GLASNIK MATEMATIČKI
Vol. 38(58)(2003), 331 – 342

BANACH-STEINHAUS THEOREMS FOR BOUNDED LINEAR OPERATORS WITH VALUES IN A GENERALIZED 2-NORMED SPACE

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Abstract. In this paper we will prove Banach-Steinhaus Theorems for some families of bounded linear operators from a normed space into a generalized 2-normed space.

1. Introduction

In 1964 S.Gähler introduced the concept of linear 2-normed spaces and he has investigated many important properties and examples for the above spaces ([1, 2]).

Definition 1.1 ([1]). *Let X be a real linear space of dimension greater than 1 and let $\| \cdot , \cdot \|$ be a real valued function on $X \times X$ satisfying the following four properties:*

(G1) $\|x, y\| = 0$ *if and only if the vectors x and y are linearly dependent;*
(G2) $\|x, y\| = \|y, x\|$*;*
(G3) $\|x, \alpha y\| = | \alpha | \cdot \|x, y\|$ *for every real number α;*
(G4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ *for every $x, y, z \in X$.*

The function $\| \cdot , \cdot \|$ will be called a 2-norm on X and the pair $(X, \| \cdot , \cdot \|)$ a linear 2-normed space.

In [3] and [4] we gave a generalization of the Gähler's 2-normed space. Namely a generalized 2-norm need not be symmetric and satisfy the first condition of the above definition.

2000 *Mathematics Subject Classification.* 46A99, 46A32.
Key words and phrases. 2-normed space, Banach-Steinhaus theorems, 2-norm in the space of linear operators.

Definition 1.2 ([3]). *Let X and Y be real linear spaces. Denote by $\mathcal{D}$ a non-empty subset of $X \times Y$ such that for every $x \in X$, $y \in Y$ the sets $\mathcal{D}_x = \{y \in Y;\ (x,y) \in \mathcal{D}\}$ and $\mathcal{D}^{\,y} = \{x \in X;\ (x,y) \in \mathcal{D}\}$ are linear subspaces of the space Y and X, respectively.*

A function $\| \cdot\,, \cdot \| \colon \mathcal{D} \to [0,\infty)$ will be called a generalized 2-norm on $\mathcal{D}$ if it satisfies the following conditions:

- (N1) $\|x, \alpha y\| = |\,\alpha\,| \cdot \|x,y\| = \|\alpha x, y\|$ *for any real number α and all $(x,y) \in \mathcal{D}$;*
- (N2) $\|x, y+z\| \le \|x,y\| + \|x,z\|$ *for $x \in X$, $y,z \in Y$ such that (x,y), $(x,z) \in \mathcal{D}$;*
- (N3) $\|x+y, z\| \le \|x,z\| + \|y,z\|$ *for $x,y \in X$, $z \in Y$ such that (x,z), $(y,z) \in \mathcal{D}$.*

The set $\mathcal{D}$ is called a 2-normed set.

In particular, if $\mathcal{D} = X \times Y$, the function $\| \cdot\,, \cdot \|$ will be called a generalized 2-norm on $X \times Y$ and the pair $(X \times Y, \| \cdot\,, \cdot \|)$ a generalized 2-normed space. Moreover, if $X = Y$, then the generalized 2-normed space will be denoted by $(X, \| \cdot\,, \cdot \|)$.

In [3] and [4] we considered properties of generalized 2-normed spaces on $X \times Y$. In what follows we shall use the following results:

Theorem 1.3 ([3]). *Let $(X \times Y, \| \cdot\,, \cdot \|)$ be a generalized 2-normed space. Then the family $\mathcal{B}$ of all sets defined by*

$$\bigcap_{i=1}^{n} \{x \in X;\ \|x, y_i\| < \varepsilon\},$$

where $y_1, y_2, ..., y_n \in Y, n \in N$ and $\varepsilon > 0$, forms a complete system of neighborhoods of zero for a locally convex topology in X.

We will denote it by the symbol $\mathcal{T}(X,Y)$. Similarly, we have the preceding theorem for a topology $\mathcal{T}(Y,X)$ in the space Y. In the case when $X = Y$ we will denote the above topologies as follows: $\mathcal{T}_1(X) = \mathcal{T}(X,Y)$ and $\mathcal{T}_2(X) = \mathcal{T}(Y,X)$.

Theorem 1.4 ([4]). *Let $(X \times Y, \| \cdot\,, \cdot \|)$ be a generalized 2-normed space. Let Σ be a directed set.*

- (a) *A net $\{x_\sigma; \sigma \in \Sigma\}$ is convergent to $x_o \in X$ in $(X, \mathcal{T}(X,Y))$ if and only if for all $y \in Y$ and $\varepsilon > 0$ there exists $\sigma_o \in \Sigma$ such that $\|x_\sigma - x_o, y\| < \varepsilon$ for all $\sigma \ge \sigma_o$.*
- (b) *A net $\{y_\sigma; \sigma \in \Sigma\}$ is convergent to $y_o \in Y$ in $(Y, \mathcal{T}(Y,X))$ if and only if for all $x \in X$ and $\varepsilon > 0$ there exists $\sigma_o \in \Sigma$ such that $\|x, y_\sigma - y_o\| < \varepsilon$ for all $\sigma \ge \sigma_o$.*

Theorem 1.5 ([4]). *Let $(X \times Y, \| \cdot\,, \cdot \|)$ be a generalized 2-normed space. If the generalized 2-norm $\| \cdot\,, \cdot \| \colon X \times Y \to [0,\infty)$ is jointly continuous and* *a sequence* $\{(x_n, y_n); n \in N\} \subset X \times Y$ *is convergent, then the sequence of 2-norms* $\{\|x_n, y_n\|;\ n \in N\}$ *is bounded.*

Definition 1.6 ([4]). *Let* $(X \times Y, \|\cdot, \cdot\|)$ *be a generalized 2-normed space. A sequence* $\{x_n; n \in N\} \subset X$ *is called a Cauchy sequence if for every* $y \in Y$ *and* $\varepsilon > 0$ *there exists a number* $n_o \in N$ *such that inequality* $n, m > n_o$ *implies* $\|x_n - x_m, y\| < \varepsilon$.

Definition 1.7 ([4]). *Let* $(X \times Y, \|\cdot, \cdot\|)$ *be a generalized 2-normed space. A space* $(X, \mathcal{T}(X, Y))$ *is called sequentially complete if every Cauchy sequence in* X *is convergent in this space.*

By analogy we obtain definitions of a Cauchy sequence in the space Y and the sequential completeness of the space $(Y, \mathcal{T}(Y, X))$.

In what follows $L(X, Y)$ stands for the linear space of all linear operators from X with values in Y, where X, Y are real linear spaces.

Definition 1.8 ([5]). *Let* X *be a real normed space and* $\mathcal{Y} \subset Y \times Y$ *be a 2-normed set, where* Y *denotes a real linear space. A set* $\mathcal{M}$ *is defined as follows:*

$$\begin{aligned}\mathcal{M} \quad = \quad & \{(f,g) \in L(X,Y)^2; \forall_{x \in X}(f(x), g(x)) \in \mathcal{Y} \\ & \wedge \exists_{M>0} \forall_{x \in X} \|f(x), g(x)\| \leq M \cdot \|x\|^2\}.\end{aligned}$$

The set $\mathcal{M}$ defined in Definition 1.8 has the following property:
For every $f, g \in L(X, Y)$ the sets

$$\mathcal{M}^g = \{f' \in L(X,Y); (f', g) \in \mathcal{M}\} \text{ and } \mathcal{M}_f = \{g' \in L(X,Y); (f, g') \in \mathcal{M}\}$$

are linear subspaces of the space $L(X, Y)$.

For $(f, g) \in \mathcal{M}$ we introduce the number

$$\|f, g\| = \inf\{M > 0; \forall_{x \in X} \|f(x), g(x)\| \leq M \cdot \|x\|^2\}. \tag{1.1}$$

Then

$$\|f(x), g(x)\| \leq \|f, g\| \cdot \|x\|^2 \text{ for all } x \in X; \tag{1.2}$$

$$\begin{aligned}\|f, g\| &= \sup\{\|f(x), g(x)\|;\ x \in X \wedge \|x\| = 1\} \\ &= \sup\{\|f(x), g(x)\|;\ x \in X \wedge \|x\| \leq 1\} \\ &= \sup\left\{\frac{\|f(x), g(x)\|}{\|x\|^2};\ x \in X \wedge \|x\| \neq 0\right\}.\end{aligned} \tag{1.3}$$

Moreover, the set $\mathcal{M}$ is a 2-normed set with the 2-norm defined by the formula (1.1) (cf. [5]).

Definition 1.9 ([5]). *Let X be a real normed space and $\mathcal{Y} \subset Y \times Y$ be a 2-normed set, where Y denotes a real linear space. A set $\mathcal{N}$ is defined as follows:*

$$\mathcal{N} = \Big\{(f,g) \in L(X,Y)^2; \forall_{x,y\in X}(f(x),g(y)) \in \mathcal{Y} \wedge \exists_{M>0}\forall_{x,y\in X}\|f(x),g(y)\| \le M\cdot\|x\|\cdot\|y\|\Big\}.$$

The set $\mathcal{N}$ defined in Definition 1.9 has similar properties:
For every $f, g \in L(X,Y)$ the sets

$$\mathcal{N}^g = \{f^{'} \in L(X,Y); (f^{'},g) \in \mathcal{N}\} \text{ and } \mathcal{N}_f = \{g^{'} \in L(X,Y); (f,g^{'}) \in \mathcal{N}\}$$

are linear subspaces of the space $L(X,Y)$.

For $(f,g) \in \mathcal{N}$ we introduce the number

$$\|f,g\| = \inf\{M > 0; \forall_{x,y\in X}\|f(x),g(y)\| \le M\cdot\|x\|\cdot\|y\|\}. \tag{1.4}$$

Then

$$\|f(x),g(y)\| \le \|f,g\|\cdot\|x\|\cdot\|y\| \text{ for all } x,y \in X; \tag{1.5}$$

$$\begin{aligned}\|f,g\| &= \sup\{\|f(x),g(y)\|;\ x,y\in X \wedge \|x\| = \|y\| = 1\}\\ &= \sup\{\|f(x),g(y)\|;\ x,y\in X \wedge \|x\| \le 1,\ \|y\| \le 1\}\\ &= \sup\Big\{\frac{\|f(x),g(y)\|}{\|x\|\cdot\|y\|};\ x,y\in X \wedge \|x\| \ne 0, \|y\| \ne 0\Big\}.\end{aligned} \tag{1.6}$$

Moreover, the set $\mathcal{N}$ is a 2-normed set with the 2-norm defined by the formula (1.4) (cf. [5]).

2. Banach-Steinhaus Theorems for bounded linear operators

In this section we will consider properties of sequences of operators, which are contained in $\mathcal{M}^g, \mathcal{M}_f$ or $\mathcal{N}^g, \mathcal{N}_f$ for some $f, g \in L(X,Y)$. Moreover we will investigate sequences $\{(f_n, g_n); n \in N\}$ from $\mathcal{M}$ or $\mathcal{N}$. In every case we will formulate Banach-Steinhaus Theorems. Because any theorem for sequences of operators from $\mathcal{M}^g$ or $\mathcal{N}^g$ is also true (after making necessary changes) for sequences of operators from $\mathcal{M}_f$ or $\mathcal{N}_f$, we will give only one version of theorems.

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a normed space, $(Y, \|\cdot,\cdot\|)$ a generalized 2-normed space and $g \in L(X,Y)$. Then:*

(a) *If a sequence $\{f_n, n \in N\} \subset \mathcal{M}^g$ and the sequence of 2 -norms $\{\|f_n, g\|; n \in N\}$ is bounded, then for every $x \in X$ the sequence $\{\|f_n(x), g(x)\|, n \in N\}$ is bounded.*

(b) *If a sequence $\{f_n, n \in N\} \subset \mathcal{N}^g$ and the sequence of 2 -norms $\{\|f_n, g\|; n \in N\}$ is bounded, then for every $x, y \in X$ the sequence $\{\|f_n(x), g(y)\|, n \in N\}$ is bounded.*

Proof. (a) Let $\|f_n, g\| \le M$ for every $n \in N$. Then for $x \in X$ we have

$$\|f_n(x), g(x)\| \le \|f_n, g\| \cdot \|x\|^2 \le M \cdot \|x\|^2.$$

Hence for every $x \in X$ the sequence $\{\|f_n(x), g(x)\|; n \in N\}$ is bounded by the number $M \cdot \|x\|^2$.
(b) If $\|f_n, g\| \le M$ for every $n \in N$, then for $x, y \in X$ we have

$$\|f_n(x), g(y)\| \le \|f_n, g\| \cdot \|x\| \cdot \|y\| \le M \cdot \|x\| \cdot \|y\|.$$

Thus for every $x, y \in X$ the sequence $\{\|f_n(x), g(y)\|; n \in N\}$ is bounded by the number $M \cdot \|x\| \cdot \|y\|$. □

Theorem 2.2. *Let $(X, \|\cdot\|)$ be a Banach space, $(Y, \|\cdot, \cdot\|)$ a generalized 2-normed space and $\{f_n; n \in N\}$ a sequence of elements from $\mathcal{N}^g$ for some $g \in L(X, Y)$. Then the following conditions are equivalent:*

(a) *The sequence of 2-norms $\{\|f_n, g\|; n \in N\}$ is bounded;*
(b) $\exists_{M>0} \forall_{x,y \in X, \|x\| \le 1, \|y\| \le 1} \forall_{n \in N} \|f_n(x), g(y)\| \le M$;
(c) *The following conditions are true:*
  (i) $\forall_{x \in X} \exists_{M_x > 0} \forall_{y \in X, \|y\| \le 1} \forall_{n \in N} \|f_n(x), g(y)\| \le M_x$;
  (ii) $\forall_{y \in X} \exists_{M_y > 0} \forall_{x \in X, \|x\| \le 1} \forall_{n \in N} \|f_n(x), g(y)\| \le M_y$.

Proof. At first let us suppose that the sequence of 2-norms $\{\|f_n, g\|; n \in N\}$ is bounded. From this it follows that there exists a positive number M such that $\|f_n, g\| \le M$ for each $n \in N$. Thus for $x, y \in X, \|x\| \le 1, \|y\| \le 1$ and $n \in N$ we have $\|f_n(x), g(y)\| \le \|f_n, g\| \cdot \|x\| \cdot \|y\| \le M$.

Now, let the condition (b) be satisfied. We fix $x \in X \setminus \{0\}$. Then for each $y \in X, \|y\| \le 1$ and $n \in N$ we obtain the inequalities:

$$\|f_n(x), g(y)\| = \Big\|f_n\Big(\frac{x}{\|x\|} \cdot \|x\|\Big), g(y)\Big\| = \|x\| \cdot \Big\|f_n\Big(\frac{x}{\|x\|}\Big), g(y)\Big\| \le M \cdot \|x\|.$$

If we choose $M_x = M \cdot \|x\|$, then we have the condition (i). Moreover, for $x = 0$ the condition (i) is satisfied for every positive number M_x. Analogously, taking $M_y = M \cdot \|y\|$ for each $y \in X \setminus \{0\}$ and any positive number for $y = 0$ we obtain (ii).

Conversely, let (i) and (ii) be satisfied. In $X \times X$ let us define a norm by the formula:

$$\|(x, y)\|_\star = \|x\| + \|y\| \text{ for each } (x, y) \in X \times X.$$

It is easy to verify that $(X \times X, \|\cdot\|_\star)$ is a Banach space. Put

$$A_{nm} = \{(x, y) \in X \times X; \|f_n(x), g(y)\| \le m\}$$

and

$$B_m = \bigcap_{n=1}^{\infty} A_{nm}$$

for $m, n \in N$. We shall show that sets B_m are closed in $(X \times X, \|\cdot\|_\star)$ for each $m \in N$.

At first we shall show that sets A_{nm} are closed in this space. Let $m, n \in N$ and let $\{(x_k, y_k); k \in N\} \subset A_{nm}$ be a sequence converging to $(x^{'}, y^{'}) \in X \times X$. Then

$$\|f_n(x_k), g(y_k)\| \leq m \text{ and } \|(x_k, y_k) - (x^{'}, y^{'})\|_\star \longrightarrow 0, k \to \infty.$$

The last condition is equivalent to the following: $\|x_k - x^{'}\| \to 0$ and $\|y_k - y^{'}\| \to 0$, which implies the convergence of the sequences $\{x_k; k \in N\}$, $\{y_k; k \in N\}$. As a consequence these sequences are bounded. There exists $K > 0$ such that the inequalities $\|x_k\| \leq K$, $\|y_k\| \leq K$ are true for each $k \in N$. Using these results we get

$$\|f_n(x^{'}), g(y^{'})\| \leq m + K \cdot \|f_n, g\| \cdot \|x_k - x^{'}\| + K \cdot \|f_n, g\| \cdot \|y_k - y^{'}\| \\ + \|f_n, g\| \cdot \|x_k - x^{'}\| \cdot \|y_k - y^{'}\|.$$

Letting $k \to \infty$ we obtain $\|f_n(x^{'}), g(y^{'})\| \leq m$, which means that $(x^{'}, y^{'}) \in A_{nm}$. Therefore the sets A_{nm} are closed for each $n, m \in N$, and hence the sets B_m are also closed in $(X \times X, \| \cdot \|_\star)$.

Now, we shall show that the equality

$$X \times X = \bigcup_{m=1}^{\infty} B_m$$

is true. Let $x, y \in X, x \neq 0$. Then $\|\frac{x}{\|x\|}\| = 1$. By virtue (ii) there exists $M_y > 0$ such that

$$\left\|f_n\Big(\frac{x}{\|x\|}\Big), g(y)\right\| \leq M_y \text{ for each } n \in N.$$

Thus $\|f_n(x), g(y)\| \leq M_y \cdot \|x\|$ for each $n \in N$.

If $x = 0$, then $\|x\| \leq 1$ and $\|f_n(x), g(y)\| = \|0, g(y)\| = 0 = M_y \cdot \|0\|$. As a consequence, for every $x, y \in X$ the sequence $\{\|f_n(x), g(y)\|; n \in N\}$ is bounded. From this it follows that for any point $(x, y) \in X \times X$ there exists $n \in N$ such that $\|f_n(x), g(y)\| \leq m$ for every $m \in N$, i.e.

$$(x, y) \in \bigcup_{m=1}^{\infty} B_m.$$

Thus

$$X \times X = \bigcup_{m=1}^{\infty} B_m.$$

By the well known Baire theorem there exists a set B_{m_o} with non-empty interior. Therefore B_{m_o} contains some closed ball with the center (x_o, y_o) and radius r. Denote it by $\mathcal{K}((x_o, y_o), r)$. Thus for each $n \in N$ and $(x, y) \in \mathcal{K}((x_o, y_o), r)$ we have $\|f_n(x), g(y)\| \leq m_o$.

Let us take $x, y \in X$ such that $\|x\| \leq \frac{r}{2}$ and $\|y\| \leq \frac{r}{2}$. Then

$$\|(x, y)\|_\star = \|x\| + \|y\| \leq r \text{ and } \|(x, y)\|_\star = \|(x + x_o, y + y_o) - (x_o, y_o)\|_\star \leq r.$$

Therefore $\|f_n(x+x_o), g(y+y_o)\| \le m_o$. In particular $\|f_n(x_o), g(y_o)\| \le m_o$. Thus

$$\begin{aligned}\|f_n(x), g(y)\| &\le \|f_n(x+x_o), g(y+y_o)\| + \|f_n(x+x_o), g(y_o)\| \\ &\quad + \|f_n(x_o), g(y+y_o)\| + \|f_n(x_o), g(y_o)\| \\ &\le 2m_o + \|f_n(x)+f_n(x_o), g(y_o)\| + \|f_n(x_o), g(y)+g(y_o)\| \\ &\le 4m_o + \|f_n(x), g(y_o)\| + \|f_n(x_o), g(y)\|.\end{aligned}$$

So we have shown that the inequalities $\|x\| \le \frac{r}{2}$ and $\|y\| \le \frac{r}{2}$ imply the condition

$$\|f_n(x), g(y)\| \le 4m_o + \|f_n(x), g(y_o)\| + \|f_n(x_o), g(y)\|.$$

Now, let $x, y \in X, \|x\| \le 1$ and $\|y\| \le 1$. Because $\|\frac{r}{2}x\| \le \frac{r}{2}$ and $\|\frac{r}{2}y\| \le \frac{r}{2}$, then

$$\|f_n(\frac{r}{2}x), g(\frac{r}{2}y)\| \le 4m_o + \|f_n(\frac{r}{2}x), g(y_o)\| + \|f_n(x_o), g(\frac{r}{2}y)\|.$$

As a consequence we obtain

$$\|f_n(x), g(y)\| \le \frac{16m_o}{r^2} + \frac{2}{r}(\|f_n(x), g(y_o)\| + \|f_n(x_o), g(y)\|)$$

for each $n \in N$. Applying (i) we have that there exists $M_{x_o} > 0$ such that for every $y \in X, \|y\| \le 1$ and $n \in N$ the inequality $\|f_n(x_o), g(y)\| \le M_{x_o}$ is true. However the assumption (ii) implies there exists $M_{y_o} > 0$ such that for every $x \in X, \|x\| \le 1$ and $n \in N$ the inequality $\|f_n(x), g(y_o)\| \le M_{y_o}$ is satisfied. So

$$\|f_n(x), g(y)\| \le \frac{16m_o}{r^2} + \frac{2}{r} \cdot (M_{y_o} + M_{x_o})$$

for each $n \in N$ and $x, y \in X$ such that $\|x\| \le 1, \|y\| \le 1$. Therefore

$$\begin{aligned}\|f_n, g\| &= \sup\{\|f_n(x), g(y)\|;\ x, y \in X \wedge \|x\| \le 1, \|y\| \le 1\} \\ &\le \frac{16m_o + 2r(M_{x_o} + M_{y_o})}{r^2}\end{aligned}$$

for each $n \in N$. So the sequence $\{\|f_n, g\|; n \in N\}$ is bounded and the proof is completed. ∎

Let $g \in L(X,Y)$. A sequence $\{f_n; n \in N\} \subset \mathcal{N}^g$ is pointwise convergent to $f \in L(X,Y)$, if

$$\forall_{x\in X}\forall_{z\in Y} \lim_{n\to\infty} \|f_n(x) - f(x), z\| = 0$$

(cf. [4]). However, if g is the operator from X on Y, then the sequence $\{f_n; n \in N\} \subset \mathcal{N}^g$ is pointwise convergent to $f \in L(X,Y)$, if

$$\forall_{x\in X}\forall_{y\in Y} \lim_{n\to\infty} \|f_n(x) - f(x), g(y)\| = 0.$$

We will use the above note in the following theorem.

Theorem 2.3. *Let $(X, \|\cdot\|)$ be a Banach space, $(Y, \|\cdot,\cdot\|)$ a generalized 2-normed space and g a linear operator from X on Y. If $\{f_n; n \in N\} \subset \mathcal{N}^g$ is pointwise convergent to $f \in L(X,Y)$ and satisfies one of the conditions (a), (b), (c) from Theorem 2.2, then $f \in \mathcal{N}^g$.*

Proof. From Theorem 2.2 the sequence of 2-norms $\{\|f_n, g\|; n \in N\}$ is bounded. Thus there exists $M > 0$ such that $\|f_n, g\| \leq M$ for each $n \in N$. For points $x, y \in X$ we have

$$\|f_n(x), g(y)\| \leq \|f_n, g\| \cdot \|x\| \cdot \|y\| \leq M \cdot \|x\| \cdot \|y\|.$$

So $\|f(x), g(y)\| \leq \|f(x) - f_n(x), g(y)\| + M \cdot \|x\| \cdot \|y\|$. Letting $n \to \infty$ in the above inequality we obtain

$$\|f(x), g(y)\| \leq M \cdot \|x\| \cdot \|y\|,$$

which implies $f \in \mathcal{N}^g$. □

Definition 2.4 ([6]). *A set A of elements of a normed space X is said to be linearly dense in X, if the set X_o of all linear combinations of elements from A is dense in X.*

Theorem 2.5. *Let A be a linearly dense set in a Banach space $(X, \|\cdot\|)$, $(Y, \|\cdot,\cdot\|)$ a generalized 2-normed space such that $(Y, \mathcal{T}_1(Y))$ is a Hausdorff sequentially complete space. Let g be a linear operator from X on Y and $\{f_n; n \in N\} \subset \mathcal{N}^g$. The following conditions are equivalent:*

(a) *The sequence $\{f_n; n \in N\}$ is pointwise convergent to $f \in L(X,Y)$ and the conditions (i),(ii) from Theorem 2.2 are satisfied.*
(b) *The sequence $\{f_n; n \in N\}$ is pointwise convergent to $f \in \mathcal{N}^g$ on the set A and the sequence of 2-norms $\{\|f_n, g\|; n \in N\}$ is bounded.*

Proof. If the sequence $\{f_n(x); n \in N\}$ is convergent to $f(x) \in Y$ for each $x \in X$, then it is convergent also for $x \in A \subset X$. Moreover - this follows from Theorem 2.2 and Theorem 2.3 - the sequence $\{\|f_n, g\|; n \in N\}$ is bounded and $f \in \mathcal{N}^g$.

Now, we will suppose that the sequence $\{f_n; n \in N\}$ is pointwise convergent to $f \in \mathcal{N}^g$ on the set A and the sequence of 2-norms $\{\|f_n, g\|; n \in N\}$ is bounded. By Theorem 2.2 the conditions (i),(ii) hold. Let X_o be the vector subspace of the Banach space X generated by A. So X_o is a normed space.

Let $x, y \in X_o$. Then $x = a_1x_1 + \cdots + a_kx_k$, $y = b_1y_1 + \cdots + b_ty_t$, where $a_i, b_j \in \mathcal{R}, x_i, y_j \in A, i = 1, 2, \ldots, k, j = 1, 2, \ldots, t, k, t \in N$. Thus, it follows from assumptions on f_n, f, g that

$$\begin{aligned}&\|f_n(x) - f(x), g(y)\| = \\ &= \|a_1(f_n(x_1) - f(x_1)) + \cdots + a_k(f_n(x_k) - f(x_k)), b_1g(y_1) + \cdots + b_tg(y_t)\|.\end{aligned}$$

Using properties of 2-norms we get:

$$\|f_n(x) - f(x), g(y)\| \leq \sum_{i=1}^{k} \sum_{j=1}^{t} | a_i b_j | \cdot \|f_n(x_i) - f(x_i), g(y_j)\|.$$

Because

$$\lim_{n\to\infty} \|f_n(x_i) - f(x_i), g(y_j)\| = 0 \text{ for each } x_i, y_j \in A,$$

then

$$\lim_{n\to\infty} \|f_n(x) - f(x), g(y)\| = 0,$$

i.e. the sequence $\{f_n; n \in N\}$ is convergent to f on X_o.

Let $\|f_n, g\| \leq M$ for every $n \in N$. Let us take a number $\varepsilon > 0, x \in X$ and $y \in X$ such that $y \neq 0$. Since X_o is a dense set in X, we can choose $x_o \in X_o, x_o \neq 0$ such that

$$\|x - x_o\| < \frac{\varepsilon}{6M \cdot \|y\|}.$$

Moreover there exists $y_o \in X_o$ with the property

$$\|y - y_o\| < \frac{\varepsilon}{6M \cdot \|x_o\|}.$$

The sequence $\{f_n(x_o); n \in N\}$ is convergent in $(Y, \mathcal{T}_1(Y))$, so it is a Cauchy sequence in this space. Therefore there exists a number n_o such that

$$\|f_n(x_o) - f_m(x_o), g(y_o)\| < \frac{\varepsilon}{3} \text{ for each } n, m \geq n_o.$$

As a consequence we obtain

$$\begin{aligned}
&\|f_n(x) - f_m(x), g(y)\| \leq \\
&\quad \leq \|f_n(x) - f_n(x_o), g(y)\| + \|f_n(x_o) - f_m(x_o), g(y)\| \\
&\quad\quad + \|f_m(x_o) - f_m(x), g(y)\| \\
&\quad \leq \|f_n, g\| \cdot \|x - x_o\| \cdot \|y\| + \|f_n(x_o) - f_m(x_o), g(y - y_o) + g(y_o)\| \\
&\quad\quad + \|f_m, g\| \cdot \|x - x_o\| \cdot \|y\| \\
&\quad \leq 2M\|x - x_o\| \cdot \|y\| + \|f_n(x_o) - f_m(x_o), g(y - y_o)\| \\
&\quad\quad + \|f_n(x_o) - f_m(x_o), g(y_o)\| \\
&\quad < 2M\|x - x_o\| \cdot \|y\| + \|f_n(x_o), g(y - y_o)\| + \|f_m(x_o), g(y - y_o)\| + \frac{\varepsilon}{3} \\
&\quad < 2M\frac{\varepsilon}{6M\|y\|}\|y\| + \|f_n, g\| \cdot \|x_o\| \cdot \|y - y_o\| \\
&\quad\quad + \|f_m, g\| \cdot \|x_o\| \cdot \|y - y_o\| + \frac{\varepsilon}{3} \\
&\quad < \frac{2}{3}\varepsilon + 2M\|x_o\| \cdot \|y - y_o\| < \frac{2}{3}\varepsilon + 2M\|x_o\|\frac{\varepsilon}{6M\|x_o\|} = \varepsilon
\end{aligned}$$

for $n, m \geq n_o$. If $y = 0$, then the inequality $\|f_n(x) - f_m(x), g(y)\| = 0 < \varepsilon$ is also true.

Hence we have shown that $\{f_n(x); n \in N\}$ is a Cauchy sequence in $(Y, \mathcal{T}_1(Y))$ for every $x \in X$. Because $(Y, \mathcal{T}_1(Y))$ is a sequentially complete space, then the sequence $\{f_n; n \in N\}$ is pointwise convergent.

Let us denote

$$h(x) = \lim_{n\to\infty} f_n(x) \text{ for every } x \in X.$$

The fact that $(Y, \mathcal{T}_1(Y))$ is a Hausdorff space implies $h(x) = f(x)$ for $x \in A$, i.e. $(h - f)(x) = 0$ for $x \in A$. The operator $h-f$ is linear, thus $(h-f)(x) = 0$ for every $x \in X_o$. Using Theorem 2.3 we see that $h \in \mathcal{N}^g$. Because $\mathcal{N}^g$ is a linear subspace, then $h - f \in \mathcal{N}^g$. Thus there exists a positive number K such that

$$\|(h-f)(x), g(y)\| \leq K \cdot \|x\| \cdot \|y\| \text{ for every } x, y \in X.$$

Let $\varepsilon > 0, x, y \in X, y \neq 0$. Since the set X_o is dense in X we can choose $x_o \in X_o$ such that

$$\|x - x_o\| < \frac{\varepsilon}{K \cdot \|y\|}.$$

Then

$$\begin{aligned} 0 \leq \|(h-f)(x), g(y)\| &= \|(h-f)(x-x_o) + (h-f)(x_o), g(y)\| = \\ &= \|(h-f)(x-x_o), g(y)\| \leq K \cdot \|x - x_o\| \cdot \|y\| < \varepsilon \end{aligned}$$

This gives $\|(h-f)(x), g(y)\| = 0$ for each $x \in X, y \in X \setminus \{0\}$. Thus $h(x) = f(x)$ for every $x \in X$. As a consequence we have shown that the sequence $\{f_n; n \in N\}$ is pointwise convergent to f, which finishes the proof. ◻

THEOREM 2.6. *Let $(X, \|\cdot\|)$ be a Banach space, $(Y, \|\cdot,\cdot\|)$ a generalized 2-normed space such that $(Y, \mathcal{T}_1(Y))$ is a Hausdorff sequentially complete space. Let g be a linear operator from X on Y. If a sequence $\{f_n; n \in N\} \subset \mathcal{N}^g$ is pointwise convergent to $f \in \mathcal{N}^g$ on a linearly dense set A in X and the sequence of 2-norms $\{\|f_n, g\|; n \in N\}$ is bounded, then $\{f_n; n \in N\}$ is pointwise convergent to f and $\|f, g\| \leq \sup\{\|f_n, g\|; n \in N\}$.*

PROOF. It follows from Theorem 2.5 that the sequence $\{f_n(x); n \in N\}$ is convergent in Y to $f(x)$ for every $x \in X$. Let us denote $M = \sup\{\|f_n, g\|; n \in N\}$. Then for every $n \in N$ and $x, y \in X$ such that $\|x\| \leq 1, \|y\| \leq 1$ we have $\|f_n(x), g(y)\| \leq M$. Thus

$$\|f(x), g(y)\| \leq \|f_n(x) - f(x), g(y)\| + \|f_n(x), g(y)\| \leq \|f_n(x) - f(x), g(y)\| + M.$$

By letting $n \to \infty$ we obtain

$$\|f(x), g(y)\| \leq M \text{ for } x, y \in X, \|x\| \leq 1, \|y\| \leq 1.$$

This implies $\|f, g\| \leq M$, which finishes the proof. ◻

Now, let us consider sequences $\{(f_n, g_n); n \in N\}$ from $\mathcal{M}$ or $\mathcal{N}$. Using analogous arguments as in proofs of the foregoing theorems we can show that the following theorems are true.

THEOREM 2.7. *Let $(X, \|\cdot\|)$ be a normed space and $(Y, \|\cdot,\cdot\|)$ a generalized 2-normed space.*

(a) *If $\{(f_n, g_n); n \in N\} \subset \mathcal{M}$ and the sequence of 2-norms $\{\|f_n, g_n\|; n \in N\}$ is bounded, then for every $x \in X$ the sequence $\{\|f_n(x), g_n(x)\|; n \in N\}$ is bounded.*

(b) *If $\{(f_n, g_n); n \in N\} \subset \mathcal{N}$ and the sequence of 2-norms $\{\|f_n, g_n\|; n \in N\}$ is bounded, then for every $x, y \in X$ the sequence $\{\|f_n(x), g_n(y)\|; n \in N\}$ is bounded.*

THEOREM 2.8. *Let $(X, \|\cdot\|)$ be a Banach space, $(Y, \|\cdot,\cdot\|)$ a generalized 2-normed space and $\{(f_n, g_n); n \in N\}$ a sequence of elements from $\mathcal{N}$. Then the following conditions are equivalent:*

(a) *The sequence of 2-norms $\{\|f_n, g_n\|; n \in N\}$ is bounded;*

(b) $\exists_{M>0} \forall_{x,y \in X, \|x\| \leq 1, \|y\| \leq 1} \forall_{n \in N} \|f_n(x), g_n(y)\| \leq M$;

(c) *The following conditions are satisfied:*

  (i) $\forall_{x \in X} \exists_{M_x > 0} \forall_{y \in X, \|y\| \leq 1} \forall_{n \in N} \|f_n(x), g_n(y)\| \leq M_x$;

  (ii) $\forall_{y \in X} \exists_{M_y > 0} \forall_{x \in X, \|x\| \leq 1} \forall_{n \in N} \|f_n(x), g_n(y)\| \leq M_y$.

THEOREM 2.9. *Let $(X, \|\cdot\|)$ be a Banach space, $(Y, \|\cdot,\cdot\|)$ a generalized 2-normed space with the continuous 2-norm. If a sequence $\{(f_n, g_n); n \in N\} \subset \mathcal{N}$ is pointwise convergent to $(f, g) \in L(X, Y)^2$ and one of three conditions (a), (b), (c) of Theorem 2.8 is true, then $(f, g) \in \mathcal{N}$.*

PROOF. Using Theorem 2.8 we have that the sequence of 2-norms $\{\|f_n, g_n\|; n \in N\}$ is bounded, i.e. there exists $M > 0$ such that $\|f_n, g_n\| \leq M$ for each $n \in N$. Let $x, y \in X$ be arbitrary. Then

$$\|f_n(x), g_n(y)\| \leq \|f_n, g_n\| \cdot \|x\| \cdot \|y\| \leq M\|x\| \cdot \|y\|.$$

Since the 2-norm is continuous, then

$$\|f(x), g(y)\| = \lim_{n \to \infty} \|f_n(x), g_n(y)\| \leq M\|x\| \cdot \|y\|,$$

i.e. $(f, g) \in \mathcal{N}$. □

From Theorem 1.5 the following follows

THEOREM 2.10. *Let $(X, \|\cdot\|)$ be a normed space, $(Y, \|\cdot,\cdot\|)$ a generalized 2-normed space. If a sequence $\{(f_n, g_n); n \in N\} \subset \mathcal{N}$ is pointwise convergent to $(f, g) \in L(X, Y) \times L(X, Y)$ and the 2-norm is continuous, then the sequence $\{\|f_n(x), g_n(y)\|; n \in N\}$ is bounded for each $x, y \in X$.*

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Received: 11.07.2002.