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ON RARELY g-CONTINUOUS FUNCTIONS

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Abstract. Popa introduced the notion of rare continuity. In this paper, we introduce a new class of functions called rarely g-continuous functions and investigate some of its fundamental properties. This type of continuity is a generalization of both rare continuity and weak continuity.

1. Introduction

Popa [11] introduced the notion of rare continuity as a generalization of weak continuity [8] which has been further investigated by Long and Herrington [10] and Jafari [6] and [7]. Levine [9] introduced the concept of generalized closed sets of a topological space and a class of topological spaces called $T_{1/2}$-spaces. Dunham [4], Dunham and Levine [5] and Caldas [2] further studied some properties of generalized closed sets and $T_{1/2}$-spaces.

The purpose of the present paper is to introduce the concept of rare g-continuity in topological spaces as a generalization of rare continuity and weak continuity. We investigate several properties of rarely g-continuous functions. The notion of $I.g$-continuity is also introduced which is weaker than g-continuity and stronger than rare g-continuity. It is shown that when the codomain of a function is regular, then the notions of rare g-continuity and $I.g$-continuity are equivalent.

2. Preliminaries

Throughout this paper, X and Y are topological spaces. Recall that a rare set is a set R such that $\mathrm{Int}(R) = \emptyset$. A nowhere dense set, is a set R which $\mathrm{Int}(\mathrm{Cl}(R)) = \emptyset$ if $\mathrm{Cl}(R)$ is codense. Levine [9] introduces the notion of g-closed sets: A set A in X is called g-closed if $\mathrm{Cl}(A) \subset G$ whenever $A \subset G$ and G is

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open in X. The complement of a g-closed set is called g-open [9]. The family of all g-open (resp. open) sets will be denoted by $GO(X)$ (resp. $O(X)$). We set $GO(X,x) = \{U \mid x \in U \in GO(X)\}$ and $O(X,x) = \{U \mid x \in U \in O(X)\}$.

Definition 2.1. *A function $f : X \to Y$ is called:*

i) *Weakly continuous* [8] *(resp. weakly-g-continuous* [3]*) if for each $x \in X$ and each open set G containing $f(x)$, there exists $U \in O(X,x)$ (resp. $U \in GO(X,x)$) such that $f(U) \subset \mathrm{Cl}(G)$.*

ii) *Rarely continuous* [11] *if for each $x \in X$ and each $G \in O(Y, f(x))$, there exist a rare set R_G with $G \cap \mathrm{Cl}(R_G) = \emptyset$ and $U \in O(X,x)$ such that $f(U) \subset G \cup R_G$.*

iii) *g-continuous* [1] *if the inverse image of every closed set in Y is g-closed in X.*

3. Rare g-continuity

Definition 3.1. *A function $f : X \to Y$ is called rarely g-continuous if for each $x \in X$ and each $G \in O(Y, f(x))$, there exist a rare set R_G with $G \cap \mathrm{Cl}(R_G) = \emptyset$ and $U \in GO(X,x)$ such that $f(U) \subset G \cup R_G$.*

Example 3.2. Let X and Y be the real line with indiscrete and discrete topologies respectively. The identity function is rarely g-continuous.

Note that, every weakly continuous function is rarely continuous and every rarely continuous function is rarely g-continuous.

Question 1. Is there any example showing that a function is rarely g-continuous but not rarely continuous?

Theorem 3.3. *The following statements are equivalent for a function $f : X \to Y$:*

(1) *The function f is rarely g-continuous at $x \in X$.*

(2) *For each set $G \in O(Y, f(x))$, there exists $U \in GO(X,x)$ such that $\mathrm{Int}[f(U) \cap (Y \setminus G)] = \emptyset$.*

(3) *For each set $G \in O(Y, f(x))$, there exists $U \in GO(X,x)$ such that $\mathrm{Int}[f(U)] \subset \mathrm{Cl}(G)$.*

Proof. (1) $\Rightarrow$ (2): Let $G \in O(Y, f(x))$. By $f(x) \in G \subset \mathrm{Int}(\mathrm{Cl}(G))$ and the fact that $\mathrm{Int}(\mathrm{Cl}(G)) \in O(Y, f(x))$, there exist a rare set R_G with $\mathrm{Int}(\mathrm{Cl}(G)) \cap \mathrm{Cl}(R_G) = \emptyset$ and a g-open set $U \subset X$ containing x such that $f(U) \subset \mathrm{Int}(\mathrm{Cl}(G)) \cup R_G$. We have $\mathrm{Int}[f(U) \cap (Y - G)] = \mathrm{Int}[f(U)] \cap \mathrm{Int}(Y - G) \subset \mathrm{Int}[\mathrm{Cl}(G) \cup R_G] \cap (Y - \mathrm{Cl}(G)) \subset (\mathrm{Cl}(G) \cup \mathrm{Int}(R_G)) \cap (Y - \mathrm{Cl}(G)) = \emptyset$.

(2) $\Rightarrow$ (3): It is straightforward.

(3) $\Rightarrow$ (1): Let $G \in O(Y, f(x))$. Then by (3), there exists $U \in GO(X,x)$ such that $\mathrm{Int}[f(U)] \subset \mathrm{Cl}(G)$. We have $f(U) = [f(U) - \mathrm{Int}(f(U))] \cup \mathrm{Int}(f(U)) \subset [f(U) - \mathrm{Int}(f(U))] \cup \mathrm{Cl}(G) = [f(U) - \mathrm{Int}(f(U))] \cup G \cup (\mathrm{Cl}(G) - G) = [f(U) - \mathrm{Int}(f(U))] \cap (Y - G) \cup G \cup (\mathrm{Cl}(G) - G)$.

Set $R^* = [f(U) - \mathrm{Int}(f(U))] \cap (Y - G)$ and $R^{**} = (\mathrm{Cl}(G) - G)$. Then R^* and R^{**} are rare sets. More $R_G = R^* \cup R^{**}$ is a rare set such that $\mathrm{Cl}(R_G) \cap G = \emptyset$ and $f(U) \subset G \cup R_G$. This shows that f is rarely-g-continuous. ∎

We define the following notion which is a new generalization of g-continuity.

DEFINITION 3.4. *A function $f : X \to Y$ is $I.g$-continuous at $x \in X$ if for each set $G \in O(Y, f(x))$, there exists $U \in GO(X, x)$ such that* $\mathrm{Int}[f(U)] \subset G$.

If f has this property at each point $x \in X$, then we say that f is $I.g$-continuous on X.

EXAMPLE 3.5. Let $X = Y = \{a, b, c\}$ and $\tau = \sigma = \{X, \emptyset, \{a\}\}$. Then a function $f : X \to Y$ defined by $f(a) = f(b) = a$ and $f(c) = c$ is $I.g$-continuous.

Question 2. Are there examples showing that a function is $I.g$-continuous but not g-continuous and a function is rarely g-continuous but not $I.g$-continuous?

REMARK 3.6. Since, if $f : X \to Y$ is g-continuous, then for each point $x \in X$ and each open set V containing $f(x)$, there exists $U \in GO(X, x)$ such that $f(U) \subset V$ ([1], Proposition 2). Then, it should be noted that $I.g$-continuity is weaker than g-continuity and stronger than rare g-continuity.

THEOREM 3.7. *Let Y be a regular space. Then the function $f : X \to Y$ is $I.g$-continuous on X if and only if f is rarely g-continuous on X.*

PROOF. We prove only the sufficient condition since the necessity condition is evident (Remark 3.6).

Let f be rarely g-continuous on X and $x \in X$. Suppose that $f(x) \in G$, where G is an open set in Y. By the regularity of Y, there exists an open set $G_1 \in O(Y, f(x))$ such that $\mathrm{Cl}(G_1) \subset G$. Since f is rarely g-continuous, then there exists $U \in GO(X, x)$ such that $\mathrm{Int}[f(U)] \subset \mathrm{Cl}(G_1)$ (Theorem 3.3). This implies that $\mathrm{Int}[f(U)] \subset G$ and therefore f is $I.g$-continuous on X. ∎

We say that a function $f : X \to Y$ is $r.g$-open if the image of a g-open set is open.

THEOREM 3.8. *If $f : X \to Y$ is an $r.g$-open rarely g-continuous function, then f is weakly g-continuous.*

PROOF. Suppose that $x \in X$ and $G \in O(Y, f(x))$. Since f is rarely g-continuous, there exist a rare set R_G with $\mathrm{Cl}(R_G) \cap U = \emptyset$ and $U \in GO(X, x)$ such that $f(U) \subset G \cup R_G$. This means that $(f(U) \cap (Y \setminus \mathrm{Cl}(G)) \subset R_G$. Since the function f is $r.g$-open, then $f(U) \cap (Y \setminus \mathrm{Cl}(G))$ is open. But the rare set R_G has no interior points. Then $f(U) \cap (Y \setminus \mathrm{Cl}(G)) = \emptyset$. This implies that $f(U) \subset \mathrm{Cl}(G)$ and thus f is weakly g-continuous. ∎

THEOREM 3.9. *If $f : X \to Y$ is rarely g-continuous function, then the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for every x in X is rarely g-continuous.*

PROOF. Suppose that $x \in X$ and W is any open set containing $g(x)$. It follows that there exist open sets U and V in X and Y, respectively, such that $(x, f(x)) \in U \times V \subset W$. Since f is rarely g-continuous, there exists $G \in GO(X, x)$ such that $\text{Int}[f(G)] \subset \text{Cl}(V)$. Let $E = U \cap G$. It follows that $E \in GO(X, x)$ and we have $\text{Int}[g(E)] \subset \text{Int}(U \times f(G)) \subset U \times \text{Cl}(V) \subset \text{Cl}(W)$. Therefore, g is rarely g-continuous. □

DEFINITION 3.10. *Let $A = \{G_i\}$ be a class of subsets of X. By rarely union sets* [6] *of A we mean $\{G_i \cup R_{G_i}\}$, where each R_{G_i} is a rare set such that each of $\{G_i \cap \text{Cl}(R_{G_i})\}$ is empty.*

Recall that, a subset B of X is said to be rarely almost compact relative to X [6] if every open cover of B by open sets of X, there exists a finite subfamily whose rarely union sets cover B.

A topological space X is said to be rarely almost compact [6] if the set X is rarely almost compact relative to X.

A topological space X is called GO-compact [1] if every cover of X by g-open sets has a finite subcover.

Question 3. Characterize the notion of rarely almost compactness. Find example/s to show the relation of compactness and rarely almost compactness.

THEOREM 3.11. *Let $f : X \to Y$ be rarely g-continuous and K a GO-compact set relative to X. Then $f(K)$ is rarely almost compact subset relative to Y.*

PROOF. Suppose that Ω is an open cover of $f(K)$. Let B be the set of all V in Ω such that $V \cap f(K) \neq \emptyset$. Then B is a open cover of $f(K)$. Hence for each $k \in K$, there is some $V_k \in B$ such that $f(k) \in V_k$. Since f is rarely g-continuous there exist a rare set R_{V_k} with $V_k \cap \text{Cl}(R_{V_k}) = \emptyset$ and a g-open set U_k containing k such that $f(U_k) \subset V_k \cup R_{V_k}$. Hence there is a finite subfamily $\{U_k\}_{k \in \triangle}$ which covers K, where $\triangle$ is a finite subset of K. The subfamily $\{V_k \cup R_{V_k}\}_{k \in \triangle}$ also covers $f(K)$. □

Recall that a space X is called $T_{1/2}$-space [4] if every g-closed set in X is closed in X.

THEOREM 3.12. *Let $f : X \to Y$ be rarely g-continuous and X a $T_{1/2}$-space. Then f is rarely continuous.*

A space X is called a door space if every subset of X is either open or closed.

W. Dunham [[4] Corollary 3.7] proved the following result:

LEMMA 3.13. *A door space is a $T_{1/2}$-space.*

THEOREM 3.14. *Let $f : X \to Y$ be a rarely g-continuous and X be a door space. Then f is rarely continuous.*

PROOF. It is an immediate consequence of Lemma 3.13 and Theorem 3.12. □

LEMMA 3.15 (Long and Herrington [10]). *If $g : Y \to Z$ is continuous and one-to-one, then g preserves rare sets.*

THEOREM 3.16. *If $f : X \to Y$ is rarely g-continuous and $g : Y \to Z$ is continuous and one-to-one, then $g \circ f : X \to Z$ is rarely g-continuous.*

PROOF. Suppose that $x \in X$ and $(g \circ f)(x) \in V$, where V is an open set in Z. By hypothesis, g is continuous, therefore there exists an open set $G \subset Y$ containing $f(x)$ such that $g(G) \subset V$. Since f is rarely g-continuous, there exist a rare set R_G with $G \cap \mathrm{Cl}(R_G) = \emptyset$ and a g-open set U containing x such that $f(U) \subset G \cup R_G$. It follows from Lemma 3.15 that $g(R_G)$ is a rare set in Z. Since R_G is a subset of $Y \setminus G$ and g is injective, we have $\mathrm{Cl}(g(R_G)) \cap V = \emptyset$. This implies that $(g \circ f)(U) \subset V \cup g(R_G)$. Hence the result. □

Recall that a function $f : X \to Y$ is called pre-g-open if $f(U)$ is g-open in Y for every g-open set U of X.

THEOREM 3.17. *Let $f : X \to Y$ be a pre-g-open surjection and $g : Y \to Z$ a function such that $g \circ f : X \to Z$ is rarely g-continuous. Then g is rarely g-continuous.*

PROOF. Let $y \in Y$ and $x \in X$ such that $f(x) = y$. Let $G \in O(Z, (g \circ f)(x))$. Since $g \circ f$ is rarely g-continuous, there exist a rare set R_G with $G \cap \mathrm{Cl}(R_G) = \emptyset$ and $U \in GO(X, x)$ such that $(g \circ f)(U) \subset G \cup R_G$. But $f(U)$ (say V) is a g-open set containing $f(x)$. Therefore, there exist a rare set R_G with $G \cap \mathrm{Cl}(R_G) = \emptyset$ and $V \in GO(Y, y)$ such that $g(V) \subset G \cup R_G$, i.e., g is rarely g-continuous. □

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