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THE DEMEYER-KANZAKI GALOIS EXTENSION AND ITS SKEW GROUP RING

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Abstract. Several characterizations are given for a ring B being a DeMeyer-Kanzaki Galois extension with Galois group G in terms of the skew group ring $B * G$. Consequently, the results of S. Ikehata on commutative Galois algebras are generalized.

1. Introduction

In [5], the class of commutative Galois algebras B with Galois group G was characterized in terms of the Azumaya skew group ring $B * G$ over B^G and the H-separable skew group ring $B * G$ of B respectively, where $B^G = \{a \in B \,|\, g(a) = a \text{ for all } g \in G\}$. In [3], a broader class of DeMeyer-Kanzaki Galois extensions B with Galois group G was investigated where B is called a DeMeyer-Kanzaki Galois extension with Galois group G if B is an Azumaya algebra over its center C and C is a Galois algebra with Galois group induced by and isomorphic with G. Further generalizations to Azumaya Galois extensions and to Hopf Azumaya Galois extensions were also given (see [2, 7]). The purpose of the present paper is to generalize the characterizations of a commutative Galois algebra B in terms of the skew group ring $B * G$ as given by S. Ikehata (see [5]). We shall show the following equivalent statements:

(1) B is a DeMeyer-Kanzaki Galois extension of B^G with Galois group G.

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(2) The skew group ring $B*G$ is an Azumaya C^G-algebra and C is a maximal commutative separable subalgebra of $V_{B*G}(B^G)$, the commutator subring of B^G in $B*G$, over C^G.
(3) The skew group ring $B*G$ is an H-separable extension of B (= the Harata separable), B is a separable algebra over C^G, and $J_g = \{0\}$ for each $g \neq 1$ in G where $J_g = \{b \in B \mid bx = g(x)b$ for all $x \in B\}$ for each $g \in G$.
(4) B is a separable C^G-algebra, C^G is a direct summand of C as a C^G-submodule, and $C \otimes_{C^G} (B*G) \cong M_n(B)$ where $M_n(B)$ is the matrix ring of order n over B and n is the order of G.
(5) B is a separable C^G-algebra, C^G is a direct summand of C as a C^G-submodule, and $C \otimes_{C^G} V_{B*G}(B^G) \cong M_n(C)$ where $M_n(C)$ is the matrix ring of order n over C and n is the order of G.

2. Basic definitions and notations

Throughout, B will represent a ring with 1, C the center of B, G a finite automorphism group of B of order n for some integer n, B^G the set of elements fixed under each element in G, and $J_g = \{b \in B \mid bx = g(x)b$ for all $x \in B\}$ for each $g \in G$. For a subring A of B with the same identity 1, we denote the commutator subring of A in B by $V_B(A)$. Following the definitions given in [10], we call B a separable extension of A if there exist $\{a_i, b_i$ in B, $i = 1, 2, \ldots, m$ for some integer $m\}$ such that $\sum a_i b_i = 1$, and $\sum ba_i \otimes b_i = \sum a_i \otimes b_i b$ for all b in B where $\otimes$ is over A. An Azumaya algebra is a separable extension of its center. A ring B is called an H-separable extension of A if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of B as a B-bimodule. B is called a Galois extension of B^G with Galois group G if there exist elements $\{a_i, b_i$ in B, $i = 1, 2, \ldots, m$ for some integer $m\}$ such that $\sum_{i=1}^m a_i g(b_i) = \delta_{1,g}$ for each $g \in G$. A Galois extension B with Galois group G is called an Azumaya Galois extension if B^G is an Azumaya algebra over C^G (see [2, 7]), and a DeMeyer-Kanzaki Galois extension if B is an Azumaya algebra over C which is a Galois algebra over C^G with Galois group induced by and isomorphic with G (see [3, 6]).

Let P be a finitely generated and projective module over a commutative ring R. Then for a prime ideal p of R, P_p (= $P \otimes_R R_p$) is a free module over R_p (= the local ring of R at p), and the rank of P_p over R_p is the number of copies of R_p in P_p, that is, $\text{rank}_{R_p}(P_p) = m$ for some integer m. It is known that the $\text{rank}_R(P)$ is a continuous function ($\text{rank}_R(P)(p) = m$) from $\text{Spec}(R)$ to the set of nonnegative integers with the discrete topology (see [4, Corollary 4.11, page 31]). We shall use the $\text{rank}_R(P)$-function for a finitely generated and projective module P over a commutative ring R.

3. Characterizations

In this section, keeping all notations as given in section 2, we shall generalize the characterizations of a commutative Galois algebra as given by S. Ikehata (see [5]) to a DeMeyer-Kanzaki Galois extension B with Galois group G in terms of the skew group ring $B * G$. We begin with an equivalent condition for a commutative Galois algebra C with Galois group G.

Theorem 3.1. *Let C be a commutative ring with a finite automorphism group G. Then, C is a commutative Galois algebra with Galois group G if and only if C^G is a direct summand of C as a C^G-submodule, and $C \otimes_{C^G} (C*G) \cong M_n(C)$.*

Proof. ($\Longrightarrow$) By Corollary 1.3 on page 85 in [4], C^G is a direct summand of C as a C^G-submodule, and that $C \otimes_{C^G} (C * G) \cong M_n(C)$ is a consequence of Theorem 2 in [5].

($\Longleftarrow$) Since $C \otimes_{C^G} (C * G) \cong M_n(C)$, $C \otimes_{C^G} (C * G)$ is an Azumaya algebra over C. But C^G is a direct summand of C as a C^G-submodule by hypothesis, so $C * G$ is an Azumaya C^G-algebra (see [4, Corollary 1.10, page 45]). Hence C is a commutative Galois algebra with Galois group G (see [5, Theorem 2]). □

Next we characterize a DeMeyer-Kanzaki Galois extension B in terms of the skew group ring $B * G$.

Theorem 3.2. *The following statements are equivalent:*

(1) *B is a DeMeyer-Kanzaki Galois extension of B^G with Galois group G.*
(2) *The skew group ring $B * G$ is an Azumaya C^G-algebra and C is a maximal commutative separable subalgebra of $V_{B*G}(B^G)$ over C^G.*
(3) *The skew group ring $B * G$ is an H-separable extension of B, B is a separable algebra over C^G, and $J_g = \{0\}$ for each $g \neq 1$ in G.*

Proof. (1) $\Longrightarrow$ (2) Since B is a DeMeyer-Kanzaki Galois extension of B^G with Galois group G, $B \cong B^G \otimes_{C^G} C$ such that B^G is an Azumaya C^G-algebra (see [3, Lemma 2]). Hence B is an Azumaya Galois extension with Galois group G; and so $B * G$ is an Azumaya C^G-algebra (see [2, Theorem 1]). Moreover, C is a commutative Galois algebra with Galois group G by hypothesis, so C is a maximal commutative separable subalgebra of $C * G$ over C^G (see [5, Theorem 2]). But $V_{B*G}(B^G) = V_B(B^G) * G = C * G$, so C is a maximal commutative separable subalgebra of $V_{B*G}(B^G)$ over C^G.

(2) $\Longrightarrow$ (1) Since $B * G$ is an Azumaya C^G-algebra, B is an Azumaya Galois extension with Galois group G (see [2, Theorem 1]). Hence $V_B(B^G)$ is a Galois algebra over C^G with Galois group G (see [1, Theorem 2]). Thus $V_B(B^G) * G \cong \mathrm{Hom}_{C^G}(V_B(B^G), V_B(B^G))$. But C is a maximal commutative separable subalgebra of $V_B(B^G) * G$ $(= V_{B*G}(B^G))$ over C^G by hypothesis,

so by the proof of Theorem 5.5 on page 64 in [4],

$$C \otimes_{C^G} (V_B(B^G) * G) \cong \mathrm{Hom}_C(V_B(B^G) * G, V_B(B^G) * G).$$

Then we have

$$\begin{aligned}\mathrm{Hom}_C(V_B(B^G) * G, V_B(B^G) * G) \cong \\ &\cong C \otimes_{C^G} (V_B(B^G) * G) \\ &\cong C \otimes_{C^G} \mathrm{Hom}_{C^G}(V_B(B^G), V_B(B^G)) \\ &\cong \mathrm{Hom}_C(C \otimes_{C^G} V_B(B^G), C \otimes_{C^G} V_B(B^G)).\end{aligned}$$

Thus $V_B(B^G) * G \cong (C \otimes_{C^G} V_B(B^G)) \otimes_C P$ as a C-module for some finitely generated and projective C-module P such that $\mathrm{rank}_C(P) = 1$. Since the rank of a Galois algebra is the order of the Galois group, applying the rank function on both sides of the above isomorphism, we have that

$$\begin{aligned}\mathrm{rank}_C(V_B(B^G)) \cdot n &= \mathrm{rank}_C(V_B(B^G) * G) = \mathrm{rank}_C(C \otimes_{C^G} V_B(B^G)) \\ &= \mathrm{rank}_{C^G}(V_B(B^G)) = n.\end{aligned}$$

This implies that $\mathrm{rank}_C(V_B(B^G)) = 1$. Noting that $V_B(B^G)$ is an Azumaya C-algebra and a finitely generated projective C^G-module, we conclude that $V_B(B^G) = C$; and so C is a Galois algebra over C^G with Galois group G. Consequently, B is a DeMeyer-Kanzaki Galois extension of B^G with Galois group G because B is also an Azumaya C-algebra.

(1) $\Longrightarrow$ (3) Since B is a DeMeyer-Kanzaki Galois extension of B^G with Galois group G, $B \cong B^G \otimes_{C^G} C$ such that B^G is an Azumaya C^G-algebra and C is a Galois algebra with Galois group induced by and isomorphic with G (see [3, Lemma 2]). Hence $B * G$ is an H-separable extension of B (see [9, Lemma 3.1 and Theorem 3.2]) and B is a separable algebra over C^G. Noting that $V_B(B^G) = C = J_1$ and that $V_B(B^G) = \oplus \sum_{g \in G} J_g$ (see [6, Proposition 1]), we conclude that $J_g = \{0\}$ for each $g \neq 1$ in G.

(3) $\Longrightarrow$ (1) Since B is a separable algebra over C^G, B is an Azumaya algebra over C. Next we claim that C is a Galois algebra with Galois group induced by and isomorphic with G. In fact, since $B * G$ is an H-separable extension of B by hypothesis and B is a direct summand of $B * G$ as a left (or right) B-module, $V_{B*G}(V_{B*G}(B)) = B$ (see [8, Proposition 1.2]). This implies that the center of $B * G$ is C^G. Moreover, B is a separable algebra over C^G, so $B * G$ is a separable algebra over C^G by the transitivity of separable extensions. Thus $B * G$ is an Azumaya C^G-algebra; and so B is an Azumaya Galois extension with Galois group G (see [2, Theorem 1]). Therefore $V_B(B^G)$ is a Galois algebra over C^G with Galois group induced by and isomorphic with G (see [1, Theorem 2]). But then, by Proposition 1 in [6], $V_B(B^G) = \oplus \sum_{g \in G} J_g$. Since $J_g = \{0\}$ for each $g \neq 1$ in G by hypothesis, so $V_B(B^G) = J_1 = C$. This proves that C is a Galois algebra with Galois group induced by and isomorphic with G. Thus statement (1) holds. □

By generalizing Theorem 3.1, we obtain another two characterizations of a DeMeyer-Kanzaki Galois extension.

THEOREM 3.3. *The following statements are equivalent:*

(1) *B is a DeMeyer-Kanzaki Galois extension of B^G with Galois group G.*
(2) *B is a separable C^G-algebra, C^G is a direct summand of C as a C^G-submodule, and $C \otimes_{C^G} (B * G) \cong M_n(B)$.*
(3) *B is a separable C^G-algebra, C^G is a direct summand of C as a C^G-submodule, and $C \otimes_{C^G} V_{B*G}(B^G) \cong M_n(C)$.*

PROOF. (1) $\Longrightarrow$ (2) Since B is a DeMeyer-Kanzaki Galois extension of B^G with Galois group G, $B \cong B^G \otimes_{C^G} C$ where B^G is an Azumaya C^G-algebra and C is a Galois algebra with Galois group induced by and isomorphic with G (see [3, Lemma 2]). Hence C^G is a direct summand of C as a C^G-submodule (see [4, Corollary 1.3, page 85]), and $V_{B*G}(B^G) = C * G$ such that $C \otimes_{C^G} (C * G) \cong M_n(C)$ (see [5, Theorem 2]); and so

$$\begin{aligned} C \otimes_{C^G} (B * G) &\cong C \otimes_{C^G} (B^G \otimes_{C^G} C * G) \cong C \otimes_{C^G} (C * G) \otimes_{C^G} B^G \\ &\cong M_n(C) \otimes_{C^G} B^G \cong M_n(B). \end{aligned}$$

(2) $\Longrightarrow$ (1) Since B is a separable C^G-algebra, B is an Azumaya algebra over C. Moreover, $M_n(B) \cong B \otimes_C M_n(C)$, so $M_n(B)$ is an Azumaya C-algebra. By hypothesis, $C \otimes_{C^G} (B * G) \cong M_n(B)$, so $C \otimes_{C^G} (B * G)$ is an Azumaya algebra over C. But C contains C^G as a direct summand as a C^G-submodule by hypothesis, so $B * G$ is an Azumaya C^G-algebra (see [4, Corollary 1.10, page 45]). Hence B is an Azumaya Galois extension with Galois group G (see [2, Theorem 1]). Thus $V_B(B^G)$ is a Galois algebra over C^G with Galois group G (see [1, Theorem 2]). Therefore both B and $B^G \cdot V_B(B^G)$ are Galois extensions of B^G with Galois group G such that $B^G \cdot V_B(B^G) \subset B$. This implies that $B = B^G \cdot V_B(B^G)$ such that $V_B(B^G)$ is a Galois algebra over C^G with Galois group G; and so $V_B(B^G)$ an Azumaya C-algebra and both $V_B(B^G)$ and C are finitely generated projective modules over C^G.

Next we claim that $V_B(B^G) = C$. In fact, since $C \otimes_{C^G} (B * G) \cong M_n(B)$, $\mathrm{rank}_{C^G}(B*G) = \mathrm{rank}_C(M_n(B))$. This implies that $\mathrm{rank}_{C^G}(C) \cdot \mathrm{rank}_C(B) \cdot n = \mathrm{rank}_C(B) \cdot n^2$. Thus $\mathrm{rank}_{C^G}(C) = n$. But $V_B(B^G)$ is a Galois algebra over C^G with Galois group G, so $\mathrm{rank}_{C^G}(V_B(B^G)) = n$. Therefore $\mathrm{rank}_{C^G}(V_B(B^G)) = n = \mathrm{rank}_{C^G}(C)$. Noting that $V_B(B^G)$ is an Azumaya C-algebra and a finitely generated projective C^G-module, we conclude that $V_B(B^G) = C$; and so C is a Galois algebra with Galois group induced by and isomorphic with G. Consequently, B is a DeMeyer-Kanzaki Galois extension with Galois group G.

(1) $\Longleftrightarrow$ (3) The proof is similar to (1) $\Longleftrightarrow$ (2). ∎

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