On Fluid Dynamics of Freshwater and Seawater in Marine Systems

O dinamici miješanja slatke i morske vode u morskim sustavima

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Summary

Interaction between freshwater and seawater can significantly influence marine life and create important productive and dynamic natural habitats, such as estuaries. We study such processes and for this purpose we develop a new mathematical model with the system of coupled nonlinear conservation equations that govern the interaction of the two incompressible fluids with different properties. The system is solved by using implicit operator splitting and mixed finite elements methods. Finally, the numerical scheme has been used to provide simulations of two real world scenarios.

KEY WORDS
Navier-Stokes
estuary
numerical simulation
freshwater and saltwater

Sažetak

Interakcija svježe i morske vode može značajno utjecati na morski život i stvoriti važna produktivna i dinamična prirodna staništa, kao što su estuariji. Proučavamo takve procese i s tim ciljem razvijamo novi matematički model sa sustavom združenih nelinearnih jednadžbi koje upravljaju interakcijom dvije nestlačive tekućine s različitim osobinama. Sustav se rješava korištenjem implicitnog odvajanja i metodama ograničenih elemenata. Konačno je upotrebljen numerički prikaz da bi se simuliralo stvarno stanje stvari.

KLJUČNE RIJEČI
Navier-Stokes
estuarij
numerička simulacija
slatka i slana voda

1. INTRODUCTION / *Uvod*

Mathematical modelling is the way of transforming problems from the real world into mathematical equations whose theoretical and numerical analysis provides insight, answers and guidance useful for better understanding of the original problem. In this paper, we consider the interaction of seawater and freshwater as fluids with different properties whose interaction results in forming of important biological environments. To this end we will derive a coupled system of nonlinear partial differential equations.

As mentioned in [12], estuary is a partly enclosed coastal body of brackish water with one or more rivers or streams flowing into it, and with a connection to the open sea. It forms a transition zone between river environments and maritime environments with influences from both, such as tides, waves and the influx of saline water from the sea and also flows of freshwater and sediment produced by river. Inflows of both seawater and freshwater make estuaries among the most productive natural habitats in the world. It is no surprise that humans also recognised fecundity of estuaries and inhabited it heavily resulting with about 60% of worlds population living along estuaries and the coast [15]. Of the thirty-two largest cities in the world, twenty-two are located on estuaries (e.g. New York City, [13]). Unfortunately, many estuaries suffer from degradation and disbalance caused by human activities, such as sedimentation from soil erosion from deforestation, overgrazing, overfishing, eutrophication as a response to presence of artificial nutrients from sewage and artificial fertilizers, various pollutants, drainage and filling of wetlands, and flood control or water diversion practices such as damming or diking [15]. In order to preserve this important natural habitats it is of great interest to understand and model their processes.

Let us mention [10] where author discussed seawater penetration in river mouths with regard to the tidal range. Another example is the west end of Salmon Bay, Washington's lake Washington Ship Canal where a complex of locks called Hiram M. Chittenden Locks has been built to prevent the mixing of seawater from Puget Sound with the freshwater of the lake.

Another notable application of the proposed model is in the water pollution. When toxic substances enter a river, they will (partially) dissolve. The resulting water pollution causes the quality of the water to deteriorate and affects aquatic ecosystems in river, estuary and potentially the sea [6].

Most studies base their conclusions on experimental data that can often be very expensive to obtain ([5], [1]), or even unavailable. With this in mind, we offer a mathematical model based on some physical parameters that can help us gain better understanding of interactions between seawater and freshwater.

In this paper we are interested in two specific questions regarding marine life. We are concerned in submarine springs, typical and abundant in karstic terrain such as Adriatic, and their influence in salinity of the sea around them, particularly below the springs themselves. Because of its importance we will also study a salt wedge estuary type and see how processes develop in such interface, mainly caused by discharge of the river and physical properties of fluids in such mixture.

The paper is organized as follows. After Introduction, we present governing equations and explain corresponding modelling issues. In Section 3, we provide simulations of the mentioned mixing in the rectangular basin in two scenarios. First we simulate submarine spring by freshwater intrusion in basin full of seawater, and then salt wedge estuary by seawater intrusion in freshwater body.

2. MATHEMATICAL MODEL / *Matematički model*

As we are considering dynamics of two fluids the first equation that governs mentioned process is the Navier-Stokes equation, also called the momentum equation. That is the basic governing equation for a viscous and heat conducting fluid. It is a vector equation derived from Newton's Law of motion applied to a fluid element. It has the form

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \nu \nabla^2 \boldsymbol{u} - \frac{1}{\rho} \nabla p + \boldsymbol{f}, \quad (1)$$

where $\nabla^2 \boldsymbol{u} = (\nabla^2 u_1, \nabla^2 u_2)$, $\nabla \boldsymbol{u}$ is tensor derivative of the velocity field $\boldsymbol{u} = (u_1, u_2)$, scalar function p is the pressure, scalar ν is viscosity and function $\boldsymbol{f} = (f_1, f_2)$ is the force that includes gravity and other body forces that act on mixture of fluids. Function ρ is the density of the fluid mixture and it is given by the diffusive conservation of mass

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = \sigma \nabla^2 \rho + \delta(t, x, y) \quad (2)$$

where σ is the diffusion coefficient and function δ is a source term, for example a spring of freshwater.

Finally, to close the system, we shall assume the incompressibility condition

$$\nabla \cdot \boldsymbol{u} = \delta(t, x, y). \quad (3)$$

Since we are dealing with nonlinear partial differential equations the only way to solve the system (1), (2), (3) in general is to try to construct a numerical solution. The most challenging task here is to solve equations (1) and (3) simultaneously. For this purpose we implemented implicit operator splitting method. Let us examine the very elegant idea that lies behind this method so we will describe it in some detail. If we, as usual, start with the natural time discretization in an implicit manner,

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^{n} - \Delta t \boldsymbol{u}^{n+1} \cdot \nabla \boldsymbol{u}^{n+1} - \frac{\Delta t}{\rho} \nabla p^n + \Delta t \nu \nabla^2 \boldsymbol{u}^{n+1} - \Delta t \boldsymbol{f}^{n+1}, \quad (4)$$

there is no guarantee that computed $\boldsymbol{u}^{n+1}$ satisfies incompressibility constraint (3). Instead, as in [8], we consider velocity obtained from (4) as a prediction of correct value, denoted by $\hat{\boldsymbol{u}}$ and use the incompressibility constraint to calculate correction $\boldsymbol{u_c}$ such that $\boldsymbol{u}^{n+1} = \hat{\boldsymbol{u}} + \boldsymbol{u}_c$. For more flexibility in control of the pressure information we multiply the pressure term ∇p^n by a factor α and now have

$$\hat{\boldsymbol{u}} = \boldsymbol{u}^n - \Delta t \hat{\boldsymbol{u}} \cdot \nabla \hat{\boldsymbol{u}} - \Delta t \frac{\alpha}{\rho} \nabla p^n + \nu \Delta t \nabla^2 \hat{\boldsymbol{u}} - \Delta t \boldsymbol{f}^{n+1}, \quad (5)$$

which is nonlinear. We employ a simple linearization strategy by using $\boldsymbol{u}^n \cdot \nabla \hat{\boldsymbol{u}}$ instead of $\hat{\boldsymbol{u}} \cdot \nabla \hat{\boldsymbol{u}}$ in further computation. The velocity $\boldsymbol{u}^{n+1}$ is supposed to fulfill

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^{n} - \Delta t \boldsymbol{u}^{n} \cdot \nabla \boldsymbol{u}^{n+1} - \frac{\Delta t}{\rho} \nabla p^{n+1} + \Delta t \nu \nabla^2 \boldsymbol{u}^{n+1} - \Delta t \boldsymbol{f}^{n+1}, \quad (6)$$

with the pressure being evaluated at time level n+1. After standard computation we now obtain the correction

$$\boldsymbol{u_c} = \boldsymbol{u}^{n+1} - \hat{\boldsymbol{u}} = \Delta t(-\boldsymbol{u}^n \cdot \nabla \boldsymbol{u}_c + \nu \nabla^2 \boldsymbol{u}_c) - \frac{\Delta t}{\rho} \nabla (p^{n+1} - \alpha p^n) = \psi - \frac{\Delta t}{\rho} \nabla P \quad (7)$$

where $\psi = \Delta t(-\boldsymbol{u}^n \cdot \nabla \boldsymbol{u}_c + \nu \nabla^2 \boldsymbol{u}_c)$ and $P = p^{n+1} - \alpha p^n$. Because ψ is small we will neglect it. Keeping in mind that $\nabla \cdot \boldsymbol{u}^{n+1} = \delta(t, x, y)$ we simply take the divergence of (7) to obtain

$$\nabla \cdot \hat{\boldsymbol{u}} = \frac{\Delta t}{\rho} \nabla^2 P + \delta(t, x, y) \quad (8)$$

In general, for given $\boldsymbol{u}_0, \rho_0$ and relaxation parameter α we iterate following two steps:

(a) Compute $\boldsymbol{u}^{n+1}$ and p^{n+1}:

(a1) Compute prediction $\hat{\boldsymbol{u}}$ and P:

Solve advection-diffusion equation

$$\hat{\boldsymbol{u}} + \Delta t\, \boldsymbol{u}^n \cdot \nabla \hat{\boldsymbol{u}} - \Delta t \frac{\alpha}{\rho^n} \nabla p^n + \Delta t \nu \nabla^2 \hat{\boldsymbol{u}} + \Delta t \boldsymbol{f}^{n+1} = \boldsymbol{u}^n.$$

Solve Poisson equation

$$\nabla \cdot \hat{\boldsymbol{u}} = \frac{\Delta t}{\rho} \nabla^2 P + \delta(t, x, y)$$

(a2) Compute p^{n+1} and $\boldsymbol{u}^{n+1}$ from

$$p^{n+1} = \alpha p^n + P,$$

$$\boldsymbol{u}^{n+1} = \hat{\boldsymbol{u}} - \frac{\Delta t}{\rho} \nabla P.$$

(b) Compute ρ^{n+1}:

Solve advection-diffusion equation

$$\rho^{n+1} = \rho^n - \Delta t\, \nabla \cdot (\rho^{n+1} \boldsymbol{u}^{n+1}) + \Delta t \sigma \Delta \rho^{n+1} + \Delta t \delta(t, x, y).$$

As for space discretization we used mixed-finite elements [11], [14].

3. SIMULATIONS / *Simulacije*

In this section we are considering the problem of mixing freshwater with seawater in rectangular-shaped basin.

First we simulate freshwater intrusion in basin full of seawater to model a simple submarine spring. After some short time we close the spring and observe the dynamics of the mixture.

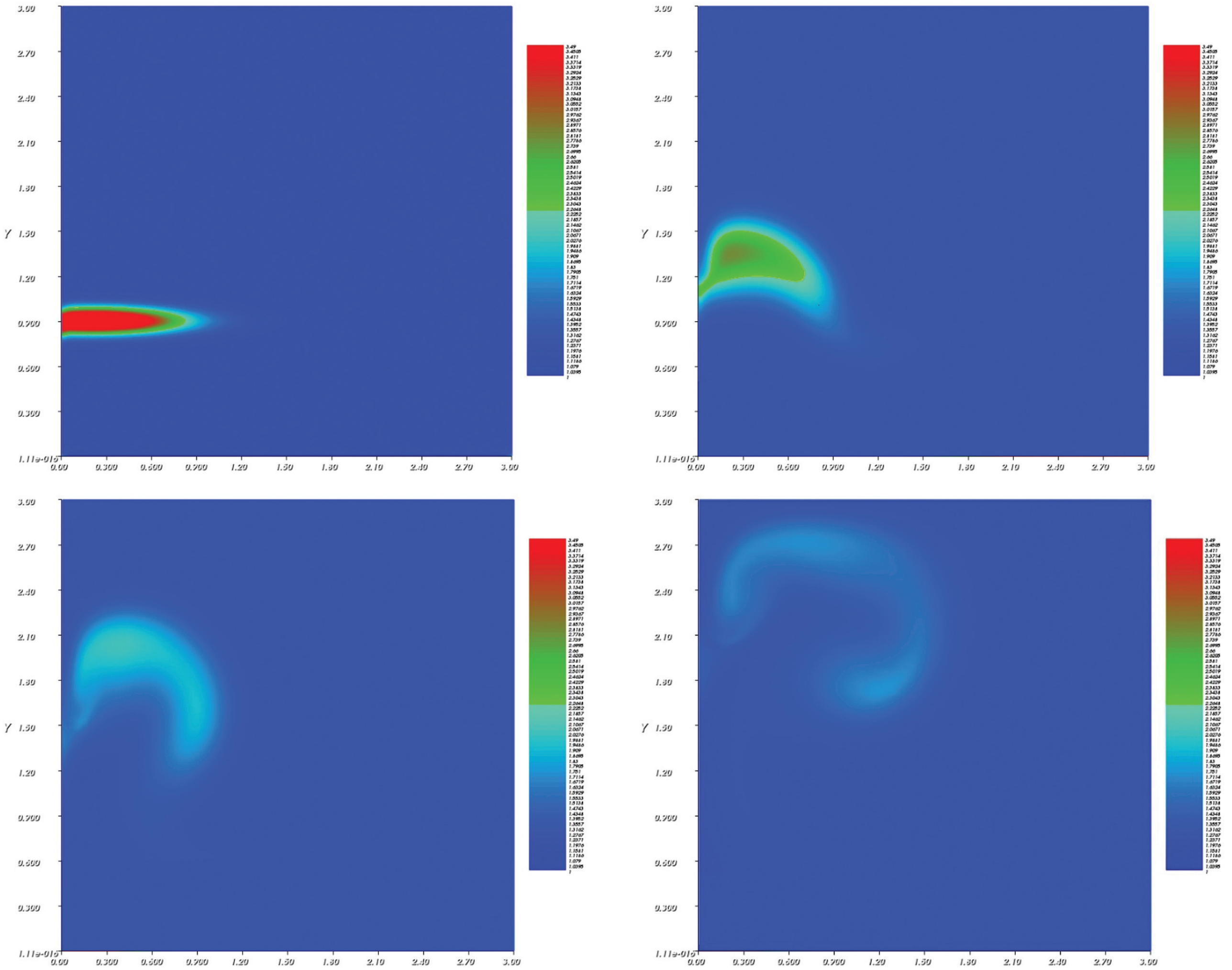

Figure 1 Dynamics of the mixture of seawater and freshwater caused by a submarine spring
Slika 1. Dinamika miješanja morske i svježe vode uzrokovana podmorskim izvorom

As we can see from Figure 1, on the left side we have spring of freshwater with smaller density than seawater so it is natural to expect that it will, while diffusing in seawater, tend to move towards surface of the basin.

Finally, we simulate process that occurs in the real-world estuary type which is in estuary classification based on water circulation known as *salt wedge*, described in [7] and with more mathematical detail in [5]. In this type of estuary, river output greatly exceeds marine input and tidal effects have a minor importance. Freshwater floats on top of the seawater in a layer that gradually thins as it moves seaward. The denser seawater moves landward along the bottom of the estuary, forming a wedge-shaped layer that is thinner as it approaches land. As a velocity difference develops between the two layers, shear forces generate internal waves at the interface, slightly mixing the seawater upward with the freshwater. Some examples of salt-wedge estuaries are the Hudson River in New York, the Columbia River in Washington and Oregon, and the Mississippi River in Louisiana.

We observe that seawater tends to penetrate into the river because of its greater density and while diffusing propagates mainly on the river bottom. Although some mixing occurs, boundary between the two water masses is cleary observable.

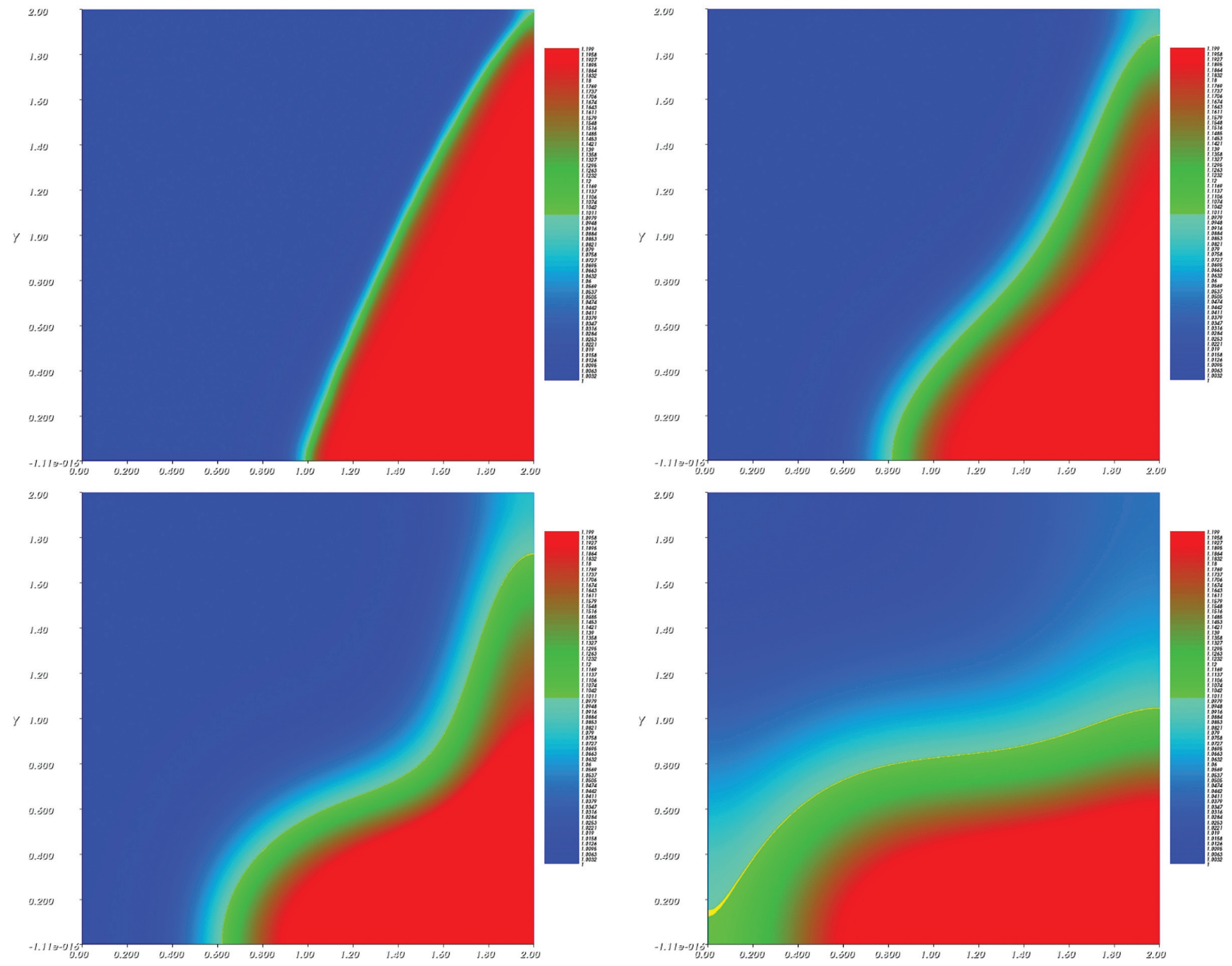

Figure 2 Dynamics of the mixture of seawater and freshwater in a salt wedge estuary
Slika 2. Dinamika mješavine morske i slatke vode u slanom klinastom estuariju

4. CONCLUSION / *Zaključak*

We have presented simple mathematical model together with two numerical experiments. We observe that submarine spring has no influence on water below because water freshwater of smaller density moves upwards and diffuses in seawater. Regarding our interest in estuary water dynamics, we see that the difference in density between freshwater and seawater causes the freshwater to float on top of the seawater, while seawater propagates on bottom of the estuary. Although presented model is simple, given the appropriate physical parameters, it can serve as a valuable base for the development of complex models that take into account more physical parameters, describe natural occurrences more precisely, and help in avoiding a lot of expensive field hours.

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