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TRIPLE SOLUTIONS FOR THE ONE-DIMENSIONAL p-LAPLACIAN

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Abstract. We give conditions on f involving pairs of lower and upper solutions which lead to the existence of at least three solutions to the two point boundary value problem $\left(|u'|^{p-2}\,u'\right) = q\,(t)\,f\,(t,u,u')$ on $(0,1)\,,\; u\,(0) = u\,(1) = 0.$

1. Introduction

In this paper we consider a two point boundary value problem for the one-dimensional $p-$Laplace equation of the form

$$(\varphi_p\,(u'))' = q\,(t)\,f\,(t,u,u')\,,\quad 0 < t < 1, \tag{1.1}$$

$$u\,(0) = u\,(1) = 0; \tag{1.2}$$

here $\varphi_p\,(s) = |s|^{p-2}\,s,\; p > 1$, and we assume the following two conditions hold:

$(H1)$ $q \in C\,(0,1)$ with $q > 0$ on $(0,1)$ and $\int_0^1 q\,(s)\,ds < \infty$, and
$(H2)$ $f : [0,1] \times R^2 \to R$ is continuous.

By a solution of (1.1)–(1.2) we mean a function $u \in C^1\,[0,1]$, with $\varphi_p\,(u') \in C^1\,(0,1)$, satisfying (1.1) on $(0,1)$ and $u(0) = u(1) = 0$. In this paper we assume there exists two lower solutions α_1, α_2 and two upper solutions β_1, β_2 for problem (1.1) and (1.2) satisfying $\alpha_1 \leq \alpha_2,\; \beta_1 \leq \beta_2$ and we show that there are three solutions. For the special case $f\,(t,u,u') = f\,(u) \geq 0$ we give growth conditions on f which lead to the existence of three positive

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solutions. In [1], J. Henderson and H. B. Thompson considered (1.1)–(1.2) with $p = 2$.

In this paper $C^k(J)$ will denote the space of functions $f : J \to R$ which are $k-$times continuously differentiable. For $u \in C[0,1]$, $\|u\|_\infty = \max_{t\in[0,1]} |u(t)|$, while for $u \in C^1[0,1]$, $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$.

Definition 1.1. *A function $\alpha \in C^1[0,1]$, $\varphi_p(\alpha') \in C^1(0,1)$ will be called a lower solution of (1.1)–(1.2) if $(\varphi_p(\alpha'))' \geq q(t) f(t, \alpha(t), \alpha'(t))$ for $t \in (0,1)$, with $\alpha(0) \leq 0$, $\alpha(1) \leq 0$.*

A function $\beta \in C^1[0,1]$, $\varphi_p(\beta') \in C^1(0,1)$ is a upper solution of (1.1)–(1.2) if the reverse inequalities hold.

Definition 1.2. *We say that f satisfies a Nagumo condition relative to the pair α and β, with $\alpha, \beta \in C[0,1]$, $\alpha \leq \beta$ in $[0,1]$, if there exists a function $\Psi : [0,\infty) \to (0,\infty)$ continuous, such that*

$$|f(t,y,z)| \leq \Psi(|z|) \quad \text{for all } (t,y,z) \in E, \tag{1.3}$$

where $E = \left\{(t,y,z) \in [0,1] \times R^2 : \alpha(t) \leq y \leq \beta(t)\right\}$, and also that

$$\int_{\varphi_p(v)}^{\infty} \frac{du}{\Psi\left(\varphi_p^{-1}(u)\right)} > \int_0^1 q(t)\, dt; \tag{1.4}$$

here

$$v = \max\{|\beta(0) - \alpha(1)|, |\beta(1) - \alpha(0)|\}.$$

2. General Results

Theorem 2.1. *Suppose $(H1)$ and $(H2)$ are satisfied. Assume that there exist two lower solutions α_1 and α_2 and two upper solutions β_1 and β_2 for problem (1.1)–(1.2) satisfying*

(i) $\alpha_1 \leq \alpha_2 \leq \beta_2$,
(ii) $\alpha_1 \leq \beta_1 \leq \beta_2$,
(iii) $\alpha_2 \not\leq \beta_1$,
(iv) *if u is a solution of (1.1)–(1.2) with $u \geq \alpha_2$, then $u > \alpha_2$ on $(0,1)$, and*
(v) *if u is a solution of (1.1)–(1.2) with $u \leq \beta_1$, then $u < \beta_1$ on $(0,1)$.*

If f satisfies the Bernstern-Nagumo condition with respect to α_1, β_2, then problem (1.1)–(1.2) has at least three solutions u_1, u_2 and u_3 satisfying

$$\alpha_1 \leq u_1 \leq \beta_1,\ \alpha_2 \leq u_2 \leq \beta_2,\ \text{and } u_3 \not\leq \beta_1 \text{ and } u_3 \not\geq \alpha_2.$$

Suppose that hypotheses $(H1)$, $(H2)$ and the Nagumo condition relative to a lower solution α_1 and upper solution β_2 are satisfied. We start with the construction of the modified problem. Define

$$P_{\alpha\beta}(t,x) = \max\{\alpha(t), \min\{x, \beta(t)\}\} \quad \text{for all } x \in R.$$

One can find the next result, with its proof, in [5].

LEMMA 2.2. *For each $u \in C^1[0,1]$ the next two properties hold:*

(a) $\dfrac{dP_{\alpha\beta}(t,u(t))}{dt}$ *exists for a.e. $t \in [0,1]$, and*

(b) *if $u, u_m \in C^1[0,1]$ and $u_m \to u$ in $C^1[0,1]$ then*

$$\frac{d}{dt}P_{\alpha\beta}(t,u_m(t)) \to \frac{d}{dt}P_{\alpha\beta}(t,u(t)), \quad \text{for a.e. } t \in [0,1].$$

From Definition 1.2, we can find a real number, $L > 0$, such that

$$0 \le v < L,\ -L < \alpha_1'(t), \beta_2'(t) < L \ \text{ for all } t \in [0,1]$$

and

$$\int_{\varphi_p(v)}^{\varphi_p(L)} \frac{du}{\Psi\left(\varphi_p^{-1}(u)\right)} > \int_0^1 q(t)\,dt.$$

We consider the following modified problem,

$$(\varphi_p(u'))' = q(t)\,k\left(t,u,\frac{d}{dt}P_{\alpha_1\beta_2}(t,u(t))\right), \quad 0<t<1, \tag{2.1}$$

$$u(0) = u(1) = 0, \tag{2.2}$$

with

$$k(t,x,y) = f(t, P_{\alpha_1\beta_2}(t,x), h(y)) + \tanh(x - P_{\alpha_1\beta_2}(t,x)),$$

where h is defined by

$$h(y) = \max\{-L, \min\{y,L\}\} \quad \text{for all } y \in R.$$

Thus k is a continuous function on $[0,1]\times R^2$ and satisfies

$$|k(t,x,y)| \le \Psi(|y|) + \frac{\pi}{2}, \ \text{ for } \ |y| \le L, \ \text{ and} \tag{2.3}$$

$$|k(t,x,y)| \le M, \ \text{ for } \ (t,x,y) \in [0,1]\times R^2, \tag{2.4}$$

for some constant M. Moreover, we may choose M so that $\|\alpha_1\|_\infty, \|\beta_2\|_\infty < M$.

First, we show that every solution of (2.1)–(2.2) is a solution (1.1)–(1.2).

LEMMA 2.3. *If u is a solution of (2.1)–(2.2), then $u \in [\alpha_1, \beta_2]$.*

PROOF. We prove $\alpha_1(t) \le u(t)$ for $t \in [0,1]$. Similar reasoning shows $u(t) \le \beta_2(t)$ for $t \in [0,1]$.

By definition of α_1 and β_2 we have that $\alpha_1(0) \le u(0) \le \beta_2(0)$ and $\alpha_1(1) \le u(1) \le \beta_2(1)$. If there exists $t_0 \in (0,1)$ such that

$$u(t_0) - \alpha_1(t_0) = \min_{t\in[0,1]}\{(u-\alpha_1)(t)\} < 0,$$

then, since $u-\alpha_1 \in C^1[0,1]$, we have $(u-\alpha_1)'(t_0)=0$. Furthermore, there exists $0 \le t_1 < t_0 < t_2 \le 1$ such that $u < \alpha_1$ in (t_1, t_2) and $(u-\alpha_1)(t_1) = (u-\alpha_1)(t_2) = 0$. Thus,

$$\begin{aligned} (\varphi_p(u'(t)))' - (\varphi_p(\alpha_1'(t)))' &\le q(t) f(t, \alpha_1(t), \alpha_1'(t)) \\ &\quad + q(t)\tanh[u(t)-\alpha_1(t)] \\ &\quad - q(t) f(t, \alpha_1(t), \alpha_1'(t)) \\ &= q(t)\tanh[u(t)-\alpha_1(t)] \\ &< 0, \end{aligned}$$

for all $t \in (t_1, t_2)$. As a result $\varphi_p(u'(t)) - \varphi_p(\alpha_1'(t)) < \varphi_p(u'(t_0)) - \varphi_p(\alpha_1'(t_0)) = 0$ for all $t \in (t_1, t_2)$, so

$$u'(t) < \alpha_1'(t) \quad \text{for all } t \in (t_0, t_2).$$

Thus $(u-\alpha_1)(t_2) < (u-\alpha_1)(t_0) < 0$, which is a contradiction. □

LEMMA 2.4. *If u is a solution of (2.1)–(2.2) then $-L < u'(t) < L$ for every $t \in [0,1]$.*

PROOF. Let $u \in C^1[0,1]$ be a solution of (2.1)–(2.2). From Lemma 2.3 we have $u \in [\alpha_1, \beta_2]$, and so

$$(\varphi_p(u'(t)))' = q(t) f(t, u(t), h(u'(t))) \quad \text{for } t \in (0,1).$$

By the mean-value theorem, there exists $t_0 \in (0,1)$ with

$$u'(t_0) = u(1) - u(0)$$

and as a result

$$-L < -v \le \alpha_1(1) - \beta_2(0) \le u'(t_0) \le \beta_2(1) - \alpha_1(0) \le v < L.$$

Let $v_0 = |u'(t_0)|$. Suppose that there exists a point in the interval $[0,1]$ for which $u' > L$ or $u' < -L$. From the continuity of u' we can choose $t_1 \in [0,1]$ such that one of the following situations hold:

(*i*) $u'(t_0) = v_0$, $u'(t_1) = L$ and $v_0 \le u'(t) \le L$ for all $t \in (t_0, t_1)$,
(*ii*) $u'(t_1) = L$, $u'(t_0) = v_0$ and $v_0 \le u'(t) \le L$ for all $t \in (t_1, t_0)$,
(*iii*) $u'(t_0) = -v_0$, $u'(t_1) = -L$ and $-L \le u'(t) \le -v_0$ for all $t \in (t_0, t_1)$, and
(*iv*) $u'(t_1) = -L$, $u'(t_0) = -v_0$ and $-L \le u'(t) \le -v_0$ for all $t \in (t_1, t_0)$.

Without loss of generality, suppose $-L \le v_0 \le u'(t) \le L$ for all $t \in (t_0, t_1)$. Then

$$\begin{aligned} (\varphi_p(u'(t)))' &= q(t) f(t, u(t), h(u'(t))) \\ &= q(t) f(t, u(t), u'(t)) \quad \text{for } t \in (t_0, t_1), \end{aligned}$$

and so

$$\begin{aligned}\left|(\varphi_p(u'(t)))'\right| &= |q(t) f(t, u(t), u'(t))| \\ &\leq q(t)\Psi(|u'(t)|) \quad \text{for } t \in (t_0, t_1).\end{aligned}$$

As a result

$$\begin{aligned}\int_{\varphi_p(v_0)}^{\varphi_p(L)} \frac{du}{\Psi\left(\varphi_p^{-1}(u)\right)} &= \int_{t_0}^{t_1} \frac{\left|(\varphi_p(u'(t)))'\right|}{\Psi(|u'(t)|)} dt \\ &\leq \int_{t_0}^{t_1} q(t)\, dt.\end{aligned}$$

Note also that $\varphi_p^{-1}(s) \geq 0$ for $s \in [\varphi_p(v_0), \varphi_p(L)]$, so we have $v_0 \leq v$ and thus $\varphi_p(v_0) \leq \varphi_p(v)$, which leads

$$\begin{aligned}\int_{\varphi_p(v_0)}^{\varphi_p(L)} \frac{du}{\Psi\left(\varphi_p^{-1}(u)\right)} &\geq \int_{\varphi_p(v)}^{\varphi_p(L)} \frac{\varphi_p^{-1}(u)}{\Psi\left(\varphi_p^{-1}(u)\right)} du \\ &> \int_0^1 q(t)\, dt,\end{aligned}$$

a contradiction. □

PROOF OF THE THEOREM 2.1. From Lemma's 2.3-2.4 it is enough to show (2.1)–(2.2) has three solutions as described in the statement of Theorem 2.1. Solving (2.1)–(2.2) is equivalent to finding a $u \in C^1[0,1]$ which satisfies

$$u(t) = \int_0^t \varphi_p^{-1}\left(A_u - \int_s^1 q(\tau) k_u(\tau)\, d\tau\right) ds, \tag{2.5}$$

where $k_u(\tau) \equiv k\left(\tau, u, \frac{d}{d\tau} P_{\alpha_1\beta_2}(\tau, u(\tau))\right)$ for a.e. $\tau \in [0,1]$, and A_u satisfies

$$\int_0^1 \varphi_p^{-1}\left(A_u - \int_s^1 q(\tau) k_u(\tau)\, d\tau\right) ds = 0. \tag{2.6}$$

The argument in [2] guarantees that A_u exists and is unique for $u \in C^1[0,1]$.

Now define the following operator $T : C^1[0,1] \to C^1[0,1]$ (here $u \in C^1[0,1]$ and $t \in [0,1]$) by

$$(Tu)(t) = \int_0^t \varphi_p^{-1}\left(A_u - \int_s^1 q(\tau) k_u(\tau)\, d\tau\right) ds, \tag{2.7}$$

where A_u satisfies (2.6).

We claim that $T : C^1[0,1] \to C^1[0,1]$ is continuous. Suppose $u_n \to u$ in $C^1[0,1]$. Let A_{u_n} correspond to u_n and A_u correspond to u, and we will now show that $\lim_{n\to\infty} A_{u_n} = A_u$. We know

$$\int_0^1 \varphi_p^{-1}\left(A_{u_n} - \int_s^1 q(\tau)\, k_{u_n}(\tau)\, d\tau\right) ds - \int_0^1 \varphi_p^{-1}\left(A_u - \int_s^1 q(\tau)\, k_u(\tau)\, d\tau\right) ds = 0. \tag{2.8}$$

The mean value theorem implies that there exists $\eta_n \in (0,1)$ such that

$$\varphi_p^{-1}\left(A_{u_n} - \int_{\eta_n}^1 q(\tau)\, k_{u_n}(\tau)\, d\tau\right) - \varphi_p^{-1}\left(A_u - \int_{\eta_n}^1 q(\tau)\, k_u(\tau)\, d\tau\right) = 0, \tag{2.9}$$

and so

$$A_{u_n} - A_u = \int_{\eta_n}^1 q(\tau)\left(k_{u_n}(\tau) - k_u(\tau)\right) d\tau. \tag{2.10}$$

On the other hand, since $u_n \to u$ in $C^1[0,1]$ and k is a continuous function we have from Lemma 2.2 that

$$k_{u_n}(t) \to k_u(t) \quad \text{for a.e. } t \in [0,1],$$

so (2.4) and the dominated convergence theorem yields

$$qk_{u_n} \to qk_u \ \text{ in } L^1(0,1).$$

Moreover,

$$\begin{aligned} 0 &\le \int_{\eta_n}^1 q(\tau)\left|k_{u_n}(\tau) - k_u(\tau)\right| d\tau \\ &\le \left\|q\left(k_{u_n} - k_u\right)\right\|_{L^1} \quad \text{for all } n \in N, \end{aligned}$$

and so

$$\lim_{n\to\infty} \int_{\eta_n}^1 q(\tau)\left(k_{u_n}(\tau) - k_u(\tau)\right) d\tau = 0. \tag{2.11}$$

This together with (2.10) yields

$$\lim_{n\to\infty} A_{u_n} = A_u.$$

Furthermore,

$$A_{u_n} - \int_t^1 q(\tau)\, k_{u_n}(\tau)\, d\tau \to A_u - \int_t^1 q(\tau)\, k_u(\tau)\, d\tau \ \text{ for all } t \in [0,1].$$

Also since

$$\begin{aligned} &\left|\left(A_{u_n} - \int_t^1 q(\tau)\, k_{u_n}(\tau)\, d\tau\right) - \left(A_u - \int_t^1 q(\tau)\, k_u(\tau)\, d\tau\right)\right| \\ &\quad \le \left|A_{u_n} - A_u\right| + \left\|q\left(k_{u_n} - k_u\right)\right\|_{L^1} \ \text{ for all } t \in [0,1], \end{aligned}$$

the convergence is uniform in $[0,1]$. In addition the uniform continuity of φ_p^{-1} on compact intervals yields

$$(Tu_n)' \to (Tu)' \quad \text{uniformly on } [0,1]$$

and as a result

$$Tu_n \to Tu \quad \text{uniformly on } [0,1].$$

We next claim that $T\left(C^1[0,1]\right)$ is a relatively compact set in $C^1[0,1]$. We first show that there exists a constant N^* with

$$|A_u| \leq N^* \quad \text{for all } u \in C^1[0,1].$$

Since

$$\int_0^1 \varphi_p^{-1}\left(A_u - \int_s^1 q(\tau)\, k_u(\tau)\, d\tau\right) ds = 0,$$

the Mean Value theorem for integrals implies that there exists $\xi \in [0,1]$ with

$$\varphi_p^{-1}\left(A_u - \int_\xi^1 q(\tau)\, k_u(\tau)\, d\tau\right) = 0.$$

Consequently,

$$A_u = \int_\xi^1 q(\tau)\, k_u(\tau)\, d\tau,$$

which implies

$$|A_u| \leq M \int_0^1 q(\tau)\, \tau \equiv N^*$$

where M is defined in (2.4).

Next we show that $T\left(C^1[0,1]\right)$ is bounded. This follows from the following inequalities:

$$|Tu(t)| \leq \int_0^1 J(s)\, ds \quad \text{and} \quad \left|(Tu)'(t)\right| \leq J(0) \quad \text{for } t \in [0,1]$$

or

$$\varphi_p^{-1}(Q) \leq (Tu)'(t) \leq \varphi_p^{-1}(-Q) \quad \text{for } t \in [0,1],$$

where

$$J(s) = \max\left\{\left|\varphi_p^{-1}\left(-N^* - M\int_s^1 q(u)\, du\right)\right|,\ \left|\varphi_p^{-1}\left(N^* + M\int_s^1 q(u)\, du\right)\right|\right\}$$

and

$$Q = N^* + M\int_0^1 q(u)\, du.$$

We next show the equicontinuity of $T\left(C^1[0,1]\right)$ on $[0,1]$. For $u \in C^1[0,1]$ and $t, s \in [0,1]$ we have

$$|Tu(t) - Tu(s)| \leq \left|\int_s^t J(v)\, dv\right|.$$

Finally, to see that $\left(T\left(C^1[0,1]\right)\right)' = \left\{y' : y \in T\left(C^1[0,1]\right)\right\}$ is equicontinuous on $[0,1]$, we use the fact that φ_p^{-1} is uniformly continuous on $[-Q,Q]$ and (2.4).

By Arzela-Ascoli Theorem, $T : C^1[0,1] \to C^1[0,1]$ is compact. Let

$$\Omega = \left\{u \in C_0^1[0,1] : ||u|| < M + L + J(0) + \int_0^1 J(s)\,ds\right\}.$$

It is immediate from the argument above that $T\left(\overline{\Omega}\right) \subset \Omega$. Thus

$$d(I-T,\Omega,0) = 1.$$

Let

$$\Omega_{\alpha_2} = \{u \in \Omega : u > \alpha_2 \text{ on } (0,1)\} \quad \text{and} \quad \Omega^{\beta_1} = \{u \in \Omega : u < \beta_1 \text{ on } (0,1)\}.$$

Since $\alpha_2 \not\leq \beta_1$, $\alpha_2 > -M$, and $\beta_1 < M$ (i.e. we choose M with $\|\alpha_2\|_\infty, \|\beta_1\|_\infty < M$) it follows that $\Omega^{\beta_1} \neq \emptyset \neq \Omega_{\alpha_2}$, $\Omega^{\beta_1} \cap \Omega_{\alpha_2} = \emptyset$, and $\Omega \backslash \overline{\{\Omega^{\beta_1} \cup \Omega_{\alpha_2}\}} \neq \emptyset$.

By assumptions (iv) and (v), there are no solutions in $\partial\Omega^{\beta_1} \cup \partial\Omega_{\alpha_2}$. Thus

$$\begin{aligned} d(I-T,\Omega,0) &= d\left(I-T,\Omega\backslash\overline{\{\Omega^{\beta_1} \cup \Omega_{\alpha_2}\}},0\right) \\ &\quad + d(I-T,\Omega_{\alpha_2},0) + d\left(I-T,\Omega^{\beta_1},0\right). \end{aligned}$$

We show that $d(I-T,\Omega_{\alpha_2},0) = d\left(I-T,\Omega^{\beta_1},0\right) = 1$. Then

$$d\left(I-T,\Omega\backslash\overline{\{\Omega^{\beta_1} \cup \Omega_{\alpha_2}\}},0\right) = -1,$$

and there are solution in $\Omega\backslash\overline{\{\Omega^{\beta_1} \cup \Omega_{\alpha_2}\}}$, Ω_{α_2} and Ω^{β_1}, as required.

We show $d(I-T,\Omega_{\alpha_2},0) = 1$. The proof that $d\left(I-T,\Omega^{\beta_1},0\right) = 1$ is similar and hence omitted. We define $I-W$, the extension to $\overline{\Omega}$ of the restriction of $I-T$ to $\overline{\Omega}_{\alpha_2}$ as follows. Let

$$w(t,x,y) = f\left(t, P_{\alpha_2\beta_2}(t,x), h(y)\right) + \tanh\left(x - P_{\alpha_2\beta_2}(t,x)\right),$$

where $P_{\alpha_2\beta_2}$ (replace α_1 by α_2) and h are defined previously. Thus w is a continuous function on $[0,1] \times R^2$ and satisfies

$$\begin{aligned} &|w(t,x,y)| \leq \Psi(|y|) + \frac{\pi}{2}, \quad \text{for } |y| \leq L, \text{ and} \\ &|w(t,x,y)| \leq M_1, \quad \text{for } (t,x,y) \in [0,1] \times R^2, \end{aligned}$$

for some constant M_1. Moreover, we may choose M_1 so that $\|\alpha_2\|_\infty, \|\beta_2\|_\infty < M_1$.

Consider the problem:

$$(\varphi_p(u'))' = q(t)\, w\left(t, u, \frac{d}{dt}P_{\alpha_2\beta_2}(t,u(t))\right), \quad 0 < t < 1, \tag{2.12}$$

$$u(0) = u(1) = 0. \tag{2.13}$$

Solving (2.12)–(2.13) is equivalent to finding a $u \in C^1[0,1]$ which satisfies

$$u(t) = \int_0^t \varphi_p^{-1}\left(B_u - \int_s^1 q(\tau)\, w_u(\tau)\, d\tau\right) ds,$$

where $w_u(\tau) \equiv w\left(\tau, u, \frac{d}{dt} P_{\alpha_2\beta_2}(t, u(t))\right)$ for a.e. $\tau \in [0,1]$, and B_u satisfies

$$\int_0^1 \varphi_p^{-1}\left(B_u - \int_s^1 q(\tau)\, w_u(\tau)\, d\tau\right) ds = 0. \tag{2.14}$$

As before B_u exists and is unique for $u \in C^1[0,1]$.

Now define the following operator $W : C^1[0,1] \to C^1[0,1]$ (here $u \in C^1[0,1]$ and $t \in [0,1]$) by

$$(Wu)(t) = \int_0^t \varphi_p^{-1}\left(B_u - \int_s^1 q(\tau)\, w_u(\tau)\, d\tau\right) ds,$$

where B_u satisfies (2.14).

Again it is easy to check (from a previous argument and (v)) that u is a solution of (2.12)–(2.13) if $u \in \Omega_{\alpha_2}$ and $Wu = u$ (note $W : C^1[0,1] \to C^1[0,1]$ is compact.

Thus $d\left(I - W,\ \Omega\backslash\overline{\Omega}_{\alpha_2}, 0\right) = 0$. Moreover it is easy to see that $W\left(\overline{\Omega}\right) \subset \Omega$. By assumptions (iv) and (v), there are no solutions in $\partial\Omega_{\alpha_2} \cap \partial\Omega^{\beta_1}$. Thus

$$\begin{aligned} d(I - T, \Omega_{\alpha_2}, 0) &= d(I - W, \Omega_{\alpha_2}, 0) \\ &= d\left(I - W, \Omega\backslash\overline{\Omega}_{\alpha_2}, 0\right) + d(I - W, \Omega_{\alpha_2}, 0) \\ &= d(I - W, \Omega, 0) \\ &= 1. \end{aligned}$$

Thus there are three solutions, as required. □

A slight modification of the argument in Theorem 2.1 yields the next result.

Theorem 2.5. *Suppose* $(H1)$ *and* $(H2)$ *satisfied. Assume that there exists two lower solutions* α_1 *and* α_2 *and two upper solutions* β_1 *and* β_2 *for problem (1.1)–(1.2) satisfying*

- (i) $\alpha_1 < \alpha_2 \le \beta_2$,
- (ii) $\alpha_1 \le \beta_1 < \beta_2$,
- (iii) *there exist* $0 < \overline{\varepsilon} < \min_{t\in[0,1]}\{\alpha_2(t) - \alpha_1(t),\ \beta_2(t) - \beta_1(t)\}$ *such that all* $\varepsilon \in (0, \overline{\varepsilon}]$, *the function* $\alpha_2(t) - \varepsilon$ *and* $\beta_1 + \varepsilon$ *are, respectively, lower and upper solution of (1.1)–(1.2), and*
- (iv) $\alpha_2 - \overline{\varepsilon} \nleq \beta_1 + \overline{\varepsilon}$.

If f *satisfies the Bernstern-Nagumo condition with respect to* α_1, β_2, *then problem (1.1)–(1.2) has at least three solution* u_1, u_2 *and* u_3 *satisfying*

$$\alpha_1 \le u_1 \le \beta_1,\ \alpha_2 \le u_2 \le \beta_2,\ \text{ and } u_3 \nleq \beta_1 \text{ and } u_3 \ngeq \alpha_2.$$

PROOF. In the proof of Theorem 2.1, define

$$\Omega_{\alpha_2} = \{u \in \Omega : u > \alpha_2 - \overline{\varepsilon} \ \text{ on } \ (0,1)\}$$

and

$$\Omega^{\beta_1} = \{u \in \Omega : u < \beta_1 + \overline{\varepsilon} \ \text{ on } \ (0,1)\},$$

where Ω is defined in Theorem 2.1. ∎

Consider the problem

$$(\varphi_p(u'))' + f(u) = 0, \quad \text{for all } t \in [0,1], \tag{2.15}$$

$$u(0) = u(1) = 0. \tag{2.16}$$

THEOREM 2.6. *Assume there exist real numbers a, b, c with $0 < a < b$, $0 < a < c$ and*

$$c > \max\left\{b + \frac{p-1}{p}\frac{b}{M}\left(\frac{\rho}{2}\right)^{\frac{1}{p-1}}\left(\frac{1}{2} - e\right)^{\frac{p}{p-1}},\ \frac{Mb}{h(e)}\right\}$$

and suppose there is a continuous nonnegative function f such that

(i) $f(y) < \left(\frac{a}{M}\right)^{p-1}$, $y \in [0,a]$,

(ii) $f(y) \geq \frac{\rho}{2}\left(\frac{b}{M}\right)^{p-1}$, $y \in \left[b, b + \frac{p-1}{p}\rho^{\frac{1}{p-1}}\left(\frac{1}{2}-e\right)^{\frac{p}{p-1}}\right]$, *and*

(iii) $f(y) \leq \left(\frac{c}{M}\right)^{p-1}$, $y \in [0,c]$.

Then problem (2.15)–(2.16) has at least three solution u_1, u_2 and u_3 satisfying $\|u_1\|_\infty < a$, $\alpha_2 \leq u_2$, and $\|u_3\|_\infty > a$ and $u_3 \ngeq \alpha_2$, where α_2 is given by

$$\alpha_2(t) = \begin{cases} \left(\frac{1}{2}\right)^{\frac{1}{p-1}} \frac{b}{M} t, & \text{for all } t \in [0,e], \\ b - \frac{p-1}{p}\frac{b}{M}\left(\frac{\rho}{2}\right)^{\frac{1}{p-1}}\left[\left(\frac{1}{2}-t\right)^{\frac{p}{p-1}} - \left(\frac{1}{2}-e\right)^{\frac{p}{p-1}}\right], & \text{for all } t \in \left[e, \frac{1}{2}\right], \\ b - \frac{p-1}{p}\frac{b}{M}\left(\frac{\rho}{2}\right)^{\frac{1}{p-1}}\left[\left(t-\frac{1}{2}\right)^{\frac{p}{p-1}} - \left(\frac{1}{2}-e\right)^{\frac{p}{p-1}}\right], & \text{for all } t \in \left[\frac{1}{2}, 1-e\right], \\ \left(\frac{1}{2}\right)^{\frac{1}{p-1}} \frac{b}{M}(1-t), & \text{for all } t \in [1-e, 1]; \end{cases}$$

here $e = \frac{p-1}{2p}$, $\rho = \frac{2b}{1-2e}$ and $M = \max_{t\in[0,1]} h(t)$ where

$$h(t) = \begin{cases} \frac{p-1}{p}\left[\left(\frac{1}{2}\right)^{\frac{p}{p-1}} - \left(\frac{1}{2}-t\right)^{\frac{p}{p-1}}\right], & \text{for } t \in \left[0, \frac{1}{2}\right], \\ \frac{p-1}{p}\left[\left(\frac{1}{2}\right)^{\frac{p}{p-1}} - \left(t-\frac{1}{2}\right)^{\frac{p}{p-1}}\right], & \text{for } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

PROOF. It is easily proved that h satisfies

$$(\varphi_p(u'))' + 1 = 0, \quad t \in [0,1],$$
$$u(0) = u(1) = 0.$$

Let $\alpha_1(t) \equiv 0$, $\beta_1(t) = \frac{a}{M}h(t)$, and $\beta_2(t) = \frac{c}{M}h(t)$ for $0 \leq t \leq 1$, and let α_2 be as above.

It is easy to check that $0 \leq \beta_2(t) \leq c$ and $(\varphi_p(\beta_2'))' = -\left(\frac{c}{M}\right)^{p-1}$ for $0 \leq t \leq 1$. It follows that β_1 is a strict upper solution and β_2 is an upper solution for problem (2.15)–(2.16) with $\beta_1 < \beta_2$ on $(0,1)$.

Since α_2 is symmetric in $t = \frac{1}{2}$, $\alpha_2(e) = \alpha_2(1-e)$, and $\alpha_2'(e) = \alpha_2'(1-e)$, so it follows that α_2 is in $C^1[0,1]$. Moreover α_2 satisfies $(\varphi_p(\alpha_2'))' = 0 \geq -f(\alpha_2)$ on $(0,e) \cup (1-e,1)$, and $(\varphi_p(\alpha_2'))' = -\frac{\rho}{2}\left(\frac{b}{M}\right)^{p-1} \geq -f(\alpha_2)$ on $(e, 1-e)$, so α_2 is a lower solution for problem (2.15)–(2.16). Moreover $\alpha_2\left(\frac{1}{2}\right) = b + \frac{p-1}{p}\frac{b}{M}\left(\frac{\rho}{2}\right)^{\frac{1}{p-1}}\left(\frac{1}{2}-e\right)^{\frac{p}{p-1}} > b > a = \beta_1\left(\frac{1}{2}\right)$.

Also since

$$\alpha_2'(0) = \left(\frac{1}{2}\right)^{\frac{1}{p-1}} \frac{b}{M} \leq \left(\frac{1}{2}\right)^{\frac{1}{p-1}} \frac{c}{M} = \beta_2'(0),$$

$$\alpha_2(e) = b \leq \frac{ch(e)}{M} = \beta_2(e),$$

$$\alpha_2\left(\frac{1}{2}\right) = b + \frac{p-1}{p}\frac{b}{M}\left(\frac{\rho}{2}\right)^{\frac{1}{p-1}}\left(\frac{1}{2}-e\right)^{\frac{p}{p-1}} < c = \beta_2\left(\frac{1}{2}\right),$$

it follows that $\alpha_2 < \beta_2$ on $(0,1)$.

We show that there is no solution u of problem (2.15)–(2.16) with $u \geq \alpha_2$ on $[0,1]$ and $u(t) = \alpha_2(t)$ for some $t \in (0,1)$. Assume this is false and that there is such a solution. Consider the case $t \in (0,e)$. Since $u'(t) = \alpha_2'(t)$ and $u \geq \alpha_2$ and $(\varphi_p(u'))' \leq (\varphi_p(\alpha_2'))'$ on $[0,e]$, it follows that $u = \alpha_2$ for all $t \in [0,e]$. Thus $0 = (\varphi_p(u'))'(e) = -f(u(e)) = -f(\alpha_2(e)) = -f(b)$, a contradiction, so $t \notin (0,e)$. Similarly $t \in [1-e,1)$ leads to the contradiction that $(\varphi_p(u'))'(1-e) = 0$, so $t \notin [1-e,1)$. Assume that $t \in [e,1-e)$. Again $u'(t) = \alpha_2'(t)$ and $y \geq \alpha_2$ and $(\varphi_p(u'))' \leq -\frac{\rho}{2}\left(\frac{b}{M}\right)^{p-1} = (\varphi_p(\alpha_2'))'$ on $[e,1-e]$. Thus $u = \alpha_2$ on $[e,1-e]$ so that $(\varphi_p(u'))'(1-e) \leq -\frac{\rho}{2}\left(\frac{b}{M}\right)^{p-1} < 0$ and $u'(1-e) = \left(\frac{1}{2}\right)^{\frac{1}{p-1}}\frac{b}{M}$. It follows that $u(x) < \alpha_2(x)$ for any $x \in (1-e, 1-e+\delta)$ for some $\delta > 0$, a contradiction. Thus $u(t) \neq \alpha_2(t)$ for any $t \in (0,1)$, as required.

Thus the conditions of Theorem 2.1 are satisfied and there are three solutions of problem (2.15)–(2.16), as required. ∎

References

[1] J. Henderson and H. B. Thompson, *Existence of multiple solutions for second order boundary value problems*, J. Diff. Equ. **166** (2000), 443–454.

[2] D. O'Regan, *Some general existence principles and results for* $(\phi(y'))' = qf(t,y,y')$, $0 < t < 1$, SIAM J. Math. Anal. **24** (1993), 648–668.

[3] D. O'Regan, *Existence theory for* $(\phi(y'))' = qf(t,y,y')$ $0 < t < 1$, Commun. Appl. Anal. **1** (1997), 33–52.

[4] C. De Coster, *Pairs of positive solutions for the one-dimensional p−Laplacian*, Nonl. Anal. **23** (1994), 669–681.

[5] M. X. Wang, A. Cabada and J. Nieto, *Monotone methods for nonlinear second order periodic boundary value problems with Carathéodory functions*, Ann. Polon. Math. **58** (1993), 221–235.

[6] Q. Yao and H. Lü, *Positive solutions of one-dimensional singular p−Laplace equation*, Acta Mathematica Sinica **41** (1998), 1255–1264.

[7] H. Lü and D. O'Regan, *A general existence theorem for singular equation* $(\varphi_p(y'))' + g(t,y) = 0$, Math. Ineq. Appl. **5** (2000), 69–78.

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