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PROJECTION-INVARIANTS, GRAM-SCHMIDT OPERATORS, AND WAVELETS

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Abstract. We introduce some projection-invariants for a normalized sequence in a Hilbert space, based on the smallness of the mutual projections of its elements. We then establish conditions to have the original sequence equivalent to its Gram-Schmidt orthonormalization. In many problems of wavelet-decomposition and reconstruction, the use of orthogonal bases cannot be implemented in the construction of certain filters and other practical features. Then, a quasiorthonormal structure for representation may be the next best alternative by achieving new constraints while we can still arbitrarily approximate the powerful classical orthogonal results.

1. Introduction

In a Hilbert space H, a (normalized) sequence is said to be orthogonal (orthonormal) if the scalar product $\langle \phi_n, \phi_k \rangle$of any two distinct elements is zero (and $\|\phi_n\| = 1$). In this case, many classical theorems are proved and extensively used in problems of decompositions, multiresolution representations, ... Starting with any normalized sequence $\{\phi_n\}$ of linearly independent vectors, a Gram-Schmidt orthonormalization $\left\{\phi_n^\perp\right\}$ always exists, but is in general topologically different from the original sequence. From stability point of view, if the size of all the projections $\langle \phi_n, \phi_k \rangle$ are small enough, it is natural to expect $\{\phi_n\}$ to somehow be close to $\left\{\phi_n^\perp\right\}$ and thus inherit of such properties as unconditionality enjoyed by orthonormal bases. Our interest is to present a functional analytic aspect with basic linear implications of the non-linear

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invariants introduced and apply to a perturbation of Mallat-Meyer's wavelet multiresolution analysis.

2. Quasiorthonormality

2.1. *Definitions.* A sequence $\{\phi_n\}$ in a Hilbert space H is a *frame* if there exist $A, B > 0$ such that $a\|f\|^2 \leq \sum_k |\langle f, \phi_n\rangle|^2 \leq B\|f\|^2$ for all f in H. Then, A and B are called *frame bounds*. The frame is called *tight* if $A = B$ and ε*-tight* if $A = 1-\varepsilon$ and $B = 1+\varepsilon$. $\{\phi_n\}$ is *Riesz sequence* if

$$A\sum_n \lambda_n^2 \leq \left\|\sum_n \lambda_n\phi_n\right\| \leq B\sum_n \lambda_n^2,$$

for any sequence of scalars $\{\lambda_n\}$. It is a *Hilbert sequence* if for any sequence $\{\lambda_n\}$ in l^2, the series $\sum_n \lambda_n\phi_n$ converges in H. It is a *Bessel sequence* if the summability of $\{\lambda_n\}$ is a necessary condition for the convergence of the series $\sum_n \lambda_n\phi_n$. So that $\{\phi_n\}$ is a Riesz sequence if and only if it is both Bessel and Hilbert. $\{\phi_n\}$ is *complete in* H if its closed linear span $\overline{\mathrm{span}}\{\phi_n\} = H$. With linear independence and the open mapping theorem, there is equivalence between frame (*exact*) and Riesz sequence (*basis*). A frame $\{\phi_n\}$ gives rise to two somewhat related bounded linear operators:

(1) the *Bessel map* $\beta : H \to H$, defined by $\beta(f) = \sum_n \langle f, \phi_n\rangle\phi_n$.

(2) the *frame operator* $F : H \to l^2$, defined by $F(f) = \{\langle f, \phi_n\rangle\}$

and to a *dual frame* defined by $\widehat{\phi}_n = (F * F)^{-1}\phi_n$, ($F * F$ can be shown to be nonsingular) with *dual Bessel map*

$$\widehat{\beta}(f) = \sum_n \left\langle f, \widehat{\phi}_n\right\rangle \widehat{\phi}_n$$

and *dual frame operator*

$$\widehat{F}(f) = \left\{\left\langle f, \widehat{\phi}_n\right\rangle\right\}.$$

Note that the frame operator associated with an ε-tight frame is an ε*-isometry*. The one-to-oness is guaranteed by linear independence. For any sequence $\varepsilon\{\phi_n\}$ of non-null vectors in H, we let

$$\widetilde{\phi}_n = \frac{\phi_n}{\|\phi_n\|}$$

(normalization of $\{\phi_n\}$),

$$\varepsilon_p(\phi_n) = \sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1}\left|\left\langle\widetilde{\phi}_n, \widetilde{\phi}_k\right\rangle\right|^p\right)^{1/p},\ 1 \leq p < \infty$$

(the *total projection* of order p for $\{\phi_n\}$),

$$\varepsilon_\infty(\phi_n) = \sup_{n \neq k} \left| \left\langle \widetilde{\phi}_n, \widetilde{\phi}_k \right\rangle \right|$$

(the *essential projection* for $\{\phi_n\}$) and let

$$\varepsilon_{\infty,p}(\phi_n) = \sup_n \left(\sum_{k=1}^{n-1} \left| \left\langle \widetilde{\phi}_n, \widetilde{\phi}_k \right\rangle \right|^p \right)^{1/p},$$

$$\varepsilon_{p,\infty}(\phi_n) = \left(\sum_{n=2}^{\infty} \max_{1 \le k < n} \left| \left\langle \widetilde{\phi}_n, \widetilde{\phi}_k \right\rangle \right|^p \right)^{1/p},$$

$$\omega_p(\phi_n) = \left(\sum_{n=2}^{\infty} \left| \left\langle \widetilde{\phi}_n, \frac{1}{n-1} \sum_{k=1}^{n-1} \widetilde{\phi}_k \right\rangle \right|^p \right)^{1/p},$$

$$\omega_\infty(\phi_n) = \sup_{n \ge 2} \left| \left\langle \widetilde{\phi}_n, \frac{1}{n-1} \sum_{k=1}^{n-1} \widetilde{\phi}_k \right\rangle \right|.$$

In absence of any ambiguity, we simply denote $\varepsilon_p(\phi_n) = \varepsilon_p$. Note that $0 \le \varepsilon_\infty \le \min\{\omega_p, \varepsilon_{\infty,p}\} \le \omega_p \le \varepsilon_{p,\infty} \le \max\{\varepsilon_{\infty,p}, \varepsilon_{p,\infty}\} \le \varepsilon_p \le \varepsilon_1$. We say that $\{\phi_n\}$ is *quasiorthogonal* (of order p) if $\varepsilon_p < \infty$. It is quasiorthonormal, if in addition it is normalized. Note that each of these projection-invariants represents some index that measures how far $\{\phi_n\}$ is from orthogonal. For example, $\{\phi_n\}$ is orthogonal if and only if $\varepsilon_p = 0$ and $\varepsilon_p = \infty$ if and only if there exists an infinite subsequence $\left\{\phi_{n_j}\right\}$ such that $\mathrm{Inf}\left|\left\langle \phi_{n_j}, \phi_{n_k} \right\rangle\right| > 0$.

In what follows, we focus on the quadratic total projection only. In particular, we simplify notations with $\varepsilon_2 \equiv \varepsilon$.

2.2. *A Quasiorthormalization Algorithm.* We exhibit the existence of intrinsic quasiorthonormal sequences by describing a more general procedure to construct such structures from any arbitrary linearly independent system.

THEOREM 2.1. *Let $\{\psi_n\}$ be a linearly independent sequence in a Hilbert space H. Then, for any $\delta > 0$, there exists a quasiorthonormal sequence $\{\phi_n\}$ generated by $\{\psi_n\}$ such that $\varepsilon(\phi_n) = \delta$.*

PROOF. First choose an orthonormalization $\{\psi_n^\perp\}$ of $\{\psi_n\}$ and $\delta_2 > \delta_3 > \cdots > 0$ such that $\sum\limits_{n=2}^{\infty} \delta_n = \delta$. We then inductively define $\{\phi_n\}$ as follows:

$\phi_1 = \psi_1^\perp$

$\phi_2 = \sqrt{1-\delta_2^2}\,\psi_2^\perp + \delta_2 \psi_1^\perp$, whence $\|\phi_2\| = \|\phi_1\| = 1$ and $|\langle \phi_2, \phi_1 \rangle| = \delta_2$.

Assume that $\phi_1, \ldots, \phi_q$ have already been defined by pairs of nonnegative coefficients $\{a_1, b_1\}, \ldots, \{a_q, b_q\}$ such that $\phi_n = a_n \psi_n^\perp + b_n \sum\limits_{k=1}^{n-1} \psi_k^\perp$, $\|\phi_n\| = 1$, and $\sum_{k=1}^{n-1} |\langle\phi_n, \phi_k\rangle|^2 = \delta_n^2$ for all $n = 2, \ldots, q$; $a_1 = 1$, $b_1 = 0$. We let

$$\begin{aligned} b_{q+1} &= \frac{\delta_{q+1}}{\sqrt{\sum_{n=1}^{q}[a_n + (n-1)b_n]^2}}, \\ a_{q+1} &= \sqrt{1 - qb_{q+1}^2}, \\ \phi_{q+1} &= a_{q+1}\psi_{q+1}^{\perp} + b_{q+1}\sum_{n=1}^{q}\psi_n^{\perp}. \end{aligned}$$

Then, it is easy to check that $\|\phi_{q+1}\| = 1$ and $\sum_{n=1}^{q} |\langle\phi_{q+1}, \phi_n\rangle|^2 = \delta_{q+1}^2$. Hence,

$$\varepsilon(\phi_n) = \sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1}\left|\left\langle\widetilde{\phi}_n, \widetilde{\phi}_k\right\rangle\right|^2\right)^{1/2} = \sum_{n=2}^{\infty}\delta_n = \delta.$$

Hence, $\{\phi_n\}$ is a non-orthogonal, quasiorthonormal sequence. Note that $\{\phi_n\}$ inherits of all the topological properties of $\{\psi_n\}$; $\operatorname{span}\{\phi_n\} = \operatorname{span}\{\psi_n\}$, and $\{\phi_n\}$ is a basis if and only if $\{\psi_n\}$ is a basis. $\square$

2.3. *Some Properties of Quasiorthonormal Sequences.* To prove our key lemma, we first recall a classical stability theorem of Krein-Milman-Rutman for Schauder bases, stating its orthonormal version only.

THEOREM 2.2. *Let $\{\psi_n\}$ denote an orthonormal basis and $\{\phi_n\}$ a normalized sequence in H. If $\sum_{n=1}^{\infty} \|\psi_n - \phi_n\| < \frac{1}{2}$, then $\{\phi_n\}$ is a Riesz basis equivalent to $\{\psi_n\}$.*

This theorem shows that all essential properties of a Schauder basis survive to small perturbations. In the sequel, we denote by $\{\phi_n^{\perp}\}$ the usual Gram-Schmidt orthonormalization of $\{\phi_n\}$; that is $\phi_1^{\perp} = \phi_1$,

$$\phi_n^{\perp} = \frac{\phi_n - \sum_{k=1}^{n-1}\langle\phi_n, \phi_k^{\perp}\rangle}{\Delta_n}, \text{ where } \Delta_n = \left\|\phi_n - \sum_{k=1}^{n-1}\langle\phi_n, \phi_k^{\perp}\rangle\right\| \text{ for } n = 2, 3, \ldots$$

Also note that

$$\Delta_n^2 = 1 - \sum_{k=1}^{n-1}\left|\left\langle\phi_n, \phi_k^{\perp}\right\rangle\right|^2, \quad \text{for } n = 2, 3, \ldots$$

LEMMA 2.3. *If $\varepsilon(\phi_n) < \frac{1}{6}\sqrt{2}$, then*

$$\sum_{n=2}^{N}\left(\sum_{k=1}^{n-1}\left|\left\langle\phi_n, \phi_k^{\perp}\right\rangle\right|^2\right)^{1/2} \le 2\sum_{n=2}^{N}\left(\sum_{k=1}^{n-1}|\langle\phi_n, \phi_k\rangle|^2\right)^{1/2}, \quad \textit{for } N \ge 2.$$

PROOF. For sake of simplicity, we do the calculations only in the real inner product case, the complex extension being natural. Since $\sup_{n\neq k} |\langle \phi_n, \phi_k\rangle| \leq \varepsilon < 1$, we inductively use relatively short Taylor expansions to get, for $n = 2, 3, \ldots$

$$\begin{aligned}
\left|\left\langle \phi_n, \phi_2^{\perp}\right\rangle\right|^2 &= \left(1 - |\langle \phi_n, \phi_2\rangle|^2\right)^{-1} |\langle \phi_n, \phi_2 - \langle \phi_2, \phi_1\rangle \phi_1\rangle|^2 \\
&= |\langle \phi_n, \phi_2\rangle|^2 + 2\langle \phi_n, \phi_2\rangle \langle \phi_n, \phi_1\rangle \langle \phi_2, \phi_1\rangle \\
&\quad + \left[|\langle \phi_n, \phi_2\rangle|^2 + |\langle \phi_n, \phi_1\rangle|^2 + 2\langle \phi_n, \phi_2\rangle \langle \phi_n, \phi_1\rangle\right] |\langle \phi_2, \phi_1\rangle|^2 \\
&\quad + \left[|\langle \phi_n, \phi_2\rangle|^2 + |\langle \phi_n, \phi_1\rangle|^2\right] |\langle \phi_2, \phi_1\rangle|^4 + \sum_{p=7}^{\infty} H_{n,2}^p
\end{aligned}$$

where $H_{n,k}^p$ denotes the sum of all the terms of order p in $\left|\left\langle \phi_n, \phi_k^{\perp}\right\rangle\right|^2$. Similarly, for $n = 3, 4, \ldots,$

$$\begin{aligned}
\left|\left\langle \phi_n, \phi_3^{\perp}\right\rangle\right|^2 &= |\langle \phi_n, \phi_3\rangle|^2 + 2\langle \phi_n, \phi_3\rangle \langle \phi_n, \phi_2\rangle \langle \phi_3, \phi_2\rangle \\
&\quad +2\langle \phi_n, \phi_3\rangle \langle \phi_n, \phi_1\rangle \langle \phi_3, \phi_1\rangle \\
&\quad + \left[|\langle \phi_n, \phi_3\rangle|^2 + |\langle \phi_n, \phi_2\rangle|^2\right] |\langle \phi_3, \phi_2\rangle|^2 \\
&\quad + \left[|\langle \phi_n, \phi_3\rangle|^2 + |\langle \phi_n, \phi_1\rangle|^2\right] |\langle \phi_3, \phi_1\rangle|^2 \\
&\quad +2\langle \phi_n, \phi_3\rangle \langle \phi_n, \phi_2\rangle \langle \phi_3, \phi_1\rangle \langle \phi_2, \phi_1\rangle \\
&\quad +2\langle \phi_n, \phi_3\rangle \langle \phi_n, \phi_1\rangle \langle \phi_3, \phi_2\rangle \langle \phi_2, \phi_1\rangle \\
&\quad +2\langle \phi_n, \phi_2\rangle \langle \phi_n, \phi_1\rangle \langle \phi_3, \phi_2\rangle \langle \phi_3, \phi_1\rangle + \sum_{p=5}^{\infty} H_{n,3}^p.
\end{aligned}$$

And more generally, for $n > k$

$$
\begin{aligned}
\left|\left\langle \phi_n, \phi_k^{\perp} \right\rangle\right|^2 &= \left|\langle \phi_n, \phi_k \rangle\right|^2 + 2\sum_{p=1}^{k-1} \langle \phi_n, \phi_k \rangle \langle \phi_n, \phi_p \rangle \langle \phi_k, \phi_p \rangle \\
&+ \left[\sum_{q=1}^{k-1} \left|\langle \phi_k, \phi_q \rangle\right|^2\right] \left|\langle \phi_n, \phi_k \rangle\right|^2 + \sum_{p=1}^{k-1} \left|\langle \phi_k, \phi_p \rangle\right|^2 \left|\langle \phi_n, \phi_p \rangle\right|^2 \\
&+2\sum_{p=2}^{k-1}\sum_{q=1}^{p-1} \langle \phi_n, \phi_k \rangle \langle \phi_n, \phi_q \rangle \langle \phi_k, \phi_p \rangle \langle \phi_p, \phi_q \rangle \\
&+2\sum_{p=2}^{k-1}\sum_{q=1}^{p-1} \langle \phi_n, \phi_k \rangle \langle \phi_n, \phi_p \rangle \langle \phi_k, \phi_q \rangle \langle \phi_p, \phi_q \rangle \\
&+2\sum_{p=2}^{k-2}\sum_{q=p+1}^{k-1} \langle \phi_n, \phi_p \rangle \langle \phi_n, \phi_q \rangle \langle \phi_k, \phi_p \rangle \langle \phi_p, \phi_q \rangle + \sum_{m=5}^{\infty} H_{n,k}^m.
\end{aligned}
$$

Hence, for any fixed $n > 1$,

$$
\begin{aligned}
\sum_{k=1}^{n-1} \left|\left\langle \phi_n, \phi_k^{\perp} \right\rangle\right|^2 &= \sum_{k=1}^{n-1} \left|\langle \phi_n, \phi_k \rangle\right|^2 + 2\sum_{k=2}^{n-1}\sum_{p=1}^{k-1} \langle \phi_n, \phi_k \rangle \langle \phi_n, \phi_p \rangle \langle \phi_k, \phi_p \rangle \\
&+ \sum_{m=4}^{\infty}\sum_{k=1}^{n-1} H_{n,k}^m.
\end{aligned}
$$

Now let $\varepsilon_N = \sum_{n=2}^{N} \sqrt{\sum_{k=1}^{n-1} \left|\langle \phi_n, \phi_k \rangle\right|^2}$, for $N = 2, 3, \ldots$ Note that $\varepsilon_N \uparrow \varepsilon$, as $N \to \infty$. We also note the following:

(1) $\left(\sum_{k=2}^{n-1} \left|\langle \phi_n, \phi_k \rangle\right|^2\right)^{1/2} \leq \varepsilon_n$

(2) $\sum_{k=2}^{n-1} \left(\sum_{p=1}^{k-1} \left|\langle \phi_k, \phi_p \rangle\right|^2\right) \leq \left[\sum_{k=2}^{n-1} \left(\sum_{p=1}^{k-1} \left|\langle \phi_k, \phi_p \rangle\right|^2\right)^{1/2}\right]^2 \leq \varepsilon_n^2.$

Applying Holder's inequality, we get

$$\begin{aligned}
\left|H_{n,k}^3\right| &= 2\left|\sum_{p=1}^{k-1}\langle\phi_n,\phi_k\rangle\langle\phi_n,\phi_p\rangle\langle\phi_k,\phi_p\rangle\right| \\
&\leq 2\left(\sum_{j=1}^{n-1}\left|\langle\phi_n,\phi_j\rangle\right|^2\right)^{1/2}\sum_{p=1}^{k-1}\left|\langle\phi_n,\phi_p\rangle\right|\left|\langle\phi_k,\phi_p\rangle\right\} \\
&\leq 2\left(\sum_{q=1}^{n-1}\left|\langle\phi_n,\phi_q\rangle\right|^2\right)^{1/2}\left(\sum_{p=1}^{k-1}\left|\langle\phi_n,\phi_p\rangle\right|^2\right)^{1/2}\left(\sum_{p=1}^{k-1}\left|\langle\phi_k,\phi_p\rangle\right|^2\right)^{1/2} \\
&\leq 2\left(\sum_{p=1}^{n-1}\left|\langle\phi_n,\phi_p\rangle\right|^2\right)\left(\sum_{p=1}^{k-1}\left|\langle\phi_k,\phi_p\rangle\right|^2\right)^{1/2}.
\end{aligned}$$

Hence,

$$\sum_{k=1}^{n-1}\left|H_{n,k}^3\right| \leq 2\left(\sum_{k=1}^{n-1}\left|\langle\phi_n,\phi_k\rangle\right|^2\right)\varepsilon_n,$$

and in particular

$$\sum_{k=1}^{n-1}\left|H_{n,k}^3\right| \leq 2\left(\sum_{k=1}^{n-1}\left|\langle\phi_n,\phi_k\rangle\right|^2\right)^{1/2}\varepsilon_n^2.$$

More generally, similar reasoning yields both

$$\sum_{k=1}^{n-1}\left|H_{n,k}^m\right| \leq 2\left(\sum_{k=1}^{n-1}\left|\langle\phi_n,\phi_k\rangle\right|^2\right)\varepsilon_n^{m-2},$$

$$\sum_{k=1}^{n-1}\left|H_{n,k}^m\right| \leq 2\left(\sum_{k=1}^{n-1}\left|\langle\phi_n,\phi_k\rangle\right|^2\right)^{1/2}\varepsilon_n^{m-1}$$

for any positive integer $m > 3$, and

$$\left|\sum_{m=4}^{\infty}\sum_{k=1}^{n-1}H_{n,k}^m\right| \leq 2\left(\sum_{k=1}^{n-1}\left|\langle\phi_n,\phi_k\rangle\right|^2\right)^{1/2}\sum_{m=4}^{\infty}\varepsilon_n^{m-1} < \frac{2\varepsilon_n^4}{1-\varepsilon_n} < \varepsilon_n^3,$$

since each $\varepsilon_n \leq \varepsilon < \frac{1}{6}\sqrt{2} < \frac{1}{3}$. On the other hand, we can apply the mean value theorem to $f(x) = \sqrt{\sum_{k=1}^{n-1}\left|\langle\phi_n,\phi_k\rangle\right|^2 + x}$ to pick some

$$0 < \eta < \sum_{k=2}^{n-1}\sum_{p=1}^{k-1}\langle\phi_n,\phi_k\rangle\langle\phi_n,\phi_p\rangle\langle\phi_k,\phi_p\rangle + \sum_{m=4}^{\infty}\sum_{k=1}^{n-1}H_{n,k}^m$$

without loss of generality such that

$$\begin{aligned}
\left(\sum_{k=1}^{n-1}\left|\left\langle\phi_n,\phi_k^{\perp}\right\rangle\right|^2\right)^{1/2} &= \Bigg[\sum_{k=1}^{n-1}\left|\langle\phi_n,\phi_k\rangle\right|^2+2\sum_{k=2}^{n-1}\sum_{p=1}^{k-1}\langle\phi_n,\phi_k\rangle\langle\phi_n,\phi_p\rangle\langle\phi_k,\phi_p\rangle \\
&\quad+\sum_{m=4}^{\infty}\sum_{k=1}^{n-1}H_{n,k}^m\Bigg]^{1/2} \\
&= \left[\sum_{k=1}^{n-1}\left|\langle\phi_n,\phi_k\rangle\right|^2\right]^{1/2}+\left[\sum_{k=1}^{n-1}\left|\langle\phi_n,\phi_k\rangle\right|^2+\eta\right]^{-1/2}\cdot \\
&\quad\cdot\left[\sum_{k=2}^{n-1}\sum_{p=1}^{k-1}\langle\phi_n,\phi_k\rangle\langle\phi_n,\phi_p\rangle\langle\phi_k,\phi_p\rangle+\sum_{m=4}^{\infty}\sum_{k=1}^{n-1}H_{n,k}^m\right]^{1/2}
\end{aligned}$$

This time we use the slightly sharper estimate

$$\begin{aligned}
\left|\sum_{m=4}^{\infty}\sum_{k=1}^{n-1}H_{n,k}^m\right| &\le 2\left(\sum_{k=1}^{n-1}\left|\langle\phi_n,\phi_k\rangle\right|^2\right)\sum_{m=4}^{\infty}\varepsilon_n^{m-2} \\
&< \frac{2\varepsilon_n^2}{1-\varepsilon_n}\left(\sum_{k=1}^{n-1}\left|\langle\phi_n,\phi_k\rangle\right|^2\right)<\varepsilon\left(\sum_{k=1}^{n-1}\left|\langle\phi_n,\phi_k\rangle\right|^2\right).
\end{aligned}$$

Since each $\varepsilon_n \le \varepsilon < \frac{1}{6\sqrt{2}} < \frac{1}{3}$, in order to write

$$\begin{aligned}
\left(\sum_{k=1}^{n-1}\left|\left\langle\phi_n,\phi_k^{\perp}\right\rangle\right|^2\right)^{1/2} &\le \left[\sum_{k=1}^{n-1}\left|\langle\phi_n,\phi_k\rangle\right|^2\right]^{1/2}+\left[\sum_{k=1}^{n-1}\left|\langle\phi_n,\phi_k\rangle\right|^2\right]^{-1/2}\cdot \\
&\quad\cdot\Bigg[\left(\sum_{k=1}^{n-1}\left|\langle\phi_n,\phi_k\rangle\right|^2\right)\left(\sum_{k=2}^{n-1}\left[\sum_{p=1}^{k-1}\left|\langle\phi_k,\phi_p\rangle\right|^2\right]^{1/2}\right) \\
&\qquad+\varepsilon\left(\sum_{k=1}^{n-1}\left|\langle\phi_n,\phi_k\rangle\right|^2\right)\Bigg] \\
&\le \left(\sum_{k=1}^{n-1}\left|\langle\phi_n,\phi_k\rangle\right|^2\right)^{1/2}(1+2\varepsilon).
\end{aligned}$$

Hence,

$$\sum_{n=2}^{N}\left(\sum_{k=1}^{n-1}\left|\left\langle\phi_n,\phi_k^{\perp}\right\rangle\right|^2\right)^{1/2} \le (1+2\varepsilon)\sum_{n=2}^{N}\left(\sum_{k=1}^{n-1}\left|\langle\phi_n,\phi_k\rangle\right|^2\right)^{1/2}.$$

We apply the condition $2\varepsilon < 1$ to complete the proof of the lemma. $\square$

THEOREM 2.4. *Let $\{\phi_n\}$ be a normalized linearly independent sequence in a Hilbert space H. If*

$$\varepsilon = \sum_{n=2}^{\infty} \sqrt{\sum_{k=1}^{n-1} \left|\langle \phi_n, \phi_k \rangle\right|^2} < \frac{1}{6\sqrt{2}},$$

then $\{\phi_n\}$ forms a $4\sqrt{2}\varepsilon$-tight frame in H with a $4\sqrt{2}\varepsilon$-isometric frame operator.

PROOF. Let $\left\{\phi_n^{\perp}\right\}$ denote the Gram-Schmidt orthonormalization of $\{\phi_n\}$. Then,

$$\left\|\phi_n^{\perp} - \phi_n\right\| = 2 - 2\left[1 - \sum_{k=1}^{n-1} \left|\left\langle \phi_n, \phi_k^{\perp} \right\rangle\right|^2\right]^{1/2}$$

for all $n \geq 2$, and using Lemma 2.3 for

$$0 < \xi < \sum_{k=1}^{n-1} \left|\left\langle \phi_n, \phi_k^{\perp} \right\rangle\right|^2$$

we can write

$$\begin{aligned}
\left\|\phi_n^{\perp} - \phi_n\right\|^2 &= \sum_{k=1}^{n-1} \left|\left\langle \phi_n, \phi_k^{\perp} \right\rangle\right|^2 + \frac{1}{4}(1-\xi)^{-3/2} \left(\sum_{k=1}^{n-1} \left|\left\langle \phi_n, \phi_k^{\perp} \right\rangle\right|^2\right)^{1/2} \\
&\leq \left[1 + \frac{1}{4} \frac{1}{\sqrt{\left(1 - \sum_{k=1}^{n-1} \left|\left\langle \phi_n, \phi_k^{\perp} \right\rangle\right|^2\right)^3}}\right] \sum_{k=1}^{n-1} \left|\left\langle \phi_n, \phi_k^{\perp} \right\rangle\right|^2 \\
&\leq \left[1 + \frac{1}{4} \frac{1}{\sqrt{(1-4\varepsilon^2)^3}}\right] \sum_{k=1}^{n-1} \left|\left\langle \phi_n, \phi_k^{\perp} \right\rangle\right|^2 \\
&\leq 2 \sum_{k=1}^{n-1} \left|\left\langle \phi_n, \phi_k^{\perp} \right\rangle\right|^2,
\end{aligned}$$

since $\varepsilon < \frac{1}{6\sqrt{2}} < \frac{\sqrt{2}}{6}$. Hence,

$$\left\|\phi_n^{\perp} - \phi_n\right\| \leq \sqrt{2} \left(\sum_{k=1}^{n-1} \left|\left\langle \phi_n, \phi_k^{\perp} \right\rangle\right|^2\right)^{1/2}.$$

Hence,

$$\begin{aligned}\sum_{n=1}^{N} \left\|\phi_n^{\perp} - \phi_n\right\| &\le \sqrt{2}\sum_{n=2}^{N}\left(\sum_{k=1}^{n-1}\left|\left\langle \phi_n, \phi_k^{\perp}\right\rangle\right|^2\right)^{1/2} \\ &\le 2\sqrt{2}\sum_{n=2}^{N}\left(\sum_{k=1}^{n-1}\left|\langle \phi_n, \phi_k\rangle\right|^2\right)^{1/2}.\end{aligned}$$

We finally let $N \to \infty$ to get $\sum\limits_{n=1}^{N} \left\|\phi_n^{\perp} - \phi_n\right\| \le 2\sqrt{2}\varepsilon(\phi_n) < \frac{1}{2}$, since $\varepsilon(\phi_n) < \frac{1}{6\sqrt{2}} < \frac{1}{4\sqrt{2}}$. Hence, all the conditions of the Krein-Milman-Rutman theorem are satisfied for $\{\phi_n\}$ to be equivalent to its Gram-Schmidt orthonormalization. In order to prove the $4\sqrt{2}\varepsilon$-tightness, fix any positive integer $N > 1$ and f in H; then,

$$\begin{aligned}\sum_{n=1}^{N}\left|\langle f, \phi_n\rangle\right|^2 - \sum_{n=1}^{N}\left|\left\langle f, \phi_n^{\perp}\right\rangle\right|^2 &\le 2\|f\|\sum_{n=1}^{N}\left|\left\langle f, \phi_n - \phi_n^{\perp}\right\rangle\right| \\ &\le 2\|f\|^2\sum_{n=1}^{\infty}\left\|\phi_n, \phi_n^{\perp}\right\| \le 4\sqrt{2}\varepsilon\|f\|^2.\end{aligned}$$

Hence, apply Parseval's identity and let $N \to \infty$ in order to get

$$(1 - 4\sqrt{2}\varepsilon)\|f\|^2 \le \sum_{n=1}^{\infty}\left|\langle f, \phi_n\rangle\right|^2 \equiv \|F(f)\|^2 \le (1 + 4\sqrt{2}\varepsilon)\|f\|^2.$$

In particular, $\|F\| \le \sqrt{1 + 4\sqrt{2}\varepsilon}$. ∎

Corollary 2.5. *Every quasiorthonormal basis $\{\phi_n\}$ contains a $4\sqrt{2}\varepsilon(\phi_n)$ –tight frame basic subsequence.*

Proof. By Theorem 2.4, it suffices to show that if

$$\sum_{n=2}^{N}\left(\sum_{k=1}^{n-1}\left|\langle \phi_n, \phi_k\rangle\right|^2\right)^{1/2} < \infty,$$

then there is $n_1 < n_2 < \ldots$ such that

$$\sum_{j=2}^{\infty}\left(\sum_{i=1}^{j-1}\left|\left\langle \phi_{n_j}, \phi_{n_i}\right\rangle\right|^2\right)^{1/2} < \frac{1}{6\sqrt{2}}.$$

Indeed, choose $\varepsilon_1 > \varepsilon_2 > \ldots \downarrow 0$ such that $\sum\limits_{j=1}^{\infty} \varepsilon_j < \frac{1}{6\sqrt{2}}$; choose n_1 such that $\sum\limits_{k=1}^{n_1-1} \left|\langle \phi_{n_1}, \phi_k\rangle\right|^2 < \varepsilon_1^2$. Let $N_1 = \{1, 2, \ldots, n_1\}$. Then choose $n_2 > n_1$ such that $\sum\limits_{k=1}^{n_2-1} |\langle \phi_{n_2}, \phi_k \rangle|^2 < \varepsilon_2^2$. Let $N_2 = N_1 \cup \{n_2\}$. Then choose $n_3 > n_2$ such that $\sum\limits_{k=1}^{n_3-1} |\langle \phi_{n_3}, \phi_k \rangle|^2 < \varepsilon_3^2$. Let $N_3 = N_2 \cup \{n_3\}$. Inductively continue this process indefinitely to get $\bigcup\limits_{j=1}^{\infty} N_j = \{n_j,\ j = 1, 2, \dots\}$ (by renaming) so that

$$\sum_{i=1}^{j-1} \left|\left\langle \phi_{n_j}, \phi_{n_i} \right\rangle\right|^2 < \sum_{k=1}^{n_j-1} \left|\left\langle \phi_{n_j}, \phi_k \right\rangle\right|^2 < \varepsilon_j^2 \text{ for all } j = 2, 3, \dots$$

And thus

$$\varepsilon(\phi_n) = \sum_{j=2}^{\infty} \left(\sum_{i=1}^{j-1} \left|\left\langle \phi_{n_j}, \phi_{n_i} \right\rangle\right|^2 \right)^{1/2} < \varepsilon_1^2 \le \sum_{j=1}^{\infty} \varepsilon_j < \frac{1}{6\sqrt{2}}.$$

□

Remark 2.6. From Corollary 2.5, it follows that a subsymmetric (equivalent to each of its infinite subsequences) quasiorthonormal basis is always a Riesz basis.

Theorem 2.7. *Let $\{\phi_n\}$ be a normalized linearly independent sequence in a Hilbert H. If $\sum\limits_{n=2}^{\infty} \sqrt{\sum\limits_{k=1}^{n-1} |\langle \phi_n, \phi_k \rangle|^2} \equiv \varepsilon < \frac{1}{6\sqrt{2}}$, then for any $\lambda \equiv \{\lambda_n\}$ in l^2, the series $\sum\limits_n \lambda_n \phi_n$ converges to an element f of H such that*

(1) $(1 - 2\sqrt{2}\varepsilon)\|\lambda\| \le \|f\| \le (1 + 2\sqrt{2}\varepsilon)\|\lambda\|$
(2) $\|\lambda - F(f)\| \le 2\sqrt{2}\varepsilon \left(1 + \sqrt{1 + 4\sqrt{2}\varepsilon}\right) \|\lambda\|.$

Proof. Let $f_n = \sum\limits_{k=1}^{n} \lambda_k \phi_k$. Then,

$$\begin{aligned} \|f_{n+p} - f_n\| &= \left\| \sum_{k=n+1}^{n+p} \lambda_k \phi_k \right\| \le \left\| \sum_{k=n+1}^{n+p} \lambda_k \phi_k^{\perp} \right\| + \left\| \sum_{k=n+1}^{n+p} \lambda_k (\phi_k - \phi_k^{\perp}) \right\| \\ &\le \left(\sum_{k=n+1}^{n+p} \lambda_k^2 \right)^{1/2} \left[1 + \sum_{k=n+1}^{n+p} \left\| \phi_k - \phi_k^{\perp} \right\| \right] \end{aligned}$$

by Parseval theorem and Cauchy-Schwarz inequality. Both series on the right converge absolutely. Hence, the Cauchy sequence of partial sums must converge to some $f = \sum\limits_n \lambda_n \phi_n$ in H. By Riesz-Fischer theorem, $f^{\perp} = \sum\limits_n \lambda_n \phi_n^{\perp}$ also converges in H and the Fourier coefficients are $\left\langle f^{\perp}, \phi_n^{\perp} \right\rangle = \lambda_n$ for all n, and $\left\|f^{\perp}\right\| = \|\lambda\| = \left(\sum_n |\lambda_n|^2\right)^{1/2}$. Hence, for any positive integer N,

$$\left\|\sum_{n=1}^{N} \lambda_n(\phi_n - \phi_n^{\perp})\right\| \le \left(\sum_{n=1}^{N} \lambda_n^2\right)^{1/2} \sum_{n=1}^{N} \|\phi_n - \phi_n^{\perp}\|.$$

Using the estimate $\sum_{n=1}^{\infty} \|\phi_n - \phi_n^{\perp}\| < 2\sqrt{2}\varepsilon$ and letting $N \to \infty$, we get $\|f - f^{\perp}\| < 2\sqrt{2}\varepsilon\|\lambda\|$ and (1) follows.

On the other hand,

$$\left|\lambda_n - \left\langle f^{\perp}, \phi_n \right\rangle\right| = \left|\left\langle f^{\perp}, \phi_n - \phi_n^{\perp} \right\rangle\right| \le \|\lambda\| \left\|\phi_n - \phi_n^{\perp}\right\|$$

and

$$\left\|\lambda - F(f^{\perp})\right\| = \left(\sum_{n=1}^{\infty} \left|\lambda_n - \left\langle f^{\perp}, \phi_n \right\rangle\right|^2\right)^{1/2} \le \|\lambda\| \sum_{n=1}^{\infty} \left\|\phi_n - \phi_n^{\perp}\right\| < 2\sqrt{2}\varepsilon\|\lambda\|.$$

Hence,

$$\|\lambda - f(f)\| \le \left\|\lambda - F(f^{\perp})\right\| + \left\|F(f - f^{\perp})\right\| \le 2\sqrt{2}\varepsilon\|\lambda\| + 2\sqrt{2}\varepsilon\sqrt{1 + 4\sqrt{2}\varepsilon}\|\lambda\|$$

since $\|F\| \le \sqrt{1 + 4\sqrt{2}\varepsilon}$. This completes the proof of the theorem. $\square$

It is a theorem of Benedetto [1] that a frame in H is exact if and only if the frame operator is a topological isomorphism. In the context of quasiorthonormality, we prove

Theorem 2.8. *A quasiorthonormal basis $\{\phi_n\}$ of H generates a bounded linear operator γ on H and a bilinear form Φ_γ on H for which $\{\phi_n\}$ is orthonormal.*

Proof. By the proof of Theorem 2.4, quasiorthonormality generates three linear maps: the frame operator $F : H \to l^2$ for $\{\phi_n\}$ defined by $F(f) = \{\langle f, \phi_n \rangle\}$, the frame operator $F^{\perp} : H \to l^2$ for $\{\phi_n^{\perp}\}$ defined by $F^{\perp}(f) = \left\{\left\langle f, \phi_n^{\perp} \right\rangle\right\}$, and a linear isomorphism $\gamma : H \to H$ we call Gram-Schmidt operator defined by $\gamma(\phi_n) = \phi_n^{\perp}$ such that $F^{\perp} \circ \gamma = F$. Clearly, under the bilinear form $\Phi_\gamma(f, g) = \langle \gamma(f), \gamma(g) \rangle$, we have $\Phi_\gamma(\phi_n, \phi_k) = \left\langle \phi_n^{\perp}, \phi_k^{\perp} \right\rangle = \delta_{n,k}$ (Kronecker). $\square$

Remark 2.9. Most classical orthogonal results are easy to establish by perturbation arguments in the case of quasiorthogonality. For example, if $\varepsilon(\phi_n) < \frac{1}{6\sqrt{2}}$ then

(1) There is equivalence between weak and strong unconditional convergence of the series $\sum_n \phi_n$ and the absolute convergence $\sum_n \|\phi_n\|^2$.

(2) For any $\{\lambda_n\} \in l^2$, $A(f) = \sum_n \lambda_n \langle f, \phi_n\rangle \phi_n$ defines a compact operator on H.

2.4. *Iterative Reconstruction Algorithms.* By the very nature of quasiorthogonality, a certain flexibility can be enjoyed in signal recovery schemes. In problems such as signal compression, edge detection, vision analysis,... we must avoid orthogonality under which many natural constraints cannot be satisfied. Then, a quasiorthogonal structure may be the very best next thing for a good enough decomposition and reconstruction. An intrinsic algorithm can be written from a general frames point of view or one would rather use an ε-perturbation of orthogonal methods. In either case, we have control over the error tolerance through quasiorthonormalization as described in section I, theorem 2.1. For instance, if $\varepsilon(\phi_n) \equiv \varepsilon < \frac{1}{6\sqrt{2}}$ then $\{\phi_n\}$ must be $4\sqrt{2}\varepsilon$-tight. Hence, we can use the bounds of the Bessel map $F^*F = \beta$, $(1-4\sqrt{2}\varepsilon)\,\mathrm{Id} \leq \beta \leq (1+4\sqrt{2}\varepsilon)\,\mathrm{Id}$ in order to get $\left\| f - \sum_n \langle f, \phi_n\rangle \phi_n \right\| < 4\sqrt{2}\varepsilon$, a near perfect decomposition and reconstruction from the frame coefficients $\langle f, \phi_n\rangle$ which are associated with the Fourier coefficients $\langle f, \phi_n^\perp\rangle$ by the global estimates

$$\sum_n \left|\langle f, \phi_n\rangle - \langle f, \phi_n^\perp\rangle\right| \leq \|f\| \sum_n \left\|\phi_n - \phi_n^\perp\right\| < 4\sqrt{2}\varepsilon\|f\|.$$

Otherwise, we can follow the general frame approach as in Daubechies [4] using the bounded inverse β^{-1} of the Bessel map with bounds $\frac{1}{1+4\sqrt{2}\varepsilon}\,\mathrm{Id} \leq \beta^{-1} \leq \frac{1}{1-4\sqrt{2}\varepsilon}\,\mathrm{Id}$ to first find the dual frame $\widehat{\phi}_n = \beta^{-1}\phi_n$. In this case, we will approximate $\widehat{\phi}_n$ by $\widehat{\phi}_n^P = (\mathrm{Id} - \delta^{P+1})\widehat{\phi}_n$, where $\delta = \mathrm{Id} - \beta$ with $\|\delta\| < 4\sqrt{2}\varepsilon$. Whence, $\left\| f - \sum_n \langle f, \phi_n\rangle \widehat{\phi}_n^P \right\| \leq (4\sqrt{2}\varepsilon)^{P+1}\|f\|$ P is chosen so as to obtain any desired degree of accuracy. Iteratively, using $\widehat{\phi}_n^0 = \phi_n$ and $\widehat{\phi}_n^P = \phi_n + \delta(\widehat{\phi}_n^{P-1})$ we get

$$\begin{aligned}
\widehat{\phi}_n^1 &= \phi_n - \sum_{n_1 \neq n} \langle \phi_n, \phi_{n_1}\rangle \phi_{n_1} \\
\widehat{\phi}_n^2 &= \phi_n - \sum_{n_1 \neq n} \langle \phi_n, \phi_{n_1}\rangle \phi_{n_1} + \left(\sum_{n_1 \neq n} |\langle \phi_n, \phi_{n_1}\rangle|^2 \right) \phi_n \\
&\quad + \sum_{n_1 \neq n} \sum_{n_2 \neq n, n_1} \langle \phi_n, \phi_{n_1}\rangle \langle \phi_{n_1}, \phi_{n_2}\rangle \phi_{n_2} \cdots
\end{aligned}$$

It is easy to verify that $\|\phi_n - \widehat{\phi}_n^1\| < \varepsilon$, $\|\phi_n - \widehat{\phi}_n^2\| < \varepsilon + \varepsilon^2$, and more generally $\|\phi_n - \widehat{\phi}_n^P\| < \sum_{k=1}^{P} \varepsilon^k = \varepsilon \frac{1-\varepsilon^P}{1-\varepsilon}$ for all $P \geq 1$.

Another approach is to consider the Gram-Schmidt operator $\gamma(\phi_n) = \phi_n^\perp$ on H such that if $f = \sum_n \lambda_n \phi_n$, then $\sum_n |\lambda_n|^2 = \|\gamma(f)\|^2$ and $\|\gamma(f)\| < \frac{1}{\sqrt{1-4\sqrt{2}\varepsilon}}\|f\|$, and thus

$$\|f - \gamma(f)\| \leq \left(\sum_n \lambda_n^2\right)^{1/2} \sum_n \|\phi_n - \phi_n^\perp\| \leq \frac{2\sqrt{2}\varepsilon}{\sqrt{1-4\sqrt{2}\varepsilon}}\|f\|$$

satisfying the Feichtinger-Grochenig condition [5] for a recovery of f from $\gamma(f)$ by the following algorithm

$$\begin{aligned} &f_0 = \gamma(f) \\ &f_{n+1} = f_n + \gamma(f - f_n) \quad \text{for all } n \geq 0. \end{aligned}$$

Then, $f = \lim_{n\to\infty} f_n$ with error $\|f - f_n\| \leq \left(\frac{2\sqrt{2}\varepsilon}{\sqrt{1-4\sqrt{2}\varepsilon}}\right)^{n+1} \|f\|$ after n iterations.

3. A Related Epsilonized Multiresolution Analysis in $L^2(\mathbb{R})$

Wavelet theory can be viewed as a derivative of the more classical Fourier analysis, where the complex exponential $\psi(x) = e^{ix}$ or sinusoidal wave has been used to generate every 2π-periodic square-integrable function as a linear combination of shifts and integral dilations of $\psi(x)$. In essence, $\psi(x)$ is said to be a (dyadic) wavelet in $L^2(\mathbb{R})$ if it satisfies certain conditions that make $\psi_{n,k}(x) = 2^{-n/2}\psi(2^{-n}x - k)$ form a Riesz basis for $L^2(\mathbb{R})$. For years, there was no systematic way of finding a wavelet until the advent of multiresolution analysis in 1985-86. Loosely speaking, the multiresolution analysis is a method of construction of a wavelet basis based on subspace decomposition, where the orthogonal projections provide coarser and coarser approximations of original functions, signals, ... In its original setting as introduced by Mallat [8] and Meyer [10], a multiresolution of $L^2(\mathbb{R})$ is defined by a nested sequence $\cdots \supset V_{-2} \supset V_{-1} \supset V_0 \supset V_1 \supset \ldots$ of closed subspaces and a square integrable function ϕ such that

$$\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R}) \tag{3.1}$$

$$\bigcap_{n\in\mathbb{Z}} V_n = \{0\} \tag{3.2}$$

$$f(x) \in V_n \iff f(2^n x) \in V_0 \tag{3.3}$$

$$f(x) \in V_0 \iff f(x-k) \in V_0, \text{ for all } k \in \mathbb{Z} \tag{3.4}$$

$$\{\phi_{0,k} : k \in \mathbb{Z}\} \text{ is an orthonormal basis for } V_0. \tag{3.5}$$

Condition (3.5) guarantees the orthonormality of the basis $\{\psi_{n,k}\}$ generated. Denoting by W_n the orthogonal complement of V_n in V_{n-1}, (3.1) and (3.2) imply

$$L^2(\mathbb{R}) = \bigoplus_{n\in\mathbb{Z}} W_n \tag{3.6}$$

a decomposition of $L^2(\mathbb{R})$ into mutually orthogonal subspaces.

In what follows, we modify condition (3.5) and write a corresponding MRA for ε-perturbations of orthonormal bases, more general than quasiorthonormal.

We start with a normalized function ϕ in $L^2(\mathbb{R})$ such that:

(I) a Gram-Schmidt operator γ exists on $L^2(\mathbb{R})$ such that $\gamma(\phi_{0,n}) = \phi_n^{\perp}$ and $\|I-\gamma\| < \varepsilon$

(II) $\phi(x) = \sqrt{2}\sum_n h_n\phi(2x-n)$, where $\sum_n |h_n|^2 < \infty$

(III) $\widehat{\phi}(\omega)$ is bounded, continuous at 0 and $\widehat{\phi}(0) = 0$.

Lemma 3.1. *Under hypothesis (I), we have*

(I') $\frac{1}{1+\varepsilon} \le \sum_{l\in\mathbb{Z}} \left|\widehat{\phi}(\omega+2\pi l)\right|^2 \le 1+\varepsilon$ *a.e.*

Proof. Let γ denote the Gram-Schmidt operator for on $L^2(\mathbb{R})$. Then $\|I-\gamma\| < \varepsilon$, $\|\gamma^{-1}\|^{-1}\left\|\sum_n \lambda_n\phi_n^{\perp}\right\| \le \left\|\sum_n \lambda_n\phi_{0,n}\right\| \le \|\gamma^{-1}\|\left\|\sum_n \lambda_n\phi_n^{\perp}\right\|$ for any sequence of scalars. Note that $\frac{1}{1+\varepsilon} \le \|\gamma^{-1}\| \le 1+\varepsilon$. Hence, $\frac{1}{1+\varepsilon}\left(\sum_n |\lambda_n|^2\right)^{1/2} \le \left\|\sum_n \lambda_n\phi_{0,n}\right\| \le (1+\varepsilon)\left(\sum_n |\lambda_n|^2\right)^{1/2}$. But,

$$\left\|\sum_n \lambda_n\phi_{0,n}\right\|^2 = \int_0^{2\pi}\left|\sum_n \lambda_n e^{-in\omega}\right|^2 \sum_{l\in\mathbb{Z}}\left|\widehat{\phi}(\omega+2\pi l)\right|^2 \frac{d\omega}{2\pi},$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_n \lambda_n e^{-in\omega}\right|^2 d\omega = \sum_n |\lambda_n|^2.$$

We then use the Gaussian functions $g_\alpha(\omega) = \frac{1}{2\sqrt{\pi\alpha}}e^{-\omega^2/4\alpha}$, in place of $\sum_n \lambda_n e^{-in\omega}$ and let $\alpha \to 0$, in order to complete the proof of the lemma. $\square$

Let $V_n = \operatorname{span}\{\phi_{n,k} : k\}$. Then, (I') $\Rightarrow \bigcap_n V_n = \{0\}$ and (III) $\Rightarrow \overline{\bigcup_n V_n} = L^2(\mathbb{R})$. Note that (II) $\Leftrightarrow\ \phi \in V_{-1} \ \Leftrightarrow\ V_{n-1} \supset V_n$ for all n.

It is our goal to show how (I'), (II) and (III) generate the construction of a Riesz basis $\{\psi_{n,k} : k\}$ of $L^2(\mathbb{R})$ from a wavelet ψ which is an ε-isometric image of another wavelet, depending on the properties of ϕ. First, we establish an easy consequence of (I')

LEMMA 3.2. *Let* $\phi \in L^2(\mathbb{R})$ *satisfy (I') and (II). Let* $m_0(\omega) = \frac{1}{\sqrt{2}} \sum_n h_n e^{-in\omega}$. *Then,*

(1) $\dfrac{2\pi}{1+\varepsilon} \leq \sum_n |h_n|^2 \leq 2\pi(1+\varepsilon)$

(2) $\dfrac{1}{(1+\varepsilon)^2} \leq |m_0(\omega)|^2 + |m_0(\omega+\pi)|^2 \leq (1+\varepsilon)^2$.

PROOF. From (II), we get

$$1 = \|\phi\|^2 = 2\left\|\sum_n h_n \phi(2x-n)\right\|^2 = \int_0^{2\pi} \left|\sum_n h_n e^{-in\omega}\right|^2 \sum_{l\in\mathbb{Z}} \left|\widehat{\phi}(\omega+2\pi l)\right|^2 d\omega$$

and from (I'), it follows that

$$\frac{1}{1+\varepsilon}\int_0^{2\pi} \left|\sum_n h_n e^{-in\omega}\right|^2 d\omega \leq 1 \leq (1+\varepsilon)\int_0^{2\pi} \left|\sum_n h_n e^{-in\omega}\right|^2 d\omega$$

which yields (1) through Parseval.

For the proof of (2), we note that $\widehat{\phi}(\omega) = m_0(\omega/2)\widehat{\phi}(\omega/2)$. Hence, (I') implies

$$\frac{1}{1+\varepsilon} \leq \sum_l |m_0(\omega+\pi l)|^2 \left|\widehat{\phi}(\omega+\pi l)\right|^2 \leq 1+\varepsilon \quad \text{a.e.}$$

We then split the sum into even and odd l's, use the 2π-periodicity of m_0 and $\sum_l \left|\widehat{\phi}(\omega+2\pi l)\right|^2 = \sum_l \left|\widehat{\phi}(\omega+(2l+1)\pi)\right|^2$ a.e. in order to write $\frac{1}{1+\varepsilon} \leq \left[|m_0(\omega)|^2 + |m_0(\omega+\pi)|^2\right] \sum_l \left|\widehat{\phi}(\omega+2\pi l)\right|^2 \leq 1+\varepsilon$ a.e. Then, we apply the estimates of Lemma 3.1 to conclude. □

Define ψ by $\widehat{\psi}(\omega) = e^{i\omega/2}\overline{m_0(\omega/2+\pi)}\widehat{\phi}(\omega/2)$. Let $f = \sum_n f_n \phi_{-1,n} \in W$ and $m_f(\omega) = \frac{1}{\sqrt{2}} \sum_n f_n e^{-in\omega}$. Then the next lemma follows directly from (I) and (II)

LEMMA 3.3. *If* $\phi \in L^2(\mathbb{R})$ *satisfies (I) and (II), then*

(1) $\dfrac{\|f\|}{1+\varepsilon} \leq \left(\sum_n |f_n|^2\right) \leq \dfrac{\|f\|}{1-\varepsilon}$,

(2) $\dfrac{\|f\|}{(1+\varepsilon)\sqrt{2}} \leq \|m_f\| \leq \dfrac{\|f\|}{(1-\varepsilon)\sqrt{2}}$,

where $f = \sum_n f_n \phi_{-1,n} \in W$.

Now, since $\widehat{f}(\omega) = m_f(\omega/2)\widehat{\phi}(\omega/2)$, we essentially follow classical calculations [4] to write

$$\left[m_f(\omega/2)\overline{m_0(\omega/2)} + m_f(\omega/2+\pi)\overline{m_0(\omega/2+\pi)}\right] \sum_l \left|\widehat{\phi}(\omega/2+2\pi l)\right|^2 = 0$$

for any $f \in W_0 = V_{-1}V_0$ (i.e $f \perp \phi_{0,n}$ for all n). But $\frac{1}{1+\varepsilon} \le \sum\limits_l \left|\widehat{\phi}(\omega/2+2\pi l)\right|^2$ from Lemma 3.1. Hence,

$$m_f(\omega/2)\overline{m_0(\omega/2)} + m_f(\omega/2+\pi)\overline{m_0(\omega/2+\pi)} = 0.$$

On the other hand, since $\frac{1}{(1+\varepsilon)^2} \le |m_0(\omega)|^2 + |m_0(\omega+\pi)|^2$, $m_0(\omega)$ and $m_0(\omega+\pi)$ cannot vanish together on a set of nonzero measure; choose a 2π-periodic function $\lambda(\omega)$ such that $m_f(\omega) = \lambda(\omega)\overline{m_0(\omega+\pi)}$ a.e. and $\lambda(\omega)+\lambda(\omega+\pi) = 0$ a.e. Set $\nu(\omega) = e^{-i\omega}\lambda(\omega/2)$. Then, ν is 2π-periodic and $\widehat{f}(\omega) = e^{i\omega/2}\overline{m_0(\omega/2+\pi)}\nu(\omega)\widehat{\phi}(\omega/2)$. Hence, $\widehat{f}(\omega) = \nu(\omega)\widehat{\psi}(\omega)$ with $\int\limits_0^{2\pi} |\nu(\omega)|^2\, d\omega = 2\int\limits_0^{2\pi} |\lambda(\omega)|^2\, d\omega$.

We are now in position to prove the following result

Theorem 3.4. *Let $\phi \in L^2(\mathbb{R})$, $\|\phi\| = 1$, and $0 \le \varepsilon < 1$ satisfy (I'), (II) and (III). Then there exist $\psi, \psi^\perp \in L^2(\mathbb{R})$, and a Gram-Schmidt operator γ for $\{\psi_{0,n}\}$ such that*

(1) *$\{\psi_{n,k}\}$ is a Riesz basis for $L^2(\mathbb{R})$.*
(2) *$\psi = \gamma(\psi^\perp)$, where*

$$\psi = \sum_n (-1)^{n-1} h_{-n-1}\phi_{-1,n}, \quad \psi^\perp = \sum_n (-1)^{n-1} h_{-n-1}\phi_n^\perp$$

and $\gamma(\phi_n^\perp) = \phi_{0,n}$, with the sequence $\{h_n\}$ defined in (II).

Proof. In view of the above calculations and remarks, it remains only to show that $\{\psi_{0,n}\}$ is a Riesz basis for W_0. We prove that every $f \in W$ has a unique decomposition $f = \sum\limits_n f_n\psi_{0,n}$ where $\sum\limits_n |f_n|^2 < \infty$ or equivalently show that $\widehat{f}(\omega) = g(\omega)\widehat{\psi}(\omega)$, where g is a 2π-periodic function in $L^2(0, 2\pi)$.

Indeed, let $g \equiv \nu$ as defined above. Then,

$$\begin{aligned}
\int_0^{2\pi} |\nu(\omega)|^2 \, d\omega &\leq 2(1+\varepsilon)^2 \int_0^{2\pi} |\lambda(\omega)|^2 \left(|m_0(\omega)|^2 + |m_0(\omega+\pi)|^2 \right) d\omega \\
&\leq 2(1+\varepsilon)^2 \int_0^{2\pi} |\lambda(\omega)|^2 \, |m_0(\omega+\pi)|^2 \, d\omega \\
&= 2(1+\varepsilon)^2 \int_0^{2\pi} |m_f(\omega)|^2 \, d\omega \leq \left(\frac{1+\varepsilon}{1-\varepsilon} \right)^2 \|f\|^2
\end{aligned}$$

follows from Lemma 3.3. Finally, note that $\widehat{\psi}(\omega) = e^{i\omega/2}\overline{m_0(\omega/2+\pi)}\widehat{\phi}(\omega/2)$ is equivalent to $\psi(x) = \sqrt{2}\sum_n (-1)^{n-1} h_{-n-1}\phi(2x-n)$. Also, $\sum_n |h_n|^2 < \infty$ and the orthonormality of $\{\phi_n^{\perp}\}$ imply that $\psi^{\perp} \in L^2(\mathbb{R})$. ◻

Remark 3.5. When the admissibility condition $C_{\psi^{\perp}} = 2\pi \int \frac{|\psi^{\perp}(\omega)|^2}{|\omega|} d\omega < \infty$ is satisfied (for eg. if $\psi^{\perp} \in L^2(\mathbb{R})$ with $\left|\widehat{\psi}^{\perp}(\omega)\right| \leq K|\omega|^{\alpha}$ or equivalently $\widehat{\psi}^{\perp}(0) = 0$ or $\sum_n (-1)^{n-1} h_{-n-1}\widehat{\phi}_n^{\perp}(0) = 0$; same if $\phi_n^{\perp} = \phi_{0,n}^{\sharp}$, where $\phi^{\sharp}$ is another scaling function), then $\psi^{\perp}$ generates an orthonormal wavelet.

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