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MIX-DECOMPOSITON OF THE COMPLETE GRAPH INTO DIRECTED FACTORS OF DIAMETER 2 AND UNDIRECTED FACTORS OF DIAMETER 3

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Abstract. We estimate the values, for each k, of the smallest n such that K_n can be mix-decomposed into k undirected factors of diameter 3 and one directed factor of diameter 2. We find the asymptotic value of ratio of n and k, when k tends to infinity and generalize this result for mix-decompositions into p directed factors of diameter 2 and k undirected factors of diameter 3.

1. Introduction

In this paper, we solve the following problem. There is a system of n communication devices and there are $k+1$ communication networks. There are k communication networks that allow two-way communication (the link of this network connecting two devices allows each of them to receive and send information) and the remaining communication network allows only one-way communication (the link of this network connecting two devices allows one of them only to send information and the other only to receive them). The following requirements are given:

1) Every pair of different devices can be connected only by a single link.
2) For every pair of different devices and for every two-way communication network, there has to be a path of at most 3 links through which they can communicate.
3) For every pair of different devices a and b there has to be a path of at most 2 links of the one-way communication network through which a can send information to b and a path of at most 2 links of the one-way communication network through which b can send information to a.

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Of course, it is not always possible to meet these requirements. Take for instance the case $n = 3$ and $k = 1$. In this paper, we want to estimate, for each k, the smallest n such that this is possible

This problem is closely connected to well-known problems of decompositions of graphs. Decompositions of graphs into factors with given diameters have been extensively studied. The problem of decomposition of the factors of equal diameters, where diameter of each factor is at least three has been solved in [4]. The majority of the papers written about decomposition of the graphs are written about decompositions into the factors of diameter two. Denote by $f(k)$ the smallest natural number so that complete graph with n vertices can be decomposed into k factors of diameter 2. In [5], it was proved that

$$f(k) \leq 7k.$$

Then in [3] this was improved to

$$f(k) \leq 6k.$$

In [7], it was further proved that this upper bound is quite close to the exact value of $f(k)$ since

$$f(k) \geq 6k - 7, \; k \geq 664$$

and in [8] the correct value of $f(k)$ was given for large values of k, to be more specific it was proved that

$$f(k) = 6k, \quad k \geq 10^{17}.$$

Therefore, the most interesting problem of decompositions of graphs are decompositions into factors of small diameters.

The problem most closely related to ours, i.e. the problem of decompositions of complete graph into factors of diameter 2 and 3 was treated in [6]. We obtain few results similar to the results of that paper, but these constructions will be somewhat more complicated, since in this paper we have to deal with directed factors of diameter 2.

Also, the organization of the system of communication networks presented here is somewhat better than one given in [6], because in this system the privileged communication network is much faster; here links of privileged network allow only one-way communication, which is much faster then two-way communication.

2. Basic definitions

Let G be undirected (resp. directed graph). By $V(G)$, we denote set of its vertices, by $v(G)$ number of its vertices, by $E(G)$ set of its edges (resp. directed edges) and by $e(G)$ the number of its edges (resp. directed edges). By $d_G(x, y)$ we denote the length of the shortest path connecting x and y (resp. the length of the shortest path from x to y). We also denote $\operatorname{diam} G = \max \{d_G(x, y) : x, y \in V(G)\}$.

We shall slightly abuse the word subgraph (resp. supergraph) by saying that A is a subgraph of B (resp. B is a supergraph of A) if B contains a subgraph isomorphic to A.

For undirected graph G, we denote by $d_G(x)$ degree of vertex x, by $\delta(G)$ minimal degree of the graph and by $\Delta(G)$ maximal degree of the graph.

For a directed graph G we denote by $d^+(x)$ outdegree of vertex x and by $d^-(x)$ indegree of vertex x. By $\Delta^+(G)$, we denote maximal outdegree of graph G and by $\Delta^-(G)$ maximal indegree of graph G. By $\delta^+(G)$ we denote minimal outdegree of graph G and by $\delta^-(G)$ minimal indegree of graph G. We say that vertex x is n-accessible-from (resp. n-accessible-to) y if there is a path from y to x (resp. from x to y) of length at most n, and for any set of vertices A, we say that A is n-accessible-from (resp. n-accessible-to) x if each vertex in A is n-accessible-from (resp. n-accessible-to) x. If $(x,y) \in E(G)$, we say that y is out-neighbor of x and that x is in-neighbor of y.

Let D be a directed graph. Denote by $|D|$ a graph such that $V(|D|) = V(D)$ and

$$xy \in E(|D|) \Leftrightarrow ((x,y) \in E(D) \vee (y,x) \in E(D))$$

We also define

Definition 2.1. *Let G be undirected graph. We say that G is mix-decomposed into undirected factors $F_1, F_2, \dots, F_k$ and directed factors $D_1, D_2, \dots, D_p$ if*

$$V(F_1) = V(F_2) = \cdots = V(F_k) = V(D_1) = V(D_2) = \cdots = V(D_p) = V(G)$$

and for each edge $\{a,b\} \in E(G)$ exactly one of the following is true:

1) $\{a,b\} \in E(F_1)$
2) $\{a,b\} \in E(F_2)$

$\vdots$

k) $\{a,b\} \in E(F_k)$
k+1) $(a,b) \in E(D_1)$
k+2) $(a,b) \in E(D_2)$

$\vdots$

k+p) $(a,b) \in E(D_p)$
k+p+1) $(b,a) \in E(D_1)$
k+p+2) $(b,a) \in E(D_2)$

$\vdots$

k+2p) $(b,a) \in E(D_p)$

Definition 2.2. *Let G be an undirected graph. Mix-complement of G is any directed graph G' such that $V(G) = V(G')$ and that for each $a, b \in V(G')$ exactly one of the following statements is true:*

1) $\{a, b\} \in V(G)$
2) $(a, b) \in E(G')$
3) $(b, a) \in E(G')$.

Definition 2.3. *Let G be a directed graph. Mix-complement of G is any undirected graph G' such that $V(G) = V(G')$ and such that for each $a, b \in V(G')$ exactly one of the following statements is true:*

1) $\{a, b\} \in V(G')$
2) $(a, b) \in E(G)$
3) $(b, a) \in E(G)$.

We define the function $\phi : \mathbb{N} \to \mathbb{N}$ by $\phi(k) = n$ if and only if n is the smallest natural number such that K_n can be mix-decomposed into k undirected factors of diameter 3 and one directed factor of diameter 2. Therefore, we have to estimate the values of the function ϕ.

3. The value of $\phi(1)$

It can be easily, but tediously proved that (if the reader is interested in the details of this and other missing proofs, please send me a mail):

Lemma 3.1. *There is no mix-complement D of cycle with 7 vertices such that* $\operatorname{diam} D \leq 2$.

Now, we prove

Lemma 3.2. $\phi(1) > 7$.

Proof. Suppose to the contrary, that it is possible to mix-decompose K_n, $n \leq 7$, into undirected factor F of diameter 2 and directed factor D of diameter 3. It is easy to see that $\delta^+(D) \geqslant 2$ and $\delta^-(D) \geqslant 2$, because otherwise it would be $\delta(F) = 0$, which is impossible. If $n \leq 4$, then this is impossible. If $n = 5$, then $\Delta(F) = 0$. If $n = 6$, then $\Delta(F) \leq 1$, hence F is disconnected. If $n = 7$, then $\Delta(F) \leq 2$ and $\operatorname{diam} F \leq 3$, so F is a cycle, but this is in contradiction with the previous lemma. In all the cases we have obtained a contradiction, therefore $\phi(1) > 7$. □

Let us prove that $\phi(1) > 8$. Denote by $G_{8,1}$, $G_{8,2}$ and $G_{8,3}$ the graphs on Figures 1, 2 and 3.

By a simple analysis, we can prove:

Lemma 3.3. *Let G be a graph such that $v(G) = 8$ and* $\operatorname{diam} G \leq 3$, *$\Delta(G) \leq 3$ and $\delta(G) = 1$, then G is a supergraph of one of the graphs $G_{8,1}, G_{8,2}$ and $G_{8,3}$.*

Simple, but tedious analysis shows that:

Lemma 3.4. *There is no directed graph D such that* $\operatorname{diam} D \leq 2$ *and D is mix-complement of $G_{8,1}$.*

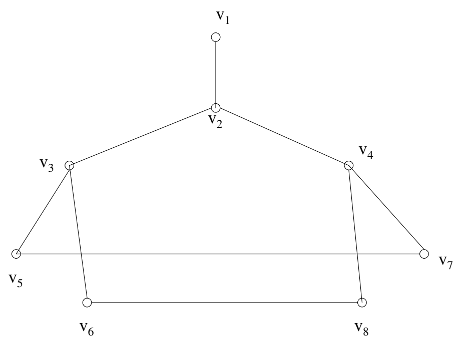

FIGURE 1

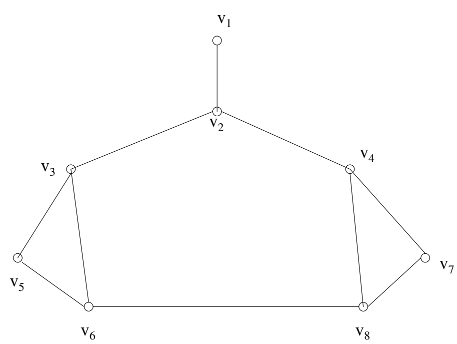

FIGURE 2

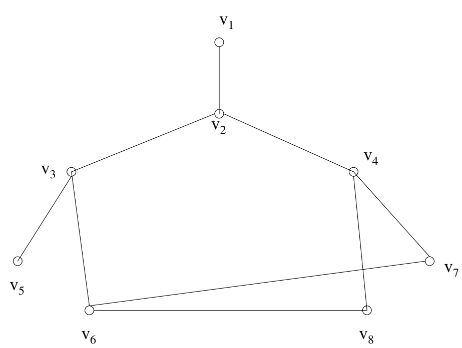

FIGURE 3

LEMMA 3.5. *There is no directed graph D such that* $\operatorname{diam} D \leq 2$ *and D is mix-complement of $G_{8,2}$.*

LEMMA 3.6. *There is no directed graph D such that* $\operatorname{diam} D \leq 2$ *and D is mix-complement of $G_{8,3}$.*

From the last four lemmas, it follows.

LEMMA 3.7. *Let K_8 be mix-decomposed into undirected factor F of diameter at most 3 and directed factor D of diameter at most 2; then $\delta(F) \geqslant 2$.*

Let us prove

LEMMA 3.8. *Let D be directed graph with 8 vertices of diameter 2 such that $\delta^+(D) \geqslant 2$, $\delta^-(D) \geqslant 2$ and sum of indegree and outdegree for at most 6 vertices is 5 and for the remaining vertices is 4, then exactly three vertices have indegree 3 and there is at most one edge between these 3 vertices, and also exactly three vertices have outdegree 3, and there is at most one edge between these vertices.*

PROOF. First let us prove that each vertex of outdegree 2 has at least one out-neighbor of outdegree 3. Suppose, to the contrary, that there is a vertex x of outdegree 2 such that its both out-neighbors have outdegree 2. In that case, there are only 7 vertices that are 2-accessible-from x and that is a contradiction.

Now let us prove that there are at least 3 vertices with outdegree 3. Suppose to the contrary, that there are at most 2 vertices with outdegree 3. Each of these vertices can be out-neighbor of at most two vertices. Since there are at least 6 vertices of outdegree 2, we have obtained a contradiction, hence there have to be at least 3 vertices of outdegree 3.

We can prove analogously that there are at least 3 vertices with indegree 3, hence there are exactly three vertices with in degree 3 and 3 vertices with outdegree 3.

At this point, we prove that there cannot be two edges between three vertices of outdegree three. Suppose to the contrary, that this is possible, but then only four vertices of outdegree two can have a neighbor of outdegree 3. Since, there are 5 vertices of outdegree 2, this is a contradiction.

Completely analogously, it can be proved that there is at most one edge between 3 vertices that have indegree 3. ∎

Denote by $G_{8,4}$, $G_{8,5}$ and $G_{8,6}$ respectively the following graphs

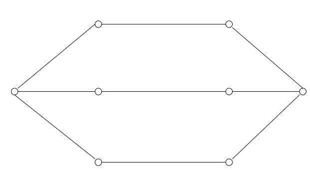

FIGURE 4

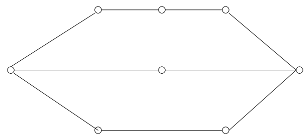

Figure 5

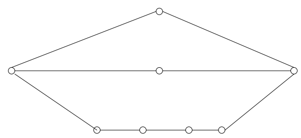

Figure 6

A simple analysis shows that:

Lemma 3.9. *Every graph G with 8 vertices such that* $\operatorname{diam} G \leq 3$, $\delta(G) = 2$, $\Delta(G) \leq 3$ *and at most two vertices have a degree larger than* 2, *is isomorphic to* $G_{8,4}$, $G_{8,5}$ *or* $G_{8,6}$.

Simple check shows that:

Lemma 3.10. *In complement of $G_{8,4}$ six vertices of degree 5 cannot be divided in two triples in such a way that each triple contains only one edge.*

Lemma 3.11. *In complement of $G_{8,5}$ six vertices of degree 5 cannot be divided in two triples in such a way that each triple contains only one edge.*

Lemma 3.12. *In complement of $G_{8,6}$ six vertices of degree 5 cannot be divided in two triples in such a way that each triple contains only one edge.*

Combining the last five lemmas we get:

Lemma 3.13. *K_8 can not be mix-decomposed into undirected factor F_1 of diameter 3 and directed factor F_2 of diameter 2, such that $\delta(F_2) \geqslant 2$.*

The last lemma and Lemma 3.7 yield the following proposition:

Proposition 3.14. $\phi(1) > 8$.

From the last Proposition and the following decomposition of K_9 (on this sketch we draw only the directed edges of directed factor D and all the missing edges are edges of undirected factor F): it follows that

Theorem 3.15. $\phi(1) = 9$.

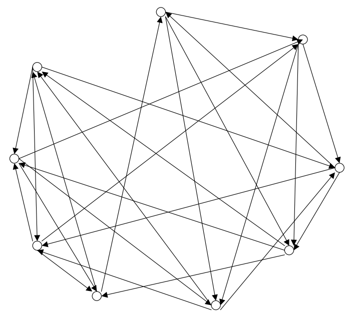

FIGURE 7

4. Estimates of $\phi(k)$ for small values of k

We start with several lemmas:

LEMMA 4.1. *If there is a directed graph D of diameter 2, such that $|D| \cong K_n$, then there is a directed graph D' such that* $\operatorname{diam} D' = 2$ *and* $|D'| \cong K_{n+2}$.

PROOF. Denote vertices of D by $v_1, v_2, \ldots, v_n$. We explicitly construct D'. Its vertices are $v_1, \ldots, v_{n+2}$ and its edges are:

1) (x, y) such that $(x, y) \in E(D)$
2) (x, v_i) such that $n+1 \leq i \leq n+2$, $(x, v_n) \in E(D)$
3) (v_i, x) such that $n+1 \leq i \leq n+2$, $(v_n, x) \in E(D)$
4) $(v_n, v_{n+1}), (v_{n+1}, v_{n+2}), (v_{n+2}, v_n)$.

□

LEMMA 4.2. *There is a directed graph $D(2,3)$ of diameter 2 such that $|D(2,3)| \cong K_3$, and there is directed graph $D(2,6)$ of diameter 2 such that $|D(2,3)| \cong K_6$.*

PROOF. Graphs with the required properties are given by the following sketches: □

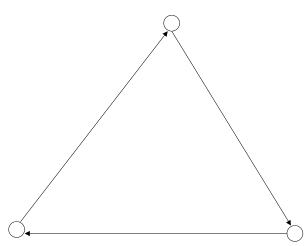

FIGURE 8

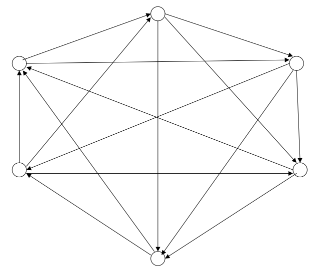

FIGURE 9

The last two Lemmas yield that there are graphs $D(2,n)$ of diameter 2 such that $|D(2,n)| \cong K_n$ for each $n \neq 1,2,4$.

Denote by $\mathcal{D}$ a mixed-decomposition of the K_{10} into undirected factor $F_{\mathcal{D}}$ and directed factor $D_{\mathcal{D}}$ given by the following sketch (on the following sketch only the edges of the factor $D_{\mathcal{D}}$ are drawn and the missing edges are edges of the factor $F_{\mathcal{D}}$):

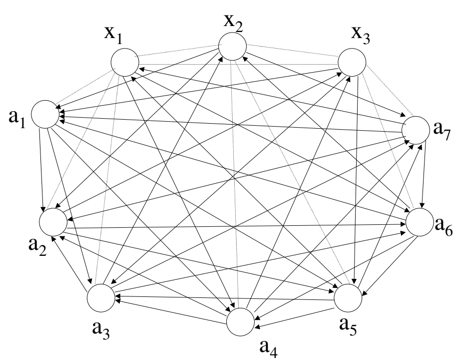

FIGURE 10

It can be easily seen that $\operatorname{diam} F_{\mathcal{D}} = 3$ and that $\operatorname{diam} D_{\mathcal{D}} = 2$. Let us prove that:

LEMMA 4.3. $\phi(k) \leq 3k+7,\ k \neq 1,2,4.$

PROOF. We explicitly give a mixed-decomposition of K_{3k+7} into undirected factors $F_1, \dots, F_k$ and directed factor D with a required properties. Denote

$$\begin{aligned} V(K_{3k+7}) &= A \cup \bigcup_{i=1}^{k} B_i \\ A &= \{a_1, a_2, \dots, a_6\},\ B_i = \{b_{i,1}, b_{i,2}, b_{i,3}\}. \end{aligned}$$

Edges of F_i, for each $1 \leq i \leq k$, are:

1) $b_{i,j}a_k$ such that $x_j a_k \in E(F_{\mathcal{D}})$
2) $b_{i,1}b_{l,2},\ b_{i,2}b_{l,3},\ b_{i,3}b_{l,1},\ 1 \leq l \leq k,\ i \neq l$
3) $b_{i,1}b_{i,2},\ b_{i,2}b_{i,3},\ b_{i,3}b_{i,1}$.

Since $b_{i,1}, b_{i,2}, b_{i,3}$ are all adjacent in F_i and each vertex is adjacent to at least one of them, it follows that $\operatorname{diam} F_i = 3$.

Denote by $g_i : \{b_{1,i}, b_{2,i}, \ldots, b_{k,i}\} \to V(D(2,k)),\ i = 1, \ldots, 3$ any bijections.

Directed edges of D are:

1) (a_i, a_j) such that $(a_i, a_j) \in V(D_{\mathcal{D}})$
2) $(b_{i,j}, a_l)$ such that $(x_j, a_l) \in V(D_{\mathcal{D}})$
3) $(a_l, b_{i,j})$ such that $(a_l, x_j) \in V(D_{\mathcal{D}})$
4) $(b_{i,j}, b_{l,j})$ such that $(g_j(b_{i,j}), g_j(b_{l,j})) \in V(D_{\mathcal{D}})$.

Let us prove that $\operatorname{diam} D \leq 2$. We need to show that $d(x,y) \leq 2$ for each $x, y \in V(D)$. Distinguish three cases:

1) $x, y \in A$ or $x \in A, y \in B_i,\ 1 \leq i \leq k$ or $x \in B_i, y \in A,\ 1 \leq i \leq k$ or $x, y \in B_i,\ 1 \leq i \leq k$.

   Note that there is a subgraph of D isomorphic to $D_{\mathcal{D}}$ that contains x and y, hence the claim follows.
2) $x = b_{i,j}, y = b_{l,m},\ i \neq l,\ j \neq m,\ 1 \leq i,j,l,m \leq k$.

   There is a vertex $a_o,\ 1 \leq o \leq 7$ such that (x_j, a_o),$(a_o, x_m) \in E(D_{\mathcal{D}})$. Therefore $(b_{i,j}, a_o), (a_o, b_{l,m}) \in E(D_{\mathcal{D}})$.
3) $x = b_{i,j}, y = b_{l,j}$.

   There is a path of length at most 2 that consists of directed edges listed in 4).

We have exhausted all the cases and we have proved our claim. ∎

LEMMA 4.4. $\phi(2) \leq 14$.

PROOF. We explicitly give a decomposition of K_{14} into undirected factors F_1 and F_2 of diameter 3 and directed factor D of diameter 2 by the following table T:

0	1	1	3	1	2	1	1	1	3	4	3	4	4
1	0	1	2	3	1	3	4	3	1	1	4	3	4
1	1	0	1	2	3	4	3	4	4	3	1	1	4
4	2	1	0	2	2	2	2	2	3	4	3	4	3
1	4	2	2	0	2	3	4	3	2	2	4	3	3
2	1	4	2	2	0	4	3	4	4	3	2	2	3
1	4	3	2	4	3	0	3	3	4	4	4	3	1
1	3	4	2	3	4	4	0	3	3	3	4	4	1
1	4	3	2	4	3	4	4	0	4	3	3	3	2
4	1	3	4	2	3	3	4	3	0	3	3	4	1
3	1	4	3	2	4	3	4	4	4	0	3	3	2

4 3 1 4 3 2 3 3 4 4 4 0 3 1
3 4 1 3 4 2 4 3 4 3 4 4 0 2
3 3 3 4 4 4 1 1 2 1 2 1 2 0

where $T_{ij} = k$ denotes $ij \in F_k$, $1 \le k \le 2$; $T_{ij} = 3$ denotes $(i,j) \in E(D)$ and $T_{ij} = 4$ denotes $(j,i) \in E(D)$. ∎

Lemma 4.5. $\phi(4) \le 20$.

Proof. We explicitly give a decomposition of K_{20} into undirected factors F_1, F_2, F_3 and F_4 of diameter 3 and directed factor D of diameter 2 by the following table T :

0 1 1 5 1 2 5 1 3 6 1 4 1 1 6 5 5 6 5 6
1 0 1 2 5 1 3 5 1 4 6 1 5 6 1 1 6 5 1 5
1 1 0 1 2 5 1 3 6 1 4 5 6 5 5 6 1 1 6 1
6 2 1 0 2 2 5 2 3 5 2 4 2 2 6 5 5 6 6 6
1 6 2 2 0 2 3 6 2 4 5 2 5 6 2 2 6 5 2 5
2 1 6 2 2 0 2 3 5 2 4 5 6 5 5 6 2 2 6 2
6 3 1 6 3 2 0 3 3 5 3 4 3 3 6 5 5 6 5 6
1 6 3 2 5 3 3 0 3 4 5 3 5 6 3 3 6 5 3 6
3 1 5 3 2 6 3 3 0 3 4 5 6 5 5 6 3 3 6 3
5 4 1 6 4 2 6 4 3 0 4 4 4 4 6 5 5 6 5 6
1 5 4 2 6 4 3 6 4 4 0 4 5 6 4 4 6 5 4 5
4 1 6 4 2 6 4 3 6 4 4 0 6 5 5 6 4 4 5 4
1 6 5 2 6 5 3 6 5 4 6 5 0 5 6 5 6 6 6 5
1 5 6 2 5 6 3 5 6 4 5 6 6 0 5 5 6 5 5 6
5 1 6 5 2 6 5 3 6 5 4 6 5 6 0 6 5 6 6 5
6 1 5 6 2 5 6 3 5 6 4 5 6 6 5 0 5 5 5 6
6 5 1 6 5 2 6 5 3 6 5 4 5 5 6 6 0 6 6 5
5 6 1 5 6 2 5 6 3 5 6 4 5 6 5 6 5 0 5 6
6 1 5 5 2 5 6 3 5 6 4 6 5 6 5 6 5 6 0 5
5 6 1 5 6 2 5 5 3 5 6 4 6 5 6 5 6 5 6 0

where $T_{ij} = k$ denotes $ij \in F_k$, $1 \le k \le 4$; $T_{ij} = 5$ denotes $(i,j) \in E(D)$ and $T_{ij} = 6$ denotes $(j,i) \in E(D)$. ∎

Lemma 4.6. *Let D be a directed graph with n vertices of diameter 2 such that $\delta^+(D) \geqslant 2$. Then D has at least $3n - 7$ edges.*

Proof. If $\delta^+(D) \geqslant 3$, the claim is trivial. If not, there is a vertex x with exactly 2 out-neighbours, say y and z. Since $\operatorname{diam} D \le 2$, it follows that $d^+(y) + d^+(z) \geqslant n - 3$. Therefore we have $\sum_{v \in V(D)} d^+(v) \geqslant 3n - 7$. ∎

The last Lemma yields

Lemma 4.7. $\phi(k) \geqslant \left\lceil \frac{2k+7+\sqrt{(2k+7)^2-56}}{2} \right\rceil$.

Proof. Suppose that K_n can be decomposed into k undirected factors each of diameter at most 3 and one directed factor of diameter 2. Each of undirected factors have at least n edges and directed graph by last Lemma has at least $3n-7$ directed edges. Therefore,

$$\begin{aligned} k \cdot n + 3n - 7 &\leq \frac{n \cdot (n-1)}{2} \\ n^2 - (2k+7)\, n + 14 &\geqslant 0, \end{aligned}$$

hence

$$n \geqslant \left\lceil \frac{2k+7+\sqrt{(2k+7)^2 - 56}}{2} \right\rceil .$$

□

So far we have shown that

Theorem 4.8.

$$\left. \begin{array}{cc} 9, & k=1 \\ \left\lceil \frac{2k+7+\sqrt{(2k+7)^2-56}}{2} \right\rceil, & k \geqslant 2 \end{array} \right\} \leq \phi(k) \leq \left\{ \begin{array}{cc} 3k+6 & k=1 \\ 3k+8 & k=2,4 \\ 3k+7 & k \neq 1,2,4 \end{array} \right. .$$

5. Estimates of $\phi(k)$ for large values of k

The upper bounds of the last theorem are quite good for small values of k, but they are bad for large values of k. Note that from the last theorem, it follows only

$$2 \leq \underline{\lim} \frac{\phi(k)}{k} \leq \overline{\lim} \frac{\phi(k)}{k} \leq 3.$$

We shall prove that

$$\lim_{k \to \infty} \frac{\phi(k)}{k} = 2.$$

This is a result analog to the result given in [6]. The techniques of proving this are similar to those of [6], but somewhat more complicated.

Lemma 5.1. *Let t, k and q be such natural numbers that*

$$\left[\left(2q + 2k + 4\left\lceil \sqrt{k} \right\rceil + 4t - 2\right) \cdot \left(2q + 2k + 4\left\lceil \sqrt{k} \right\rceil + 4t - 3\right) - 2k \cdot (2k-1) \right] \cdot \left(\frac{3}{4}\right)^q < 1$$

then there is a directed graph D' such that its vertices can be divided into five pairwise disjoint sets C_1, C_2, C_3, C_4 and C_5 in such a way that:

1) $|C_1| = q$
2) $|C_2| = q$
3) $|C_3| = k$

4) $|C_4| = k$
5) $|C_5| = 4\left\lceil\sqrt{k}\right\rceil + 4t - 2$
6) *For any two vertices* $x \in X$ *and* $y \in Y$ *either* $(x,y) \in E(D')$ *or* $(y,x) \in E(D')$, *where*

$$\{X,Y\} \in \{\{C_1,C_2\},\{C_1,C_4\},\{C_1,C_5\},\{C_2,C_3\},\{C_2,C_5\},\{C_1\},\{C_2\}\}$$

7) *For any two vertices* $x \in X$ *and* $y \in Y$ *neither* $(x,y) \in E(D')$ *nor* $(y,x) \in E(D')$, *where*

$$\{X,Y\} \in \{\{C_3\},\{C_4\},\{C_5\},\{C_1,C_3\},\{C_2,C_4\},\{C_3,C_4\},\{C_3,C_5\},\{C_4,C_5\}\}$$

8) *For each pair of vertices*

$$(x,y) \in (V(D') \times V(D')) \setminus ((C_3 \cup C_4) \times (C_3 \cup C_4)).$$

we have $d_{D'}(x,y) \leq 2$.

Proof. Let D'' be a random graph such that $V(D'') = C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5$ such that

1) For any two vertices $x \in X$ and $y \in Y$ either $(x,y) \in E(D'')$ or $(y,x) \in E(D'')$, where

$$\{X,Y\} \in \{\{C_1,C_2\},\{C_1,C_4\},\{C_1,C_5\},\{C_2,C_3\},\{C_2,C_5\},\{C_1\},\{C_2\}\}$$

and the probability of each direction is $\frac{1}{2}$.
2) For any two vertices $x \in X$ and $y \in Y$ neither $(x,y) \in E(D'')$ nor $(y,x) \in E(D'')$, where

$$\{X,Y\} \in \{\{C_3\},\{C_4\},\{C_5\},\{C_1,C_3\},\{C_2,C_4\},\{C_3,C_4\},\{C_3,C_5\},\{C_4,C_5\}\}$$

Denote the following condition by $(*)$:

For each pair of vertices

$$(x,y) \in (V(D'') \times V(D'')) \setminus ((C_3 \cup C_4) \times (C_3 \cup C_4)).$$

we have $d_{D''}(x,y) \leq 2$.

Let us calculate probability that D'' does not satisfy $(*)$. First we estimate the probability $\operatorname{prob}(x,y)$ that $d_{D''}(x,y) > 2$, where $x \neq y$ and $(x,y) \in (V(D') \times V(D')) \setminus ((C_3 \cup C_4) \times (C_3 \cup C_4))$.

The following 21 cases, described in the following table may occur:

$x \in$	$y \in$	$\operatorname{prob}(x,y)$
C_1	C_1	$\frac{1}{2}\cdot\left(\frac{3}{4}\right)^{2q+k+4\lceil\sqrt{k}\rceil+4t-4}$
C_1	C_2	$\frac{1}{2}\cdot\left(\frac{3}{4}\right)^{2q+4\lceil\sqrt{k}\rceil+4t-4}$
C_1	C_3	$\left(\frac{3}{4}\right)^{q}$
C_1	C_4	$\frac{1}{2}\cdot\left(\frac{3}{4}\right)^{q-1}$
C_1	C_5	$\frac{1}{2}\cdot\left(\frac{3}{4}\right)^{2q-1}$
C_2	C_1	$\frac{1}{2}\cdot\left(\frac{3}{4}\right)^{2q+4\lceil\sqrt{k}\rceil+4t-4}$
C_2	C_2	$\frac{1}{2}\cdot\left(\frac{3}{4}\right)^{2q+k+4\lceil\sqrt{k}\rceil+4t-4}$
C_2	C_3	$\frac{1}{2}\cdot\left(\frac{3}{4}\right)^{q-1}$
C_2	C_4	$\left(\frac{3}{4}\right)^{q}$
C_2	C_5	$\frac{1}{2}\cdot\left(\frac{3}{4}\right)^{2q-1}$
C_3	C_1	$\left(\frac{3}{4}\right)^{q}$
C_3	C_2	$\frac{1}{2}\cdot\left(\frac{3}{4}\right)^{q-1}$
C_3	C_5	$\left(\frac{3}{4}\right)^{q}$
C_4	C_1	$\frac{1}{2}\cdot\left(\frac{3}{4}\right)^{q-1}$
C_4	C_2	$\left(\frac{3}{4}\right)^{q}$
C_4	C_5	$\left(\frac{3}{4}\right)^{q}$
C_5	C_1	$\frac{1}{2}\cdot\left(\frac{3}{4}\right)^{2q-1}$
C_5	C_2	$\frac{1}{2}\cdot\left(\frac{3}{4}\right)^{2q-1}$
C_5	C_3	$\left(\frac{3}{4}\right)^{q}$
C_5	C_4	$\left(\frac{3}{4}\right)^{q}$
C_5	C_5	$\left(\frac{3}{4}\right)^{2q}$

Thus, $\operatorname{prob}(x,y) \leq \left(\frac{3}{4}\right)^q$ for any two different vertices x,y such that $(x,y) \in (V(D') \times V(D')) \setminus ((C_3 \cup C_4) \times (C_3 \cup C_4))$. Therefore, a probability that D'' does not satisfy $(*)$ is less then

$$\left[\left(2q+2k+4\left\lceil\sqrt{k}\right\rceil+4t-2\right)\cdot\left(2q+2k+4\left\lceil\sqrt{k}\right\rceil+4t-3\right) - 2k\cdot(2k-1)\right]\cdot\left(\frac{3}{4}\right)^q$$

Since the last expression is less than 1, there is a graph with the required properties. □

Now we can prove:

THEOREM 5.2. *For any* $k \in \mathbb{N}$, *we have* $\phi(k) \le 2k + 4\left\lceil \sqrt{k} \right\rceil + 4t - 2 + 2q$ *where* t *is the least natural number such that*

$$\binom{2t-1}{t-1} \ge k.$$

and q *is the least natural number such that*

$$\left[\left(2q + 2k + 4\left\lceil \sqrt{k} \right\rceil + 4t - 2\right) \cdot \left(2q + 2k + 4\left\lceil \sqrt{k} \right\rceil + 4t - 3\right) - 2k \cdot (2k-1) \right] \cdot \left(\frac{3}{4}\right)^q < 1.$$

PROOF. We will construct a decomposition of K_n, $n = 2k + 4\left\lceil \sqrt{k} \right\rceil + 4t - 2 + 2q$, into undirected factors $F_1, F_2, \ldots, F_k$ and directed factor F such that $\operatorname{diam} F = 2$ and $\operatorname{diam} F_i = 3$, $1 \le i \le k$. Let

$$V(K_n) = L \cup D \cup W \cup Z \cup U \cup U' \cup A \cup B \cup B' \cup C,$$

where

$$L = \{l_1, \ldots, l_k\}, \quad D = \{d_1, \ldots, d_k\}, \quad W = \left\{w_1, \ldots, w_{\lceil \sqrt{k} \rceil}\right\},$$
$$Z = \left\{z_1, \ldots, z_{\lceil \sqrt{k} \rceil}\right\}, \quad U = \left\{u_1, \ldots, u_{\lceil \sqrt{k} \rceil}\right\}, \quad U' = \left\{u'_1, \ldots, u'_{\lceil \sqrt{k} \rceil}\right\},$$
$$A = \{a_1, \ldots, a_q\}, \quad B = \{b_1, \ldots, b_{2t-1}\},$$
$$B' = \left\{b'_1, \ldots, b'_{2t-1}\right\}, \quad C = \{c_1, \ldots, c_q\}.$$

Let $\mathcal{B}$ be the set of all subsets of $t-1$ elements of the set $\{1, 2, \ldots, 2t-1\}$. Let f be any injection

$$f : \{1, \ldots, k\} \to \mathcal{B}.$$

Let us notice that for each $j \in \{1, \ldots, k\}$ there are unique numbers q_j and r_j so that

$$j = (q_j - 1) \cdot \left\lceil \sqrt{k} \right\rceil + r_j, \ 1 \le q_j \le \left\lceil \sqrt{k} \right\rceil, \ 1 \le r_j \le \left\lceil \sqrt{k} \right\rceil.$$

The edges of the factor $F_i, 1 \le i \le k$ are

1) $l_i d_i$
2) $l_i l_j$, $1 \le j < i$
3) $d_i l_j$, $i < j \le k$
4) $l_i d_j$, $i < j \le k$
5) $d_i d_j$, $1 \le j < i$
6) $l_i a_j$, $1 \le j \le q$
7) $d_i c_j$, $1 \le j \le q$
8) $l_i b_j, l_i b'_j$, $j \in f(i)$
9) $d_i b_j, d_i b'_j$, $j \in \{1, 2, \ldots, 2t-1\} \setminus f(i)$
10) $l_i w_j$, $1 \le j \le \left\lceil \sqrt{k} \right\rceil$

11) $d_i z_j,\ 1 \leq j \leq \left\lceil \sqrt{k} \right\rceil$
12) $w_{q_i} u_{r_i},\ z_{q_i} u_{r_i}, w_{q_i} u'_{r_i}, z_{q_i} u'_{r_i}$
13) $d_i u_j,\ d_i u'_j,\ 1 \leq j \leq \left\lceil \sqrt{k} \right\rceil,\ j \neq r_i.$

In each factor $F_i, 1 \leq i \leq k$ all vertices are adjacent to either l_i or d_i, except u_{r_i} and u'_{r_i} each of which is connected by a path of length 2 to both, l_i and d_i. Also, l_i and d_i are adjacent, and there is a path of length 2 which connects u_{r_i} and u'_{r_i}, hence we have $\operatorname{diam} F_i = 3,\ 1 \leq i \leq k$.

Let D' be a digraph with a properties required in the previous lemma and let

$$g : V(K_n) \rightarrow V(D')$$

be a bijection such that

$$\begin{aligned} g(A) &= C_1 \\ g(C) &= C_2 \\ g(L) &= C_3 \\ g(D) &= C_4 \\ g(B \cup B' \cup V \cup Z \cup U \cup U') &= C_5 \end{aligned}$$

Directed edges of F are:

1) (x, y) such that $(g(x), g(y)) \in E(D')$
2) $(l_i, u_{r_i}),\ (u_{r_i}, d_i),\ (d_i, u'_{r_i}),\ (u'_{r_i}, l_i)$
3) $(b_j, d_i),\ (d_i, b'_j),\ j \in f(i)$
4) $(l_i, b_j),\ (b'_j, l_i),\ j \in \{1, 2, \ldots, t-1\} \setminus f(i)$
5) Edges in

$$E(K_n) \setminus \left(\bigcup_{i=1}^{k} E(F_i) \cup \left\{ \begin{array}{c} xy : [(x,y) \text{ is edge of } D \text{ listed in 1)-5)}] \ \vee \\ [(y,x) \text{ is edge of } D \text{ listed in 1)-5)}] \end{array} \right\} \right)$$

with an arbitrary orientations.

It remains to prove that $\operatorname{diam} F \leq 2$, i.e. that for arbitrary x, y, we have $d(x, y) \leq 2$. Distinguish 5 cases:

1) $x = l_i,\ y = d_i,\ 1 \leq i \leq k.$
   There is a path $l_i u_{r_i} d_i$.
2) $x = d_i,\ y = l_i,\ 1 \leq i \leq k.$
   There is a path $d_i u'_{r_i} l_i$.
3) $x = l_i,\ y = d_j,\ i \neq j,\ 1 \leq i, j \leq k.$
   Since f is a bijection, there is $m \in f(j) \setminus f(i)$ and therefore, there is a path $l_i b_m d_j$.
4) $x = d_i,\ y = l_j,\ i \neq j,\ 1 \leq i, j \leq k.$
   Since f is a bijection, there is $m \in f(i) \setminus f(j)$ and therefore, there is a path $d_i b'_m l_j$.

5) $x \in V(K_n) \setminus (L \cup D)$ or $y \in V(K_n) \setminus (L \cup D)$.
There is a path of length at most 2 from x to y consisting of edges listed in 1).

All the cases are exhausted and the claim is proved. □

From the last theorem, it easily follows

COROLLARY 5.3. $\lim_{k\to\infty} \frac{\phi(k)}{k} = 2$.

PROOF. Let $k \in \mathbb{N}$ be sufficiently large. Let us find upper and lower bounds for $\phi(k)$. We have

$$k \cdot (\phi(k) - 1) \leq \binom{\phi(k)}{2} \Rightarrow k \leq \frac{\phi(k)}{2} \Rightarrow \phi(k) \geq 2k.$$

Let us notice that, for sufficiently large k, we have

$$\binom{2\left\lceil\sqrt{k}\right\rceil - 1}{\left\lceil\sqrt{k}\right\rceil - 1} \geq k,$$

thus $t \leq \left\lceil\sqrt{k}\right\rceil$. Also, for sufficiently large k, we have

$$\left[\left(2q + 2k + 8\left\lceil\sqrt{k}\right\rceil - 2\right) \cdot \left(2q + 2k + 8\left\lceil\sqrt{k}\right\rceil - 3\right) - 2k \cdot (2k-1)\right] \cdot \left(\frac{3}{4}\right)^{\left\lceil\sqrt{k}\right\rceil} < 1,$$

therefore $q \leq \left\lceil\sqrt{k}\right\rceil$. It follows that

$$2k \leq \phi(k) \leq 2k + 10\left(\sqrt{k} + 1\right) \Rightarrow 2 \leq \frac{\phi(k)}{k} \leq 2 + \frac{10}{\sqrt{k}} + \frac{10}{k}.$$

$$\Rightarrow \lim_{k\to\infty} 2 \leq \lim_{k\to\infty}\left(\frac{\phi(k)}{k}\right) \leq \lim_{k\to\infty}\left(2 + \frac{5}{\sqrt{k}} + \frac{5}{k}\right),$$

which proves the claim. □

Let us generalize the last corollary. By a simple probabilistic argument (similarly as in Lemma 5.1), we show that:

LEMMA 5.4. *Let p, r, r' and k be natural numbers such that*

$$p \cdot \left[(r' + 2r + 2k) \cdot (r' + 2r + 2k - 1) - 2k \cdot (2k-1)\right] \cdot \left(1 - \left(\frac{1}{2p}\right)^2\right)^r < 1.$$

There is a digraph $D_{r,r'}$ such that its vertices can be divided into five pairwise disjoint sets $C_{(r,r'),1}$, $C_{(r,r'),2}$, $C_{(r,r'),3}$, $C_{(r,r'),4}$, and $C_{(r,r'),5}$ in such way that:

1) $\left|C_{(r,r'),1}\right| = r$

2) $\left|C_{(r,r'),2}\right| = r$
3) $\left|C_{(r,r'),3}\right| = k$
4) $\left|C_{(r,r'),2}\right| = k$
5) $\left|C_{(r,r'),2}\right| = r'$
6) *For any two vertices* $x \in X$ *and* $y \in Y$ *either* $(x,y) \in E\left(D_{r,r'}\right)$ *or* $(y,x) \in E\left(D_{r,r'}\right)$*, where*

$$\{X,Y\} \in \left\{ \begin{array}{c} \left\{C_{(r,r'),1}, C_{(r,r'),2}\right\}, \left\{C_{(r,r'),1}, C_{(r,r'),4}\right\}, \left\{C_{(r,r'),1}, C_{(r,r'),5}\right\}, \\ \left\{C_{(r,r'),2}, C_{(r,r'),3}\right\}, \left\{C_{(r,r'),2}, C_{(r,r'),5}\right\}, \left\{C_{(r,r'),1}\right\}, \left\{C_{(r,r'),2}\right\} \end{array} \right\}$$

7) *For any two vertices* $x \in X$ *and* $y \in Y$ *neither* $(x,y) \in E\left(D_{r,r'}\right)$ *nor* $(y,x) \in E\left(D_{r,r'}\right)$*, where*

$$\{X,Y\} \in \left\{ \begin{array}{c} \left\{C_{(r,r'),1}, C_{(r,r'),3}\right\}, \left\{C_{(r,r'),2}, C_{(r,r'),4}\right\}, \left\{C_{(r,r'),3}, C_{(r,r'),4}\right\}, \\ \left\{C_{(r,r'),3}, C_{(r,r'),5}\right\}, \left\{C_{(r,r'),4}, C_{(r,r'),5}\right\}, \\ \left\{C_{(r,r'),3}\right\}, \left\{C_{(r,r'),4}\right\}, \left\{C_{(r,r'),5}\right\}, \end{array} \right\}$$

which can be decomposed into p factors $P_{(r,r'),1}, \ldots, P_{(r,r'),p}$ *such that*

8) *For each pair of vertices*

$$(x,y) \in \left(V\left(D_{r,r'}\right) \times V\left(D_{r,r'}\right)\right) \backslash \left(\left(C_{(r,r'),3} \cup C_{(r,r'),4}\right) \times \left(C_{(r,r'),3} \cup C_{(r,r'),4}\right)\right).$$

and each factor $P_{(r,r'),i}$*,* $1 \leq i \leq k$*, we have* $d_{P_{(r,r'),i}}(x,y) \leq 2$.

Proof. Let $D'_{r,r'}$ be a random digraph such that its vertices can be decomposed into five pairwise disjoint sets $C_{(r,r'),1}$, $C_{(r,r'),2}$, $C_{(r,r'),3}$, $C_{(r,r'),4}$, and $C_{(r,r'),5}$ in such way that:

1) For any two vertices $x \in X$ and $y \in Y$ either $(x,y) \in E\left(D'_{r,r'}\right)$ or $(y,x) \in E\left(D'_{r,r'}\right)$, where

$$\{X,Y\} \in \left\{ \begin{array}{c} \left\{C_{(r,r'),1}, C_{(r,r'),2}\right\}, \left\{C_{(r,r'),1}, C_{(r,r'),4}\right\}, \left\{C_{(r,r'),1}, C_{(r,r'),5}\right\}, \\ \left\{C_{(r,r'),2}, C_{(r,r'),3}\right\}, \left\{C_{(r,r'),2}, C_{(r,r'),5}\right\}, \left\{C_{(r,r'),1}\right\}, \left\{C_{(r,r'),2}\right\} \end{array} \right\}$$

and probability of each direction is $\frac{1}{2}$.

2) For any two vertices $x \in X$ and $y \in Y$ neither $(x,y) \in E\left(D'_{r,r'}\right)$ nor $(y,x) \in E\left(D'_{r,r'}\right)$, where

$$\{X,Y\} \in \left\{ \begin{array}{c} \left\{C_{(r,r'),1}, C_{(r,r'),3}\right\}, \left\{C_{(r,r'),2}, C_{(r,r'),4}\right\}, \left\{C_{(r,r'),3}, C_{(r,r'),4}\right\}, \\ \left\{C_{(r,r'),3}, C_{(r,r'),5}\right\}, \left\{C_{(r,r'),4}, C_{(r,r'),5}\right\}, \\ \left\{C_{(r,r'),3}\right\}, \left\{C_{(r,r'),4}\right\}, \left\{C_{(r,r'),5}\right\}, \end{array} \right\}$$

Let $P'_{(r,r'),1}, \ldots, P'_{(r,r'),p}$ be a random decomposition of random digraph (each directed edge of (x,y) has a probability $\frac{1}{p}$ to be a directed edge of $P'_{(r,r'),i}$, for every $i \in \{1,\ldots,p\}$). Denote by $(*)$ the following condition:

For each pair of vertices

$$(x,y) \in \left(V\left(D'_{r,r'}\right) \times V\left(D'_{r,r'}\right)\right) \backslash \left(\left(C_{(r,r'),3} \cup C_{(r,r'),4}\right) \times \left(C_{(r,r'),3} \cup C_{(r,r'),4}\right)\right).$$

and each factor$P_{(r,r'),i}$, $1 \leq i \leq k$, we have $d_{P_{(r,r'),i}}(x,y) \leq 2$.

Let us calculate the probability that $D'_{r,r'}$ does not satisfy $(*)$. First, we estimate the probability $\operatorname{prob}(x,y,i)$ that $d_{P'_{(r,r'),i}}(x,y) > 2$, where $(x,y) \in (V(D') \times V(D')) \setminus \left(\left(C_{(r,r'),3} \cup C_{(r,r'),4} \right) \times \left(C_{(r,r'),3} \cup C_{(r,r'),4} \right) \right)$ and $1 \le i \le p$. Analogously as in the Lemma 5.1, we solve this problem by observing 21 possible cases. We get

$$\operatorname{prob}(x,y,i) \le \left(1 - \left(\frac{1}{2p} \right)^2 \right)^r .$$

for each

$$(x,y) \in \left(V\left(D'_{r,r'} \right) \times V\left(D'_{r,r'} \right) \right) \setminus \left(\left(C_{(r,r'),3} \cup C_{(r,r'),4} \right) \times \left(C_{(r,r'),3} \cup C_{(r,r'),4} \right) \right) .$$

Therefore, a probability that $D^{'}_{r,r'}$ does not satisfy $(*)$ is at most

$$p \cdot \left[(r' + 2r + 2k) \cdot (r' + 2r + 2k - 1) - 2k \cdot (2k-1) \right] \cdot \left(1 - \left(\frac{1}{2p} \right)^2 \right)^r .$$

Since the last expression is less than 1, there is a graph with the required properties. □

We can now prove:

THEOREM 5.5. *Let p and k be any natural numbers. Then K_n can be mixed-decomposed into k undirected factors $F_1, F_2, \ldots, F_k$ of diameter 3 and p directed factors $D_1, D_2, \ldots, D_p$ of diameter 2 where*

$$n = 2k + 2r + 2p(2t-1) + (2p+2) \left\lceil \sqrt{k} \right\rceil ,$$

where t is the smallest integer such that

$$\binom{2t-1}{t-1} \geqslant k$$

and r is the smallest integer such that

$$p \cdot \left[(r' + 2r + 2k) \cdot (r' + 2r + 2k - 1) - 2k \cdot (2k-1) \right] \cdot \left(1 - \left(\frac{1}{2p} \right)^2 \right)^r < 1.$$

where

$$r' = 2p(2t-1) + (2p+2) \left\lceil \sqrt{k} \right\rceil .$$

PROOF. Denote

$$V(K_n) = L \cup R \cup A \cup B \cup \bigcup_{\alpha=1}^{p} C_\alpha \cup \bigcup_{\alpha=1}^{p} E_\alpha \cup X \cup Y \cup \bigcup_{\alpha=1}^{p} U_\alpha \cup \bigcup_{\alpha=1}^{p} V_\alpha ,$$

$$\begin{aligned} L &= \{l_1, \dots, l_k\}, \ R = \{r_1, \dots, r_k\}, \\ A &= \{a_1, \dots a_r\}, \ B = \{b_1, \dots, b_r\}, \\ X &= \left\{x_1, \dots, x_{\lceil \sqrt{k} \rceil}\right\}, Y = \left\{y_1, \dots, y_{\lceil \sqrt{k} \rceil}\right\}, \end{aligned}$$

and for, each $\alpha = 1, \dots, p$, denote

$$\begin{aligned} C_\alpha &= \left\{c_1^\alpha, \dots, c_{2t-1}^\alpha\right\}, \ E_\alpha = \left\{e_1^\alpha, \dots, e_{2t-1}^\alpha\right\}, \\ U_\alpha &= \left\{u_1^\alpha, \dots, u_{\lceil \sqrt{k} \rceil}^\alpha\right\}, \ V_\alpha = \left\{v_1^\alpha, \dots, v_{\lceil \sqrt{k} \rceil}^\alpha\right\}. \end{aligned}$$

Let us notice, as in the Theorem 5.2 that for each $j \in \{1, \dots, k\}$ there are unique numbers Q_j and R_j such that

$$j = (Q_j - 1) \cdot \left\lceil \sqrt{k} \right\rceil + R_j, \ 1 \le Q_j \le \left\lceil \sqrt{k} \right\rceil, \ 1 \le R_j \le \left\lceil \sqrt{k} \right\rceil.$$

Also, as in the Theorem 5.2, let $\mathcal{B}$ be the set of all subsets of $t-1$ elements of the set $\{1, 2, \dots, 2t-1\}$ and let f be any injection

$$f : \{1, \dots, k\} \to \mathcal{B}.$$

We explicitly give a mixed-decomposition with the required properties.

The edges of F_i, $i = 1, \dots, k$ are:

1) $l_i r_i$
2) $l_i l_j, \ r_i r_j, \ i < j \le k$
3) $l_i r_j, \ r_i l_j, \ 1 \le j < i$
4) $l_i a_j, \ j = 1, \dots, r$
5) $r_i b_j, \ j = 1, \dots, r$
6) $l_i c_j^\alpha, l_i e_j^\alpha, \ j \in f(i), \ \ \alpha = 1, \dots, p$
7) $r_i c_j^\alpha, r_i e_j^\alpha, \ j \in \{1, 2, \dots, 2t-1\} \setminus f(i), \ \ \alpha = 1, \dots, p$
8) $l_i x_j, r_i y_j, \ j = 1, \dots, \left\lceil \sqrt{k} \right\rceil$
9) $x_{Q_i} u_{R_i}^\alpha, \ y_{Q_i} u_{R_i}^\alpha, x_{Q_i} v_{R_i}^\alpha, y_{Q_i} v_{R_i}^\alpha, \ \alpha = 1, \dots, p$
10) $l_i u_j^\alpha, \ r_i v_j^\alpha, \ 1 \le j \le \left\lceil \sqrt{k} \right\rceil, \ j \ne R_i, \ 1 \le \alpha \le p$

It can be easily checked that $\operatorname{diam} F_i = 3$, for each $i = 1, \dots, k$.

Let

$$g : V(K_n) \to V(D_{r,r'})$$

be any bijection such that

$$\begin{aligned} g(A) &= C_{(r,r'),1} \\ g(B) &= C_{(r,r'),2} \\ g(L) &= C_{(r,r'),3} \\ g(R) &= C_{(r,r'),4} \\ g\left(X \cup Y \cup \bigcup_{\alpha=1}^{p} (C_\alpha \cup E_\alpha \cup U_\alpha \cup V_\alpha)\right) &= C_{(r,r'),5} \end{aligned}$$

Directed edges of D_α, $\alpha = 1, \dots, p$ are:

1) (x, y) such that $\left(g(x), g(x) \in P_{(r,r'),\alpha}\right)$
2) $\left(l_i, u^\alpha_{R_i}\right), \left(u^\alpha_{R_i}, r_i\right), \left(r_i, v^\alpha_{R_i}\right), \left(v^\alpha_{R_i}, l_i\right)$
3) $\left(l_i, c^\alpha_j\right), \left(e^\alpha_j, l_i\right), j \in \{1, 2, \dots, 2t-1\} \setminus f(i), i = 1, \dots, k$
4) $\left(c^\alpha_j, r_i\right), \left(r_i, e^\alpha_j\right), j \in f(i), i = 1, \dots, k.$

Edges in

$$E(K_n) \setminus \left(\begin{array}{c} \bigcup_{i=1}^{k} E(F_i) \cup \bigcup_{\alpha=2}^{p} E(|D_\alpha|) \cup \\ \cup \left\{ \begin{array}{c} xy : [(x, y) \text{ is edge of } D_1 \text{ listed in 1)-4)}] \vee \\ [(y, x) \text{ is edge of } D_1 \text{ listed in 1)-4)}] \end{array} \right\} \end{array} \right)$$

are directed edges of D_1 with an arbitrary orientations.

Let us prove that $\operatorname{diam} D_\alpha \leq 2$, $\alpha = 1, \dots, p$, i.e. that $d_{D_\alpha}(x, y) \leq 2$, for each $x, y \in V(K_n)$. Distinguish 5 cases:

1) $x = l_i$, $y = r_i$.
   There is a path $l_i u^\alpha_{R_i} r_i$.
2) $x = r_i$, $y = l_i$.
   There is a path $r_i v^\alpha_{R_i} l_i$.
3) $x = l_i,\ y = r_j,\ i \neq j$.
   Since f is a bijection, there is $m \in f(j) \setminus f(i)$ and therefore, there is a path $l_i c^\alpha_m r_j$.
4) $x = r_i,\ y = l_j,\ i \neq j$.
   Since f is a bijection, there is $m \in f(i) \setminus f(j)$ and therefore, there is a path $r_i e^\alpha_m l_j$.
5) $x \in V(K_n) \setminus (L \cup R)$ or $y \in V(K_n) \setminus (L \cup D)$.
   There is a path of length at most 2 from x to y consisting of edges listed in 1).

All the cases are exhausted and the claim is proved. □

Corollary 5.6. *Let p be a fixed natural number and let $\Phi(p, k)$ be the smallest natural number such that $K_{\Phi(p,k)}$ can be decomposed into p directed factors of diameter 2 and k undirected factors of diameter 3. Then*

$$\lim_{k \to \infty} \frac{\Phi(p, k)}{k} = 2.$$

Proof. From the last theorem, it follows that

$$\Phi(p, k) \leq 2k + 2r + 2p(2t - 1) + (2p + 2) \left\lceil \sqrt{k} \right\rceil,$$

where t is the smallest integer such that

$$\binom{2t-1}{t-1} \geqslant k$$

and r is the smallest integer such that

$$p \cdot [(r' + 2r + 2k) \cdot (r' + 2r + 2k - 1) - 2k \cdot (2k - 1)] \cdot \left(1 - \left(\frac{1}{2p}\right)^2\right)^r < 1.$$

where

$$r' = 2p\,(2t - 1) + (2p + 2)\left\lceil \sqrt{k} \right\rceil.$$

Note that, for sufficiently large k, we have $t \leq \left\lceil \sqrt{k} \right\rceil$ and $r \leq \left\lceil \sqrt{k} \right\rceil$, hence

$$\begin{aligned} \phi(k) &\leq \Phi(p,k) \leq 2k + (6p+4)\left\lceil \sqrt{k} \right\rceil \\ \lim_{k\to\infty} \frac{\phi(k)}{k} &\leq \lim_{k\to\infty} \frac{\Phi(p,k)}{k} \leq \lim_{k\to\infty} \left(2 + \frac{(6p+4)\left\lceil \sqrt{k} \right\rceil}{k}\right), \end{aligned}$$

which proves the claim. □

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