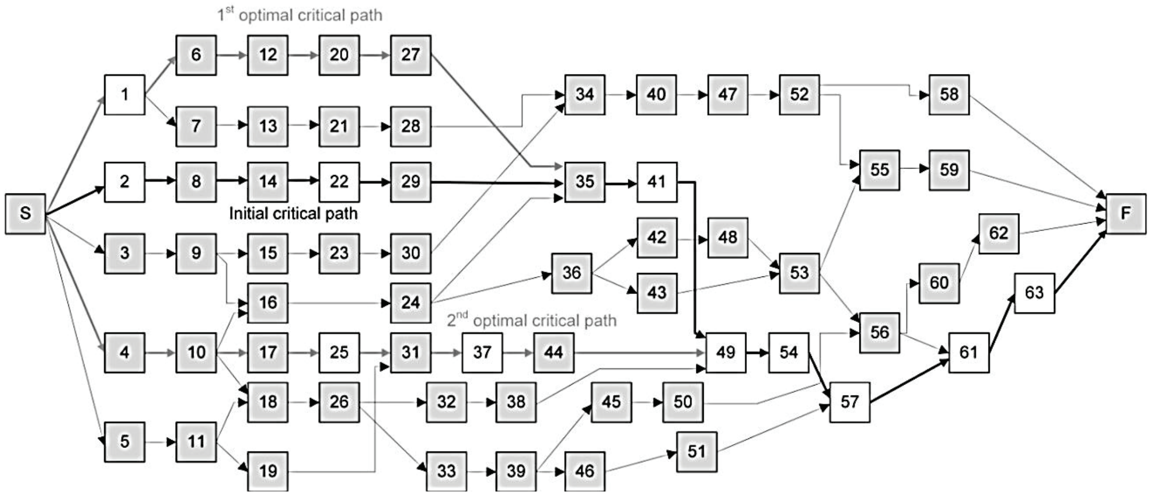

Figure 7 Critical paths of normal and optimal project duration schedules and accelerated activities

5.2 Example of construction project time-cost trade-off analysis under nonconvex penalty function

For a better presentation of the advantages of MINLP optimization, this paper also introduces an example of the construction project time-cost trade-off analysis executed under practical nonconvex penalty function. The example discusses a variant of the 29-activity construction project proposed by Sakellaropoulos and Chassiakos [26]. The precedence relationships and the lag/lead times between succeeding activities as well as the direct cost-duration options were included into the MINLP model as suggested in the reference [26]. Afterwards, the direct cost-duration functions were formulated for activities using polynomial curve-fitting calculations on given direct cost-duration options.

The target project duration was set at 75 days. Both the indirect cost and the project bonus for earlier finish were set at 150 units per day. Maximum project penalty allowed in the construction contract was set at 1000 units. The delay time period during which the value of penalty was time-dependent and raised linearly with 200 units per day was 5 days. At the end of that period, the penalty achieved the maximum value of 1000 units. The project penalty function was modelled using Eq. (1÷17).

The MINLP model presented with Eqs. (1÷17) was used to perform the project time-cost trade-off analysis. The model included an objective variable (e.g. variable *COST*), 61 continuous variables (e.g. 29 variables S_i; 29 variables D_i; variable *DP*; variable *DE*; variable *DL*), 69 discrete variables (e.g. binary variables $y_{i,e}$) and 100 constraints. The constraints contained: 25 FS + 9 SS + 1 FF precedence relations; 5 project duration constraints; 1 constraint for the relationship between the actual project duration, the delay time, the amount of time the project is early and the target project duration; 1 extra constraint for the relation between the delay time and the amount of time the project is early; and 58 logical constraints.

Table 2 Distribution of minimum total project costs

Activity	Direct costs of project activities																							
	Project duration																							
	93	92	91	90	89	88	87	86	85	84	83	82	81	80	79	78	77	76	75	74	73	72	71	70
1	2030	2030	2030	2030	2030	2030	2030	2030	2030	2030	2030	2030	2030	2030	2030	2030	2030	2030	2030	2300	2300	2300	2300	2300
2	1020	1020	1020	1020	1020	1020	1020	1020	1020	1020	1020	1020	1020	1020	1020	1020	1020	1510	1510	1510	1510	1510	1510	1510
3	1700	1700	1700	1700	1700	1700	1700	1850	1700	1850	1850	1850	1850	1850	1850	1850	2090	1850	2090	2090	2090	2090	2090	2090
4	590	590	590	590	590	590	590	590	590	590	590	590	590	590	590	590	590	590	590	590	590	590	590	590
5	90	90	90	90	90	90	90	90	90	90	90	90	90	90	90	90	90	90	90	90	90	90	90	90
6	910	910	910	910	910	910	910	910	910	910	910	910	910	910	910	910	910	910	910	910	910	910	910	910
7	250	250	250	250	250	250	250	250	250	250	250	250	250	250	250	250	250	250	250	250	250	250	250	250
8	1490	1490	1490	1490	1490	1490	1490	1490	1490	1490	1490	1650	1650	1650	1830	1830	1830	1830	1830	1830	1830	1830	1830	1830
9	520	520	520	520	520	520	520	520	520	520	520	520	520	520	520	750	750	750	750	750	750	750	750	750
10	90	90	90	90	90	90	90	90	90	90	90	90	90	90	90	90	90	90	90	90	90	90	90	90
11	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
12	3260	3260	3260	3260	3260	3260	3260	3260	3260	3260	3260	3260	3260	3260	3260	3260	3260	3260	3260	3260	3260	3260	3260	3260
13	1140	1140	1140	1140	1140	1140	1140	1140	1140	1140	1140	1140	1140	1140	1140	1140	1140	1140	1140	1140	1140	1140	1140	1140
14	300	300	300	300	300	300	300	300	300	300	450	450	450	450	450	450	450	450	450	450	450	450	450	450
15	1020	1020	1020	1020	1020	1020	1020	1020	1020	1020	1020	1020	1020	1020	1020	1020	1020	1020	1020	1020	1020	1020	1020	1020
16	790	790	790	790	790	790	790	790	790	790	790	790	790	790	790	790	790	790	790	790	790	790	790	790
17	3340	3340	3340	3340	3340	3340	3340	3340	3340	3340	3340	3340	3340	3340	3340	3340	3340	3340	3340	3340	3340	3340	3340	3340
18	470	470	470	470	470	470	470	470	470	470	470	470	470	470	470	470	470	470	470	470	470	470	470	470
19	460	460	460	460	460	460	460	460	460	460	460	460	460	460	460	460	460	460	460	460	460	460	460	460
20	1280	1280	1280	1280	1280	1280	1280	1280	1280	1280	1280	1280	1280	1280	1280	1280	1280	1280	1280	1280	1280	1280	1280	1280
21	1090	1090	1090	1090	1090	1090	1090	1090	1090	1090	1090	1090	1090	1090	1090	1090	1090	1090	1090	1090	1090	1090	1090	1090
22	900	900	1140	1140	1140	1400	1400	1400	1400	1400	1400	1400	1400	1400	1400	1400	1400	1400	1400	1400	1400	1400	1400	1400
23	2220	2220	2220	2220	2220	2220	2220	2220	2510	2510	2510	2510	2510	2690	2690	2690	2690	2690	2690	2690	2690	2690	2690	2690
24	230	230	230	230	230	230	230	230	230	230	230	230	230	230	230	230	230	230	230	230	230	230	230	230
25	1590	1590	1590	1590	1590	1590	1590	1590	1590	1590	1590	1590	1590	1590	1590	1590	1590	1590	1590	1590	1590	1590	1590	1590
26	2630	2630	2630	2630	2630	2630	2630	2630	2630	2630	2630	2630	2630	2630	2630	2630	2630	2630	2630	2630	2930	2630	2930	3240
27	2060	2060	2060	2060	2060	2060	2060	2060	2060	2060	2060	2060	2060	2060	2060	2060	2060	2060	2060	2060	2060	2660	2660	2660
28	320	440	320	320	440	320	440	440	440	440	440	440	610	610	610	610	610	610	610	610	610	610	610	610
29	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
Indirect cost:	13950	13800	13650	13500	13350	13200	13050	12900	12750	12600	12450	12300	12150	12000	11850	11700	11550	11400	11250	11100	10950	10800	10650	10500
Penalty:	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	800	600	400	200	–	–	–	–	–	–
Bonus:	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–150	–300	–450	–600	–750
Total cost:	46840	46810	46780	46630	46600	46590	46560	46560	46550	46550	46550	46560	46580	46610	46440	46320	46210	46110	46000	45970	45970	45970	45970	45980

Notes: Project durations are given in days. Direct costs of project activities, indirect costs, penalties, bonuses and total costs are assessed in cost units

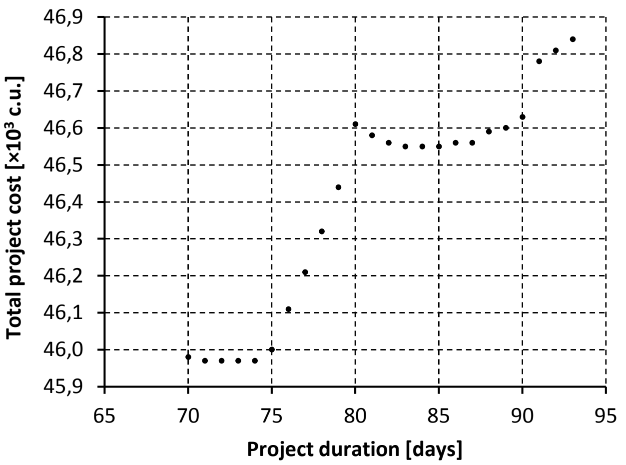

Figure 8 Time-cost curve of the 29-activity example project

Fig. 8 demonstrates gained time-cost curve of the project. Distribution of the minimum total project costs for each optimal solution is presented in Tab. 2.

The optimal scheduling solutions and their corresponding minimum total cost values shown in Fig. 8 and Tab. 2 were obtained after running the MINLP model by the SBB method for all feasible discrete project durations between normal and crashed values.The MINLP search found 24 different optimal solutions between normal (93-day) and crashed (70-day) project durations. The required total solver time for each solution was less than a second.

Tab. 2 indicates that the identical minimum total cost of 45970 units was obtained at four different solutions, namely at 71-, 72-, 73- and 74-day project durations. In this case, the optimization extended the amount time the project is early rather than its duration because such solutions reached bigger total cost savings on reduced indirect costs and achieved bonuses than the alternative solutions on lower direct costs and incurred penalties.

Further observations are related to the nonuniform shape of the project time-cost curve. There were two main reasons why the time-cost curve was found in such form. The first one was the discrete nature of the direct cost options. Tab. 2 shows that the sum of selected optimal discrete direct cost options, in the total cost, changed nonuniformly as the project duration altered. The second reason was the nonconvex relation between project duration and total cost. Namely, the direct cost-, the indirect cost- and the bonus-duration functions were defined using convex expressions while the penalty-duration function was set by nonconvex term. Although all parts of the objective function had an effect on nonuniform shape of the time-cost curve, the influence of project penalties was significant in this case.

6 Conclusions

The TCTPs in construction projects can be combinatorial and highly nonlinear network problems that are generally hard to be solved to optimality. The aim of the paper was to show advantages of the MINLP based optimal solution of the nonconvex discrete TCTP. Thus, the MINLP model was developed and applied. The ability of the MINLP approach to achieve optimal solution for a specific nonconvex and discrete network problem similar to the TCTP was investigated in the past study [31] on a set of different nonlinear discrete transportation problems. Encouraged by gained results of experiments on test problems with different nonlinear expressions (such as regular quadratic functions, square root functions, rational functions, sine functions and arc-tangent approximations of piece-wise linear functions), the motivation for this research was to make a step forward in the field of project scheduling under nonconvex costs which occur in construction practice.

The example from literature presented the capability of MINLP model to find the optimal solution for highly combinatorial nonconvex discrete TCTP from the initially generated normal project schedule by consuming a reasonable total solver time. In the next example, a time-cost trade-off analysis of a construction project with constrained linear penalties was performed employing the MINLP model for nonconvex discrete TCTP which included polynomial and arctangent functions. The results exposed the nonuniform shape of the time-cost curve which occurred due to discrete nature of direct cost options as well as the nonconvex relation between project duration and total cost.

The modelling capabilities were also identified as the advantage of the proposed approach. Namely, the MINLP model may contain a wide variety of nonlinear functions and thus avoids the need for (piece-wise) linear or discrete approximation of the nonlinear expressions. For instance, the continuous arc-tangent function was proposed in Section 4 to incorporate classical upwards limited linear penalties into the MINLP model for optimal project scheduling. Hence, the combinatorial size of the model can be reduced by using continuous variables for handling discrete scheduling parameters in nonlinear dependences instead of applying integer variables for model discretization, see e.g. variables DE and DL.

From the viewpoint of improvements of existing techniques, the capability of the MINLP optimization to handle nonconvex dependencies explicitly reduces the user's effort in dealing with large-size data and updating the model when circumstances, under which the project scheduling was done, are changed. The employment of nonlinear expressions can enable more compact model formulation as well as more rapid execution of model management tasks, such as transformation of data into model parameters and model modifications.

In this way, the present study intends to provide new information for construction management experts as well as it introduces the basis for further research in the field of optimal project scheduling. Further research effort will be focused on the exact nonlinear discrete cost optimization of project schedules under limited resources as well as the optimal multi-project scheduling.

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