GLASNIK MATEMATIČKI
Vol. 38(58)(2003), 343 – 359

EXTENSIONS OF HILBERT C^*-MODULES II

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Abstract. We describe the pullback construction in the category of Hilbert C^*-modules (with a suitable class of morphisms) in terms of pullbacks of underlying C^*-algebras. In the second section the Busby invariant for extensions of Hilbert C^*-modules is introduced and it is proved that each extension is uniquely determined, up to isomorphism, by the corresponding Busby map. The induced extensions of the underlying C^*-algebras as well as of the corresponding linking algebras are also discussed. The paper ends with a Hilbert C^*-module version of a familiar result which states that a C^*-algebra is projective if and only if it is corona projective.

1. Introduction

A (right) Hilbert C^*-module over a C^*-algebra $\mathcal{A}$ is a right $\mathcal{A}$-module V equipped with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle$ such that V is a Banach space with the norm $\|v\| = \|\langle v, v\rangle\|^{1/2}$. We refer to [6] for general facts about Hilbert C^*-modules. The reader may also consult the corresponding chapters in [10] and in [9].

Given two Hilbert C^*-modules V and W over $\mathcal{A}$ and $\mathcal{B}$, respectively, a map $\Phi : V \to W$ is called a morphism of Hilbert C^*-modules if there exists a morphism of C^*-algebras $\varphi : \mathcal{A} \to \mathcal{B}$ such that $\langle \Phi(x), \Phi(y)\rangle = \varphi(\langle x, y\rangle)$ is satisfied for all x and y in V. When the underlying morphism φ has to be specified, the map Φ is said to be a φ-morphism. It is known ([2]) that each φ-morphism is necessarily a linear contraction satisfying $\Phi(xa) = \Phi(x)\varphi(a)$, $x \in V$, $a \in \mathcal{A}$, such that $\operatorname{Ker}\Phi = V_{\operatorname{Ker}\varphi}$, i.e. the kernel of Φ is the ideal submodule of V associated to the kernel of φ. In particular, if V is full, then Φ is an injection if and only if φ is an injection.

2000 *Mathematics Subject Classification.* 46C50, 46L08.
Key words and phrases. Hilbert C^*-module, adjointable operator, extension, multiplier algebra.

Typically, morphisms of this kind arise in the following way: consider an ideal $\mathcal{I}$ of $\mathcal{A}$ and take the corresponding ideal submodule $V_{\mathcal{I}} = V\mathcal{I}$ of a Hilbert $\mathcal{A}$-module V. Then the quotient map $q : V \to V/V_{\mathcal{I}}$ is a π-morphism of Hilbert C^*-modules (with $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ denoting the quotient map on the underlying C^*-algebra). Such morphisms, together with ideal submodules, are discussed in more detail in [2]. Let us also note in passing that the class of all ideal submodules of V coincides with the class of all subbimodules of the $\boldsymbol{K}(V) - \mathcal{A}$ bimodule V (cf. Remark 1.15 in [2]). Alternatively, as it is observed in the concluding Remark in [2], this class can be recognized as the class of all ideals in the linking algebra of V.

A surjective φ-morphism $\Phi : V \to W$ is a unitary operator of Hilbert C^*-modules if the underlying morphism φ is an injection. If this is the case we say that V and W are unitarily equivalent Hilbert C^*-modules. Unitary equivalence of full Hilbert C^*-modules is an equivalence relation.

The present paper is a continuation of [3] and provides necessary tools for a systematic study of extensions of Hilbert C^*-modules. The starting point is the presence of the maximal extension V_d of a given Hilbert C^*-module V. For the convenience of the reader we briefly recall the description and properties of V_d from [3].

Let V be a full Hilbert C^*-module over a (non unital) C^*-algebra $\mathcal{A}$. Denote by $V_d = \boldsymbol{B}(\mathcal{A}, V)$ the Hilbert C^*-module over the multiplier algebra $M(\mathcal{A})$ consisting of all adjointable maps from $\mathcal{A}$ to V with the inner product $\langle r, s\rangle = r^*s$. Let $\Gamma : V \to V_d$ be defined by $\Gamma(v) = r_v$ where $r_v : \mathcal{A} \to V$ denotes the "multiplier" $r_v(a) = va$. Then $(V_d, M(\mathcal{A}), \Gamma)$ is an extension of V in the following sense: if we identify v in V with $\Gamma(v)$ in V_d, then V is an ideal submodule of V_d corresponding to the ideal $\mathcal{A}$ of $M(\mathcal{A})$. The extended module V_d has the following universal property: Let $\varphi : \mathcal{A} \to \mathcal{B}$ be an injective morphism of C^*-algebras such that $\varphi(\mathcal{A})$ is an (essential) ideal in a C^*-algebra $\mathcal{B}$ and let $\lambda : \mathcal{B} \to M(\mathcal{A})$ be the resulting morphism acting as the identity on $\mathcal{A}$. Let W be a Hilbert $\mathcal{B}$-module. Suppose that $\Phi : V \to W$ is a φ-morphism of Hilbert C^*-modules with $\Phi(V) = V\varphi(\mathcal{A})$, so that $(W, \mathcal{B}, \Phi)$ is another (essential) extension of V. Then there exists a λ-morphism $\Lambda : W \to V_d$ such that $\Lambda\Phi = \Gamma$. Since the maps λ and Λ are injections precisely when $\varphi(\mathcal{A})$ is an essential ideal in $\mathcal{B}$, this shows that V_d is the largest essential extension of V ([3], Theorem 1.1).

Further, given an essential extension $(W, \mathcal{B}, \Phi)$ of V, we define a (variant of) strict topology τ_V on W by the family of seminorms $w \mapsto \|\langle \Phi(v), w\rangle\|$, $v \in V$ and $w \mapsto \|w\varphi(a)\|$, $a \in \mathcal{A}$. It turns out that V is strictly dense in V_d and that V_d is complete with respect to the strict topology τ_V. Moreover, each essential extension of V complete with respect to τ_V is unitarily equivalent to V_d. Note that in the case $V = \mathcal{A}$ the extended module V_d is nothing else than the multiplier algebra $M(\mathcal{A})$ and τ_V coincides with the usual (C^*) strict topology on $M(\mathcal{A})$.

Finally, V_d coincides (up to the identification $v \leftrightarrow r_v$) with V whenever either $\mathcal{A}$ or $\boldsymbol{K}(V)$ is a unital C^*-algebra.

We end the list of analogies with multiplier algebras with the following property of the largest extension V_d. Given a surjective morphism of Hilbert C^*-modules $\Phi : V \to W$, it is not hard to see that there exists $\overline{\Phi} : V_d \to W_d$, an extension of Φ in the sense $\overline{\Phi}(r_x) = r_{\Phi(x)}$, $\forall x \in V$. What is more, it is proved in [1] that if V is a countably generated Hilbert C^*-module over a σ-unital C^*-algebra, then $\overline{\Phi}$ is also a surjection - a fact which serves as a Hilbert C^*-module version of the noncommutative Tietze extension theorem (cf. [6], Proposition 6.8).

All of this shows that V_d can be regarded as the Hilbert C^*-module analogue of the multiplier algebra. Hence in the sequel V_d will be referred to as the multiplier module of a Hilbert C^*-module V.

Let us now take a Hilbert $\mathcal{A}$-module V and consider an extension $(W, \mathcal{B}, \Phi)$ of V, i.e. an exact sequence of Hilbert C^*-modules and their morphisms: $0 \to V \stackrel{\Phi}{\to} W \stackrel{q}{\to} W/\mathrm{Im}\,\Phi \to 0$. If we regard the maximal extension, namely $0 \to V \stackrel{\Gamma}{\to} V_d \stackrel{q_d}{\to} Q(V) \to 0$ (with $Q(V)$ denoting the quotient $V_d/\mathrm{Im}\,\Gamma$) as fixed, then by the described property of V_d there exists a morphism $\Lambda : W \to V_d$ and it is easy to define the Busby morphism $\Delta_W : W/\mathrm{Im}\,\Phi \to Q(V)$ in terms of Λ. Following the theory of extensions of C^*-algebras one may ask: given a morphism of Hilbert C^*-modules $\Delta : Z \to Q(V)$, does there exists an extension $(W, \mathcal{B}, \Phi)$ such that the induced Busby map Δ_W coincides with Δ?

To answer this question we first describe the pullback construction for Hilbert C^*-modules. The pullback of Hilbert C^*-modules is obtained in Proposition 2.3 as their restricted direct sum over the corresponding restricted direct sum of underlying C^*-algebras. We then in Section 2 show that the Busby map is the invariant determining an extension uniquely, up to isomorphism. We also discuss, for an extension $0 \to V \stackrel{\Phi}{\to} W \stackrel{q}{\to} W/\mathrm{Im}\,\Phi \to 0$, the induced extensions $0 \to \mathcal{A} \stackrel{\phi}{\to} B \stackrel{\pi}{\to} B/\mathrm{Im}\,\phi \to 0$ and $0 \to \boldsymbol{K}(V) \stackrel{\Phi^+}{\to} \boldsymbol{K}(W) \stackrel{q^+}{\to} \boldsymbol{K}(W/\mathrm{Im}\,\Phi) \to 0$ of underlying C^*-algebras as well as the induced extension of the corresponding linking algebra.

At the end two applications are demonstrated. First, it is proved that the process of forming multiplier modules preserves pullbacks. The other is a Hilbert C^*-module version of a familiar result which states that a C^*-algebra is projective if and only if it is corona projective.

We end this introductory section by fixing some of our notations which will be used throughout the paper. If V and W are Hilbert $\mathcal{A}$-modules, we denote by $\boldsymbol{B}(V, W)$ the Banach space of all adjointable operators from V to W. The ideal of "compact" operators (generated by all operators of the form $\theta_{x,y}$, $\theta_{x,y}(v) = x\langle y, v\rangle$) from V to W is denoted by $\boldsymbol{K}(V, W)$. When $V = W$ we write $\boldsymbol{B}(V)$ and $\boldsymbol{K}(V)$ instead of $\boldsymbol{B}(V, V)$ and $\boldsymbol{K}(V, V)$, respectively. When a C^*-algebra $\mathcal{A}$ is considered as a Hilbert $\mathcal{A}$-module with the inner product $\langle a,b\rangle = a^*b$, then $\mathcal{A} \simeq \boldsymbol{K}(\mathcal{A})$ and $M(\mathcal{A}) \simeq \boldsymbol{B}(\mathcal{A})$. The corresponding identifications $a \leftrightarrow T_a$ and $m \leftrightarrow T_m$, with T_a and T_m denoting the left translations by $a \in \mathcal{A}$ resp. $m \in M(\mathcal{A})$, will be used freely. Remaining notations are defined in context.

2. Pullback diagrams of Hilbert C^*-modules

Recall from 2.2 in [8] that a commutative diagram of C^*-algebras

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\delta_2} & \mathcal{A}_2 \\ \downarrow \delta_1 & & \downarrow \varphi_2 \\ \mathcal{A}_1 & \xrightarrow{\varphi_1} & \mathcal{C} \end{array} \tag{2.1}$$

is a pullback if $\operatorname{Ker}\delta_1 \cap \operatorname{Ker}\delta_2 = \{0\}$ and if every other coherent pair of morphisms $\lambda_1 : \mathcal{B} \to \mathcal{A}_1$, $\lambda_2 : \mathcal{B} \to \mathcal{A}_2$ (coherent in the sense $\varphi_1\lambda_1 = \varphi_2\lambda_2$) from a C^*-algebra $\mathcal{B}$ factors through $\mathcal{A}$, i.e. there exists a morphism $\lambda : \mathcal{B} \to \mathcal{A}$ such that $\lambda_1 = \delta_1\lambda$ and $\lambda_2 = \delta_2\lambda$.

It is easily verified that the morphism λ is uniquely determined with the above property. Further, given a diagram

$$\begin{array}{ccc} & & \mathcal{A}_2 \\ & & \downarrow \varphi_2 \\ \mathcal{A}_1 & \xrightarrow{\varphi_1} & \mathcal{C} \end{array} \tag{2.2}$$

of C^*-algebras, it is known that there exists a C^*-algebra $\mathcal{A}$ together with maps $\delta_{1,2} : \mathcal{A} \to \mathcal{A}_{1,2}$ such that (1) is a pullback diagram. One easily verifies that $\mathcal{A}$ is necessarily unique, up to isomorphism; hence $\mathcal{A}$ is said to be the pullback for the triple $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{C})$ with linking morphisms φ_1, φ_2. The pullback $\mathcal{A}$ is isomorphic to the restricted direct sum

$$\mathcal{A}_1 \oplus_C \mathcal{A}_2 = \{(a_1,a_2) \in \mathcal{A}_1 \oplus \mathcal{A}_2 : \varphi_1(a_1) = \varphi_2(a_2)\} \tag{2.3}$$

while δ_1 and δ_2 are identified with the projections on first and second coordinates, respectively.

Suppose now that we are given a diagram

$$\begin{array}{ccc} & & V_2 \\ & & \downarrow \Phi_2 \\ V_1 & \xrightarrow{\Phi_1} & W \end{array} \tag{2.4}$$

of Hilbert C^*-modules V_1, V_2, W over C^*-algebras $\mathcal{A}_1, \mathcal{A}_2, \mathcal{C}$, respectively. The above description indicates how it can be completed to a pullback diagram of Hilbert C^*-modules. First observe that there is the induced diagram (2.2) of C^*-algebras.

Lemma 2.1. *Let $\Phi_1 : V_1 \to W$ and $\Phi_2 : V_2 \to W$ be morphisms of Hilbert C^*-modules, let $\varphi_1 : \mathcal{A}_1 \to \mathcal{C}$ and $\varphi_2 : \mathcal{A}_2 \to \mathcal{C}$ denote the corresponding morphisms of underlying C^*-algebras. Denote by $V_1 \oplus_W V_2$ the set $\{(v_1,v_2) \in V_1 \oplus V_2 : \Phi_1(a_1) = \Phi_2(a_2)\}$. Then $V_1 \oplus_W V_2$ is a Hilbert C^*-module (with* *operations inherited from a Hilbert $\mathcal{A}_1\oplus\mathcal{A}_2$-module $V_1\oplus V_2$) over the restricted direct sum $\mathcal{A}_1\oplus_C\mathcal{A}_2$.*

PROOF. Straightforward verification. □

REMARK 2.2. (a) $V_1\oplus_W V_2$ is called the restricted direct sum of Hilbert C^*-modules. If V_1 and V_2 are full, then $V_1\oplus V_2$ is a full $\mathcal{A}_1\oplus\mathcal{A}_2$-module and one easily conclude that $V_1\oplus_W V_2$ is also a full module over $\mathcal{A}_1\oplus_C\mathcal{A}_2$.

(b) Let us define $\Delta_1 : V_1\oplus_W V_2\to V_1$ and $\Delta_2 : V_1\oplus_W V_2\to V_2$ by $\Delta_1(v_1,v_2)=v_1$ and $\Delta_2(v_1,v_2)=v_2$, respectively. Then, obviously, Δ_i is a δ_i-morphism of Hilbert C^*-modules where $\delta_i:\mathcal{A}_1\oplus_C\mathcal{A}_2\to\mathcal{A}_i$, $i=1,2$ are the corresponding projections.

PROPOSITION 2.3. *Let V_1,V_2,W be Hilbert C^*-modules with linking morphisms Φ_1 and Φ_2 as in (2.4). Then*

$$\begin{array}{ccc} V_1\oplus_W V_2 & \xrightarrow{\Delta_2} & V_2\\ \downarrow{\scriptstyle\Delta_1} & & \downarrow{\scriptstyle\Phi_2}\\ V_1 & \xrightarrow{\Phi_1} & W \end{array}\tag{2.5}$$

with the maps Δ_1, Δ_2 from Remark 1.2(b) is a pullback diagram of Hilbert C^*-modules in the sense: $\Phi_1\Delta_1=\Phi_2\Delta_2$, $\operatorname{Ker}\Delta_1\cap\operatorname{Ker}\Delta_2=\{0\}$ and every other coherent pair of morphisms $\Lambda_1 : X\to V_1$, $\Lambda_2 : X\to V_2$ (where coherence means $\Phi_1\Lambda_1=\Phi_2\Lambda_2$) from a full Hilbert C^*-module X factors through $V_1\oplus_W V_2$, i.e. there exists a morphism $\Lambda : X\to V_1\oplus_W V_2$ such that $\Lambda_1=\Delta_1\Lambda$ and $\Lambda_2=\Delta_2\Lambda$.

PROOF. Assume that X is a full Hilbert $\mathcal{B}$-module and that $\Lambda_1 : X\to V_1$, $\Lambda_2 : X\to V_2$ is a coherent pair of morphisms; let the underlying morphisms of C^*-algebras be denoted by $\lambda_1:\mathcal{B}\to\mathcal{A}_1$, $\lambda_2:\mathcal{B}\to\mathcal{A}_2$, respectively. For x in X we then have $\varphi_1\lambda_1(\langle x,x\rangle)=\langle\Phi_1\Lambda_1(x),\Phi_1\Lambda_1(x)\rangle=\langle\Phi_2\Lambda_2(x),\Phi_2\Lambda_2(x)\rangle=\varphi_2\lambda_2(\langle x,x\rangle)$. Since X is by supposition full, this shows that (λ_1,λ_2) is a coherent pair of morphisms of C^*-algebras; here coherence is understood with respect to the corresponding diagram of underlying C^*-algebras. Let $\lambda:\mathcal{B}\to\mathcal{A}_1\oplus_C\mathcal{A}_2$ be the resulting morphism. Thus we have the following diagram of C^*-algebras

$$\begin{array}{ccccccc} & & & & \mathcal{A}_1 & & \\ & & & \overset{\delta_1}{\nearrow} & & \overset{\varphi_1}{\searrow} & \\ \mathcal{B} & \xrightarrow{\lambda} & \mathcal{A}_1\oplus_C\mathcal{A}_2 & & & & \mathcal{C}\\ & & & \underset{\delta_2}{\searrow} & & \underset{\varphi_2}{\nearrow} & \\ & & & & \mathcal{A}_2 & & \end{array}\tag{2.6}$$

Observe that λ is defined by $\lambda(b)=(\lambda_1(b),\lambda_2(b))$ and satisfies $\lambda_1=\delta_1\lambda$, $\lambda_2=\delta_2\lambda$. Let us now define $\Lambda : X\to V_1\oplus_W V_2$ by $\Lambda(x)=(\Lambda_1(x),\Lambda_2(x))$ to obtain a diagram corresponding to the above diagram (2.6):

$$
\begin{array}{ccccccc}
 & & & & V_1 & & \\
 & & & \overset{\Delta_1}{\nearrow} & & \overset{\Phi_1}{\searrow} & \\
X & \xrightarrow{\Lambda} & V_1 \oplus_W V_2 & & & & W \\
 & & & \overset{\Delta_2}{\searrow} & & \overset{\Phi_2}{\nearrow} & \\
 & & & & V_2 & &
\end{array}
\tag{2.7}
$$

It remains to see that Λ is a λ-morphism of Hilbert C^*-modules and to verify (evident) equalities $\Lambda_1 = \Delta_1\Lambda$ and $\Lambda_2 = \Delta_2\Lambda$. We omit the details. ∎

Remark 2.4. Since unitary equivalence is an equivalence relation only when applied to full Hilbert C^*-modules, we must assume that V_1 and V_2 are full in order to ensure uniqueness in the above construction. Indeed, if V_1 and V_2 are full Hilbert C^*-modules then the pullback $V_1 \oplus_W V_2$ for the triple (V_1, V_2, W) with linking morphisms Φ_1 and Φ_2 as in (2.4) is by Remark 2.2 also a full Hilbert C^*-module, hence uniquely determined, up to unitary equivalence.

3. The Busby invariant

Let V be a full Hilbert C^*-module. Consider an arbitrary extension $(W, \mathcal{B}, \Phi)$ of V such that W is a full Hilbert $\mathcal{B}$-module. By definition ([3]), Φ is a φ-morphism where $\varphi : \mathcal{A} \to \mathcal{B}$ is an injective morphism of C^*-algebras such that $\mathrm{Im}\,\varphi$ is an ideal in $\mathcal{B}$ and $\mathrm{Im}\,\Phi = W\mathrm{Im}\,\varphi$. Thus, we have an exact sequence of Hilbert C^*-modules $0 \to V \xrightarrow{\Phi} W \xrightarrow{q} W/\mathrm{Im}\,\Phi \to 0$ supported by the corresponding exact sequence of C^*-algebras $0 \to \mathcal{A} \xrightarrow{\varphi} \mathcal{B} \xrightarrow{\pi} \mathcal{B}/\mathrm{Im}\,\varphi \to 0$. We say that $(W, \mathcal{B}, \Phi)$ is an essential extension if $\mathrm{Im}\,\varphi$ is an essential ideal in $\mathcal{B}$. Now take the largest essential extension V_d of V; that is the sequence $0 \to V \xrightarrow{\Gamma} V_d \xrightarrow{q_d} Q(V) \to 0$ where $\Gamma(x) = r_x$, $r_x : \mathcal{A} \to V$, $r_x(a) = xa$.

By Theorem 1.1 from [3] there exists a morphism of Hilbert C^*-modules $\Lambda : W \to V_d$ satisfying $\Lambda\Phi = \Gamma$. Observe that Λ is a λ-morphism where $\lambda : \mathcal{B} \to M(\mathcal{A})$ is the unique morphism of C^*-algebras such that $\lambda\varphi$ is the identity on $\mathcal{A}$. We start this section with two comments concerned with the morphism Λ.

First, it is known that V_d need not be a full Hilbert C^*-module over $M(\mathcal{A})$. In the sequel, we shall often regard V_d as a full Hilbert C^*-module over $\langle V_d, V_d\rangle \subseteq M(\mathcal{A})$ (where $\langle V_d, V_d\rangle$ denotes the closed twosided ideal in $M(\mathcal{A})$ generated by all products $\langle r, s\rangle$, $r, s \in V_d$). However, we still may regard Λ as an λ-morphism since $\mathrm{Im}\,\lambda = \lambda(\langle W, W\rangle) = \langle \Lambda(W), \Lambda(W)\rangle \subseteq \langle V_d, V_d\rangle \subseteq M(\mathcal{A})$.

Another property of the map Λ we state in an independent lemma.

LEMMA 3.1. *Let V be a full Hilbert C^*-module, let $(W, \mathcal{B}, \Phi)$ be an extension of V and let $\Lambda : W \to V_d$ be the resulting morphism of Hilbert C^*-modules. Then Λ is uniquely determined by the property $\Lambda\Phi = \Gamma$.*

PROOF. Suppose that there is another morphism of Hilbert C^*-modules $\Lambda' : W \to V_d$ such that $\Lambda'\Phi = \Gamma$. First observe that the last equality implies that Λ' is also a λ-morphism since $\lambda : \mathcal{B} \to M(\mathcal{A})$ is the only morphism such that $\lambda\varphi$ is the identity on $\mathcal{A}$.

By Proposition 1.2 from [3] the ideal submodule $W\mathrm{Im}\,\Phi$ is strictly dense in W. Explicitely: if (e_j) is an approximate unit for $\mathcal{A}$, then $w = \lim_j w\varphi(e_j)$ is satisfied for each w in W. Now we observe that each morphism of Hilbert C^*-modules is strictly continuous (this can be seen as in the proof of Proposition 3 in [1]). Note also that, since $w\varphi(e_j)$ belongs to the ideal submodule $W\mathrm{Im}\,\varphi = \mathrm{Im}\,\Phi$ we may write $w\varphi(e_j) = \Phi(x_j)$ for (necessarily unique) $x_j \in V$. Therefore, $\Lambda'(w) = \lim_j \Lambda'(w\varphi(e_j)) = \lim_j \Lambda'(\Phi(x_j)) = \lim_j \Gamma(x_j) = \lim_j \Lambda(\Phi(x_j)) = \lim_j \Lambda(w\varphi(e_j)) = \Lambda(w)$. □

Let us now take again an arbitrary extension $(W, \mathcal{B}, \Phi)$ of a full Hilbert C^*-module V. Comparing it with the maximal extension V_d one obtains the following diagrams of Hilbert C^*-modules and C^*-algebras, respectively:

(3.1)

$$\begin{array}{ccccc} V & \xrightarrow{\Phi} & W & \xrightarrow{q} & W/\mathrm{Im}\,\Phi \\ & & \downarrow \Lambda & & \downarrow \Delta \\ V & \xrightarrow{\Gamma} & V_d & \xrightarrow{q_d} & Q(V) \end{array} \qquad \begin{array}{ccccc} \mathcal{A} & \xrightarrow{\varphi} & \mathcal{B} & \xrightarrow{\pi} & \mathcal{B}/\mathrm{Im}\,\varphi \\ & & \downarrow \lambda & & \downarrow \delta \\ \mathcal{A} & \longrightarrow & \langle V_d, V_d\rangle & \xrightarrow{\pi_d} & \langle V_d, V_d\rangle/\mathcal{A} \end{array}$$

Here $\delta : \mathcal{B}/\mathrm{Im}\,\varphi \to \langle V_d, V_d\rangle/\mathcal{A} \subseteq Q(\mathcal{A})$ denotes the Busby invariant corresponding to the extension $\mathcal{B}$ of $\mathcal{A}$ defined by $\delta(\pi(b)) = \pi_d(\lambda(b))$, $b \in \mathcal{B}$.

In an analogous way we define (belatedly) $\Delta : W/\mathrm{Im}\,\Phi \to Q(V)$ by $\Delta(q(w)) = q_d(\Lambda(w))$, $w \in W$. First observe that the definition is unambiguous: if $q(w) = 0$ then $w = \Phi(v) \in \mathrm{Im}\,\Phi$; hence $\Lambda(w) = \Lambda\Phi(v) = \Gamma(v)$ and, finally, $q_d(\Lambda(w)) = 0$. Secondly, we claim that Δ is a δ-morphism of Hilbert C^*-modules. Indeed, $\langle \Delta(q(w)), \Delta(q(w))\rangle = \langle q_d(\Lambda((w)), q_d(\Lambda((w))\rangle = \pi_d\lambda(\langle w, w\rangle) = \delta\pi(\langle w, w\rangle) = \delta(\langle q(w), q(w)\rangle)$, $\forall w \in W$.

DEFINITION 3.2. *Let V be a full Hilbert C^*-module. The morphism Δ from the preceding discussion is called the Busby invariant corresponding to an extension $(W, \mathcal{B}, \Phi)$ of V.*

PROPOSITION 3.3. *Let $(W, \mathcal{B}, \Phi)$ be an extension of V such that W is a full Hilbert $\mathcal{B}$-module. Then the Busby invariant Δ is an injection if and only if W is an essential extension of V.*

PROOF. By [3], Theorem 1.1, W is an essential extension of V if and only if Λ is an injection (and if and only if λ is an injection). Assume first that Λ is injective. Then $\Delta(q(v)) = 0 \Rightarrow q_d(\Lambda(w)) = 0 \Rightarrow \Lambda(w) = \Gamma(v)$ for some $v \in V \Rightarrow \Lambda(w) = \Lambda\Phi(w)) \Rightarrow w = \Phi(v) \Rightarrow q(w) = 0$.

Conversely, if Δ is an injection then $\Lambda(w) = 0 \Rightarrow q_d(\Lambda(w)) = 0 \Rightarrow \Delta(q(w)) = 0 \Rightarrow q(w) = 0 \Rightarrow w = \Phi(v)$ for some $v \in V$ and now $0 = \Lambda(w) = \Lambda\Phi(v) = \Gamma(v) \Rightarrow v = 0$; in particular $w = \Phi(v) = 0$. ∎

PROPOSITION 3.4. *Let V be a full Hilbert $\mathcal{A}$-module and let $\Delta : Z \to Q(V)$ be a morphism of full Hilbert C^*-modules. Then there exists an extension W of V whose Busby invariant Δ_W coincides with Δ.*

PROOF. Let us assume, just to make our notations simpler, $\langle V_d, V_d\rangle = M(\mathcal{A})$ (keeping in mind the possibility $\langle V_d, V_d\rangle \neq M(\mathcal{A})$; however it is evident that the argument below does not depend on the assumed equality). Observe that in this case the quotient $\langle V_d, V_d\rangle/\mathcal{A}$ is equal to the corona algebra $Q(\mathcal{A})$. Let Δ be a δ-morphism where $\delta : \mathcal{C} \to Q(\mathcal{A})$ (so that Z is a full Hilbert $\mathcal{C}$-module). Consider a diagram of Hilbert C^*-modules together with the corresponding diagram of underlying C^*-algebras:

$$\begin{array}{ccccccccccc} & & & & \mathcal{C} & & & & & & Z \\ (3.2) & & & & \downarrow \delta & & & & & & \downarrow \Delta \\ & \mathcal{A} & \longrightarrow & M(\mathcal{A}) & \xrightarrow{\pi_d} & Q(\mathcal{A}) & & V & \xrightarrow{\Gamma} & V_d & \xrightarrow{q_d} & Q(V) \end{array}$$

After performing the pullback constructions in both categories we get

$$\begin{array}{cccccccc} & M(\mathcal{A}) \oplus_{Q(\mathcal{A})} \mathcal{C} & \xrightarrow{\kappa_2} & \mathcal{C} & & V_d \oplus_{Q(V)} Z & \xrightarrow{p_2} & Z \\ (3.3) & \downarrow \kappa_1 & & \downarrow \delta & & \downarrow p_1 & & \downarrow \Delta \\ & M(\mathcal{A}) & \xrightarrow{\pi_d} & Q(\mathcal{A}) & & V_d & \xrightarrow{q_d} & Q(V) \end{array}$$

Now consider the extension $0 \to V \xrightarrow{\Gamma'} V_d \oplus_{Q(V)} Z \xrightarrow{p_2} Z \to 0$ with $\Gamma'(v) = (\Gamma(v), 0)$. Denote the corresponding Busby invariant by Δ'. We claim $\Delta' = \Delta$.

Let $\Lambda' : V_d \oplus_{Q(V)} Z \to V_d$ be the associated map into the multiplier module V_d. Note that $\Lambda'\Gamma' = \Gamma$ and $p_1\Gamma' = \Gamma$. It follows by Lemma 3.1 $\Lambda' = p_1$. This means that the action of Δ' is in fact described in terms of p_1. Explicitly: for $(r, z) \in V_d \oplus_{Q(V)} Z$ we have $\Delta' p_2(r, z) = \Delta'(z) = q_d p_1(r, z) = q_d(r_1) = \Delta(z)$. (The last equality is obtained by the definition of the restricted direct sum). ∎

Now we fix a Hilbert C^*-module V. Let $(W, \mathcal{B}, \Phi)$ be an extension of V and denote by Z the quotient $W/\mathrm{Im}\,\Phi$. Then we say that $(W, \mathcal{B}, \Phi)$ (or W) is an extension of V by Z. An extension $(W, \mathcal{B}, \Phi)$ is said to be full if W is a full Hilbert $\mathcal{B}$-module. We introduce an equivalence relation in the set of full extensions of V by Z in the standard way.

DEFINITION 3.5. *Let V be a full Hilbert C^*-module. We say that the full extensions $(W, \mathcal{B}, \Phi)$ and $(W', \mathcal{B}', \Phi')$ of V by Z are equivalent if there exists a unitary operator of Hilbert C^*-modules $\Psi : W \to W'$ such that the following* *diagram commutes:*

$$\begin{array}{ccccc} V & \xrightarrow{\Phi} & W & \xrightarrow{q} & Z \\ \downarrow id & & \downarrow \Psi & & \downarrow id \\ V & \xrightarrow{\Phi'} & W' & \xrightarrow{q'} & Z \end{array} \tag{3.4}$$

THEOREM 3.6. *Let V be a full Hilbert $\mathcal{A}$-module. Then the set of equivalence classes of full extensions of V by a full Hilbert C^*-module Z is in a bijective correspondence with the set of all morphisms of Hilbert C^*-modules $\Delta : Z \to Q(V)$.*

PROOF. It only remains to prove the following assertion: let $(W, \mathcal{B}, \Phi)$ be a full extension of V by Z with the Busby invariant Δ and let $0 \to V \xrightarrow{\Gamma'} V_d \oplus_{Q(V)} Z \xrightarrow{p_2} Z \to 0$ be the extension from the proof of Proposition 3.4. Then there exists a unitary operator of Hilbert C^*-modules $\Theta : W \to V_d \oplus_{Q(V)} Z$ making the following diagram commutative:

$$\begin{array}{ccccc} V & \xrightarrow{\Phi} & W & \xrightarrow{q} & Z \\ \downarrow \mathrm{id} & & \downarrow \Theta & & \downarrow \mathrm{id} \\ V & \xrightarrow{\Gamma'} & V_d \oplus_{Q(V)} Z & \xrightarrow{p_2} & Z \end{array} \tag{3.5}$$

Denote by $\mathcal{C}$ the underlying algebra of Z. Observe that we already have the corresponding commutative diagram of C^*-algebras (again the equality $\langle V_d, V_d \rangle = M(\mathcal{A})$ will be assumed for simplicity) with the isomorphism $\theta : \mathcal{B} \to M(\mathcal{A}) \oplus_{Q(\mathcal{A})} \mathcal{C}$:

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{\varphi} & \mathcal{B} & \xrightarrow{\pi} & \mathcal{C} \\ \downarrow \mathrm{id} & & \downarrow \theta & & \downarrow \mathrm{id} \\ \mathcal{A} & \xrightarrow{\iota'} & M(\mathcal{A}) \oplus_{Q(\mathcal{A})} \mathcal{C} & \xrightarrow{i_2} & \mathcal{C} \end{array} \tag{3.6}$$

If we again denote by $\lambda : B \to M(\mathcal{A})$ the only morphism such that $\lambda\varphi$ is the identity on $\mathcal{A}$, then the above isomorphism θ is defined by $\theta(b) = (\lambda(b), \pi(b))$.

Let us now define analogously $\Theta(w) = (\Lambda(w), q(w))$ (with $\Lambda : W \to V_d$ as in diagram (3.1)). This is well defined in the sense that Θ does take values in $V_d \oplus_{Q(V)} Z$ by definition of the Busby invariant: $q_d\Lambda(w) = \Delta q(w)$, $w \in W$.

Obviously, Θ is a θ-morphism and the diagram (3.5) is commutative by the definition of the restricted direct sum. Hence it only remains to see that Θ is a surjection.

Let $(r, q(v)) \in V_d \oplus_{Q(V)} Z$ be given; note $q_d(r) = \Delta q(w)$. We first claim that $r - \Lambda(w) \in \operatorname{Im}\Gamma$. This is indeed true because $\operatorname{Ker} q_d =$ and $q_d(r - \Lambda(w)) = q_d(r) - q_d\Lambda(w) = q_d(r) - \Delta q(w) = 0$. Thus $r - \Lambda(w) = \Gamma(v)$ for a uniquely determined $v \in V$. Finally, $\Theta(\Phi(v) + w) = (\Lambda(\Phi(v) + w), q(\Phi(v) + w)) = (r, q(v))$. ∎

We end the general discussion by a comment on split extensions.

Definition 3.7. *An extension W of V by Z is called a split extension if there is a morphism of Hilbert C^*-modules $\Psi : Z \to W$ such that $q\Psi = id_Z$.*

Remark 3.8. Consider a split extension W of V by Z and the corresponding diagram

$$\begin{array}{ccccc} V & \xrightarrow{\Phi} & W & \overset{\Psi \quad q}{\longleftrightarrow} & Z \\ \downarrow \mathrm{id} & & \downarrow \Lambda & & \downarrow \Delta \\ V & \xrightarrow{\Gamma} & V_d & \xrightarrow{q_d} & Q(V) \end{array} \tag{3.7}$$

Obviously, we can define $\Delta_0 : Z \to V_d$ by $\Delta_0 = \Lambda\Psi$ and it turns out that Δ_0 is a lift for Δ.

Conversely, given a morphism of Hilbert C^*-modules $\Delta_0 : Z \to V_d$, define $\Delta : Z \to Q(V)$ by $\Delta = q_d\Delta_0$. Now apply Proposition 3.4 to obtain the extension $V_d \oplus_{Q(V)} Z$ whose Busby invariant is Δ. By a routine verification (which we omit) one sees that the map $\Psi : Z \to V_d \oplus_{Q(V)} Z,\ \Psi(z) = (\Delta_0(z), z)$ makes this extension split.

One should also note that, given a full split extension of a full Hilbert C^*-module, the corresponding extension of underlying C^*-algebras is also split.

In the sequel we show that an extension of Hilbert C^*-modules produces an exact sequence of the corresponding linking algebras. We first need to describe how morphisms of Hilbert C^*-modules induce morphisms of the corresponding algebras of "compact" operators.

Consider the quotient $\mathcal{A}/\mathcal{I}$-module $V/V_{\mathcal{I}}$ of a Hilbert $\mathcal{A}$-module V over the ideal submodule $V_{\mathcal{I}} = V\mathcal{I}$ associated with an ideal $\mathcal{I}$ of $\mathcal{A}$ and denote as before by $q : V \to V/V_{\mathcal{I}}$ the quotient morphism of Hilbert C^*-modules.

Take an arbitrary adjointable operator $T \in \boldsymbol{B}(V)$. Since $V_{\mathcal{I}}$ is obviously invariant for T (because T is $\mathcal{A}$-linear), there is a well defined operator $\hat{T}$ on $V/V_{\mathcal{I}}$ given by $\hat{T}(q(v)) = q(Tv)$. Note also $\overset{\wedge}{\theta}_{x,y} = \theta_{q(x),q(y)}$, $x, y \in V$. It is proved in Corollary 1.18 from [2] that the map $\beta : \boldsymbol{B}(V) \to \boldsymbol{B}(V/V_{\mathcal{I}})$, $\beta(T) = \hat{T}$ is a morphism of C^*-algebras such that $\beta(\boldsymbol{K}(V)) = \boldsymbol{K}(V/V_{\mathcal{I}})$. In particular, if V is countably generated, then $\boldsymbol{K}(V)$ is a σ-unital C^*-algebra ([6], Proposition 6.7), and by the noncommutative Tietze extension theorem, β is a surjection. In the following remark we denote by β the restriction of the above described map to "compact" operators.

Remark 3.9. Let $V_{\mathcal{I}}$ be an ideal submodule of a Hilbert C^*-module V. Then the quotient map $q : V \to V/V_{\mathcal{I}}$ induces a surjective morphism β of the corresponding C^*-algebras of "compact" operators. By [2], Proposition 1.17, the kernel of the map $\beta : \boldsymbol{K}(V) \to \boldsymbol{K}(V/V_{\mathcal{I}})$ coincides with $\boldsymbol{K}(V_{\mathcal{I}})$.

Consequently, each surjective morphism $\Phi : V \to W$ induces a surjective morphism $\Phi^+ : \boldsymbol{K}(V) \to \boldsymbol{K}(W)$ such that $\Phi^+(\theta_{x,y}) = \theta_{\Phi(x),\Phi(y)}$, $\forall x, y \in V$.

Since unitarily equivalent Hilbert C^*-modules have naturally isomorphic C^*-algebras of "compact" operators, this follows from the preceding assertion after passing through the quotient $V/\operatorname{Ker}\Phi$.

In our next proposition the hypothesis in the concluding assertion of above remark, namely that Φ should be a surjection, will be omitted. First observe that by a result of D. Blecher (see [4], Theorem 3.8) the ideal of all "compact" operators on a Hilbert $\mathcal{A}$-module V, $\boldsymbol{K}(V)$, can be written in the form $\boldsymbol{K}(V) = V \otimes_{h\mathcal{A}} V^*$. Here $\otimes_{h\mathcal{A}}$ denotes the Haagerup tensor product and V^* stands for the antilinear version of V (cf. equation (3.15) below where V^* is identified with the space of "compact" operators $\boldsymbol{K}(V, \mathcal{A})$).

PROPOSITION 3.10. *Let $\Phi : V \to W$ be a morphism of Hilbert C^*-modules. Then there is a unique morphism $\Phi^+ : \boldsymbol{K}(V) \to \boldsymbol{K}(W)$ satisfying $\Phi^+(\theta_{x,y}) = \theta_{\Phi(x),\Phi(y)}$, $\forall x, y \in V$.*

PROOF. By the preceding observation we can write $\boldsymbol{K}(V) = V \otimes_{h\mathcal{A}} V^*$ and $\boldsymbol{K}(W) = W \otimes_{h\mathcal{B}} W^*$. Now one easily verifies that the map $\Phi^+ : V \otimes_{h\mathcal{A}} V^* \to W \otimes_{h\mathcal{B}} W^*$ given on elementary tensors by $\Phi^+ : x \otimes_{\mathcal{A}} y^* \mapsto \Phi(x) \otimes_{\mathcal{B}} \Phi(y)^*$ is well defined and contractive. □

Recall that there is a natural left Hilbert C^*-module structure on each right Hilbert C^*-module V. Namely, V is a natural left $\boldsymbol{K}(V)$-module with the $\boldsymbol{K}(V)$-valued inner product on V is defined by $[x, y] = \theta_{x,y}$. Observe that the resulting norm coincides with the original norm on V.

COROLLARY 3.11. *Let $\Phi : V \to W$ be a morphism of Hilbert C^*-modules and let Φ^+ be the morphism from the preceding proposition. Then Φ is a Φ^+-morphism of left Hilbert C^*-modules V and W. The map Φ is an injection if and only if Φ is an injection.*

PROOF. $[\Phi(x), \Phi(y)] = \Phi^+([x, y])$ is just the assertion $\Phi^+(\theta_{x,y}) = \theta_{\Phi(x),\Phi(y)}$ from the preceding proposition. In particular, we note that Φ is necessarily Φ^+-linear in the sense $\Phi(Tx) = \Phi^+(T)\Phi(x)$, $T \in \boldsymbol{K}(V), x \in V$.

Since V is a full left $\boldsymbol{K}(V)$-module, the second assertion follows from Theorem 2.3 in [2]. □

THEOREM 3.12. *Let $(W, \mathcal{B}, \Phi)$ be a full essential extension of a full Hilbert $\mathcal{A}$-module V. Consider the diagram*

$$\begin{array}{ccccc} V & \xrightarrow{\Phi} & W & \xrightarrow{q} & W/Im\,\Phi \\ \downarrow id & & \downarrow \Lambda & & \downarrow \Delta \\ V & \xrightarrow{\Gamma} & V_d & \xrightarrow{q_d} & Q(V) \end{array} \tag{3.8}$$

Then the sequence

$$0 \longrightarrow \boldsymbol{K}(V) \xrightarrow{\Phi^+} \boldsymbol{K}(W) \xrightarrow{q^+} \boldsymbol{K}(W/Im\,\Phi) \longrightarrow 0 \tag{3.9}$$

induced by the first row in (3.8) is an extension of $\boldsymbol{K}(V)$*. Further, there exists a well defined morphism* $\Delta^+ : \boldsymbol{K}(W/Im\,\Phi) \to \boldsymbol{K}(Q(V))$ *such that* $\Delta^+(\theta_{q(x),q(y)}) = \theta_{q_d\Lambda(x),q_d\Lambda(y)}$, $\forall x,y \in W$*. The map* Δ^+ *is the Busby invariant corresponding to the extension (3.9).*

Proof. Since W is an essential extension, Λ and the Busby map Δ are injections by Proposition 3.3. The induced maps Φ^+ and q^+ are ensured by Proposition 3.10.

Note that, by the definition of an extension, $\operatorname{Im}\Phi$ is an ideal submodule of W which implies that $\operatorname{Im}\Phi^+$ is an ideal in $\boldsymbol{K}(W)$; one easily concludes $\operatorname{Im}\Phi^+ = \operatorname{Ker} q^+$. Since by Corollary 3.11 Φ^+ is an injection and since q^+ is surjective by Remark 3.9, (3.9) is an exact sequence. Further, again by Remark 3.9, we may write $\boldsymbol{K}(W/\operatorname{Im}\Phi) = \boldsymbol{K}(W)/\operatorname{Im}\Phi^+$, thus (3.9) can be rewritten as

$$0 \longrightarrow \boldsymbol{K}(V) \xrightarrow{\Phi^+} \boldsymbol{K}(W) \xrightarrow{q^+} \boldsymbol{K}(W)/\operatorname{Im}\Phi^+ \longrightarrow 0. \tag{3.10}$$

Analogously, the second row in (3.8) induces the sequence

$$0 \longrightarrow \boldsymbol{K}(V) \xrightarrow{\Gamma^+} \boldsymbol{K}(V_d) \xrightarrow{q_d^+} \boldsymbol{K}(V_d)/\operatorname{Im}\Gamma^+ \longrightarrow 0. \tag{3.11}$$

We are going to adjust the above sequence in two steps. First, replace $\boldsymbol{K}(V_d)$ with $M(\boldsymbol{K}(V_d)) = \boldsymbol{B}(V_d)$. Namely, Γ^+ obviously can be regarded as the map from $\boldsymbol{K}(V)$ into $\boldsymbol{B}(V_d)$. Concerning q_d^+, we can take its canonical extension (again denoted by q_d^+) to multiplier algebras $q_d^+ : M(\boldsymbol{K}(V_d)) \to M(\boldsymbol{K}(V_d)/\operatorname{Im}\Gamma^+)$. Now observe: if $\mathcal{I}$ is an ideal in a C^*-algebra $\mathcal{A}$, then $\mathcal{A}/\mathcal{I}$ is an essential ideal in $M(\mathcal{A})/\mathcal{I}$, hence $M(\mathcal{A}/\mathcal{I}) \subseteq M(\mathcal{A})/\mathcal{I}$. Consequently, we may write $q_d^+ : M(\boldsymbol{K}(V_d)) \to M(\boldsymbol{K}(V_d))/\operatorname{Im}\Gamma^+$. After all, (3.11) can be rewritten in the form

$$0 \longrightarrow \boldsymbol{K}(V) \xrightarrow{\Gamma^+} \boldsymbol{B}(V_d) \xrightarrow{q_d^+} \boldsymbol{B}(V_d)/\operatorname{Im}\Gamma^+ \longrightarrow 0. \tag{3.12}$$

The final modification is enabled by Theorem 2.2 in [3] which asserts that for each Hilbert C^*-module V the C^*-algebras $\boldsymbol{B}(V)$ and $\boldsymbol{B}(V_d)$ are naturally isomorphic. Thus, (3.12) is recognized as (we write $\boldsymbol{B}(V)$ in the form $M(\boldsymbol{K}(V))$)

$$0 \longrightarrow \boldsymbol{K}(V) \xrightarrow{\Gamma^+} M(\boldsymbol{K}(V)) \xrightarrow{q_d^+} M(\boldsymbol{K}(V))/\operatorname{Im}\Gamma^+ \longrightarrow 0. \tag{3.13}$$

Consider now sequences (3.10) and (3.13) together; let us additionally insert suitable vertical maps.

$$\begin{array}{ccccc} \boldsymbol{K}(V) & \xrightarrow{\Phi^+} & \boldsymbol{K}(W) & \xrightarrow{q^+} & \boldsymbol{K}(W)/\operatorname{Im}\Phi^+ \\ \downarrow \mathrm{id} & & \downarrow \nu & & \downarrow \eta \\ \boldsymbol{K}(V) & \xrightarrow{\Gamma^+} & M(\boldsymbol{K}(V)) & \xrightarrow{q_d^+} & M(\boldsymbol{K}(V))/\operatorname{Im}\Gamma^+ \end{array} \tag{3.14}$$

Here η is the Busby invariant and $\nu : \boldsymbol{K}(V) \to M(\boldsymbol{K}(V))$ is the only map with the property $\nu\Phi^+ = \Gamma^+$. What is obtained is precisely the Busby picture of the sequence (3.10) (i.e (3.9)).

Now observe that the $\Lambda : W \to V_d$ from diagram (3.8) is an injection and satisfies $\Lambda\Phi = \Gamma$, hence Λ is an essential morphism. By Proposition 3.10 we have well defined (injective, by 3.11!) morphism of C^*-algebras $\Lambda^+ : \boldsymbol{K}(V) \to \boldsymbol{K}(V_d) \subseteq M(\boldsymbol{K}(V))$. Having in mind all identifications we made, we now find $\Lambda^+\Phi^+ = \Gamma^+$ (it suffices to verify this equality on all operators of the form $\theta_{x,y}$ - and this is obvious). This is enough to conclude $\nu = \Gamma^+$.

Finally, we claim $\eta = \Delta^+$. First, as before, we may write $\boldsymbol{K}(W/\mathrm{Im}\,\Phi) = \boldsymbol{K}(W)/\mathrm{Im}\,\Phi^+$. The same argument gives $\boldsymbol{K}(V_d/V) = \boldsymbol{K}(V_d)/\mathrm{Im}\,\Gamma^+ \subseteq \boldsymbol{B}(V_d)/\mathrm{Im}\,\Gamma^+$; replacing in passing $\boldsymbol{B}(V_d)$ with $\boldsymbol{B}(V) = M(\boldsymbol{K}(V))$, we shall recognize Δ^+ as the map $\Delta^+ : \boldsymbol{K}(W)/\mathrm{Im}\,\Phi^+ \to M(\boldsymbol{K}(V))/\mathrm{Im}\,\Gamma^+$. Now the desired conclusion $\eta = \Delta^+$ follows immediately from the (obvious) equality $\eta(\theta_{q(x),q(y)}) = \theta_{q_d\Lambda(x),q_d\Lambda(y)}$, $\forall x, y \in W$. ∎

We now turn to the induced extensions of linking algebras. Let V be a full Hilbert $\mathcal{A}$-module. Recall from Lemma 2.32 and Corollary 3.21 in [9] that the linking algebra $L(V)$ of V can be described in terms of a Hilbert $\mathcal{A}$-module $\mathcal{A} \oplus V$:

$$L(V) = \boldsymbol{K}(\mathcal{A} \oplus V) = \begin{bmatrix} \boldsymbol{K}(\mathcal{A}) & \boldsymbol{K}(V,\mathcal{A}) \\ \boldsymbol{K}(\mathcal{A},V) & \boldsymbol{K}(V) \end{bmatrix}. \tag{3.15}$$

Since each operator in $\boldsymbol{K}(\mathcal{A}, V)$ is of the form r_x for some $x \in V$, we have

$$L(V) = \{ \begin{bmatrix} T_a & r_y^* \\ r_x & T \end{bmatrix} : a \in \mathcal{A}, x, y \in V, T \in \boldsymbol{K}(V)\}. \tag{3.16}$$

We shall also need the following well known property of the linking algebra $L(V)$: if $V_{\mathcal{I}}$ is the ideal submodule of V corresponding to an ideal $\mathcal{I}$ in $\mathcal{A}$, then the linking algebra of the quotient $V/V_{\mathcal{I}}$ is equal to the quotient of the corresponding linking algebras:

$$L(V/V_{\mathcal{I}}) = L(V)/L(V_{\mathcal{I}}). \tag{3.17}$$

Suppose that $\Phi : V \to W$ is a φ-morphism of Hilbert C^*-modules such that there exists the induced morphism $\Phi^+ : \boldsymbol{K}(V) \to \boldsymbol{K}(W)$. Then there is a map Φ^L of the corresponding linking algebras;

$$\Phi^L : L(V) \to L(W),\ \Phi^L(\begin{bmatrix} T_a & r_y^* \\ r_x & T \end{bmatrix}) = \begin{bmatrix} T_{\varphi(a)} & r_{\Phi(y)}^* \\ r_{\Phi(x)} & \Phi^+(T) \end{bmatrix}. \tag{3.18}$$

It is proved in Theorem 2.15 in [2] that Φ^L is a morphism of C^*-algebras. In fact, it turns out that Φ^L is the restriction of the map $(\varphi \oplus \Phi)^+$ to "compact" operators, where $\varphi \oplus \Phi : \mathcal{A} \oplus V \to \mathcal{B} \oplus W$ is a φ-morphism of Hilbert C^*-modules obtained by applying φ and Φ componentwise.

Let us again fix a full essential extension $(W, \mathcal{B}, \Phi)$ of a full Hilbert $\mathcal{A}$-module V. Consider the underlying extension

$$0 \longrightarrow \mathcal{A} \xrightarrow{\varphi} \mathcal{B} \xrightarrow{\pi} \mathcal{B}/\mathrm{Im}\,\varphi \longrightarrow 0$$

and also the induced extension of $\boldsymbol{K}(V)$ from Theorem 3.12:

$$0 \longrightarrow \boldsymbol{K}(V) \xrightarrow{\Phi^+} \boldsymbol{K}(W) \xrightarrow{q^+} \boldsymbol{K}(W)/\mathrm{Im}\,\Phi^+ \longrightarrow 0.$$

By the preceding interpretation of linking algebras there is a sequence of the corresponding linking algebras

$$0 \longrightarrow L(V) \xrightarrow{\Phi^L} L(W) \xrightarrow{q^L} L(W/\mathrm{Im}\,\Phi) \longrightarrow 0. \tag{3.19}$$

Comparing (3.19) with two sequences above and using (3.17) applied to the ideal submodule $\mathrm{Im}\,\Phi$ of W, we conclude that (3.19) is an extension of the linking algebra $L(V)$:

$$0 \longrightarrow L(V) \xrightarrow{\Phi^L} L(W) \xrightarrow{q^L} L(W)/L(\mathrm{Im}\,\Phi) \longrightarrow 0. \tag{3.20}$$

Now we can state a corollary concerning induced extensions of linking algebras analogous to (and derived from) Theorem 3.12.

Corollary 3.13. *Let $(W, \mathcal{B}, \Phi)$ be a full essential extension of a full Hilbert $\mathcal{A}$-module V. Consider the diagram*

$$\begin{array}{ccccc} V & \xrightarrow{\Phi} & W & \xrightarrow{q} & W/Im\,\Phi \\ \downarrow id & & \downarrow \Lambda & & \downarrow \Delta \\ V & \xrightarrow{\Gamma} & V_d & \xrightarrow{q_d} & Q(V) \end{array}$$

Then the sequence (3.20) induced by the first row of the above diagram is an extension of $L(V)$. Further, there exists a well defined morphism $\Delta^L : L(W)/L(Im\,\Phi) \to L(Q(V))$ induced by the maps Δ and Δ^+. The map Δ^L is the Busby invariant corresponding to the extension (3.20).

Proof. Consider the induced diagram of linking algebras

$$\begin{array}{ccccc} L(V) & \xrightarrow{\Phi^L} & L(W) & \xrightarrow{q^L} & L(W)/L(\mathrm{Im}\,\Phi) \\ \downarrow \mathrm{id} & & \downarrow \Lambda^L & & \downarrow \Delta^L \\ L(V) & \xrightarrow{\Gamma^L} & L(V_d) & \xrightarrow{q_d^L} & L(V_d)/L(\mathrm{Im}\,\Gamma) \end{array} \tag{3.21}$$

and compare it with the Busby picture of the extension (3.20)

$$\begin{array}{ccccc} L(V) & \xrightarrow{\Phi^L} & L(W) & \xrightarrow{q^L} & L(W)/L(\mathrm{Im}\,\Phi) \\ \downarrow \mathrm{id} & & \downarrow \xi & & \downarrow \zeta \\ L(V) & \xrightarrow{\Gamma^L} & M(L(V)) & \xrightarrow{\pi} & Q(L(V)) \end{array} \tag{3.22}$$

We only need to introduce the map Δ^L and to show that it is equal to the above Busby map ζ. But this is already done in Theorem 3.12 - up to the following observation: since $V_d = \boldsymbol{B}(\mathcal{A}, V)$, we can write $L(V_d)$ in the form

$$L(V_d) = \begin{bmatrix} \langle V_d, V_d \rangle & \boldsymbol{B}(V,\mathcal{A}) \\ \boldsymbol{B}(\mathcal{A},V) & \boldsymbol{K}(V_d) \end{bmatrix} \subseteq \begin{bmatrix} \boldsymbol{B}(\mathcal{A}) & \boldsymbol{B}(V,\mathcal{A}) \\ \boldsymbol{B}(\mathcal{A},V) & \boldsymbol{B}(V_d) \end{bmatrix} \tag{3.23}$$

and now we once again appeal Theorem 2.2 from [3] ($\boldsymbol{B}(V_d) = \boldsymbol{B}(V)$) to write

$$L(V_d) \subseteq \begin{bmatrix} \boldsymbol{B}(\mathcal{A}) & \boldsymbol{B}(V,\mathcal{A}) \\ \boldsymbol{B}(\mathcal{A},V) & \boldsymbol{B}(V_d) \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}(\mathcal{A}) & \boldsymbol{B}(V,\mathcal{A}) \\ \boldsymbol{B}(\mathcal{A},V) & \boldsymbol{B}(V) \end{bmatrix} = \boldsymbol{B}(\mathcal{A} \oplus V) =$$

$$= M(\boldsymbol{K}(\mathcal{A} \oplus V)) = M(L(V)). \tag{3.24}$$

This shows that Λ^L can be regarded as a map from $L(W)$ into $M(L(V))$. Now we conclude $\Lambda^L = \xi$ since ξ is the only morphism satisfying $\xi\Phi^L = \Gamma^L$. The remaining equality, namely $\Delta^L = \zeta$, then follows easily. ∎

We end the paper with two additional results concerned with pullbacks of Hilbert C^*-modules. The first one shows that process of forming multiplier modules preserves pullbacks, thus serves as a Hilbert C^*-module version of Proposition 7.2 from [8].

Proposition 3.14. *Let*

$$\begin{array}{ccc} V & \xrightarrow{\Delta_2} & V_2 \\ \downarrow \Delta_1 & & \downarrow \Phi_2 \\ V_1 & \xrightarrow{\Phi_1} & W \end{array}$$

be a pullback diagram of full Hilbert C^*-modules in which all morphisms are surjective. Then the extended diagram on the multiplier modules

$$\begin{array}{ccc} V_d & \xrightarrow{\overline{\Delta_2}} & (V_2)_d \\ \downarrow \overline{\Delta_1} & & \downarrow \overline{\Phi_2} \\ (V_1)_d & \xrightarrow{\overline{\Phi_1}} & W_d \end{array}$$

is also a pullback diagram of Hilbert C^*-modules.

Proof. Recall from the proof of Proposition 1 in [1] that, given a surjective morphism of Hilbert C^*-modules $\Psi : X \to Y$ over a morphism $\psi : \mathcal{A} \to \mathcal{B}$ of the underlying C^*-algebras, the extension $\overline{\Psi} : X_d \to Y_d$ is given by $\overline{\Psi}(r)(\psi(a)) = \Psi(r(a))$, $r \in X_d$. Therefore $\operatorname{Ker} \overline{\Psi} = \{r \in X_d : \operatorname{Im} r \subseteq \operatorname{Ker} \Psi\}$.

Another general observation we need is that X_d is the largest essential extension of a Hilbert C^*-module X.

Now the proof follows by repeating, mutatis mutandis, the argument from 7.2 in [8]. ∎

The following theorem may be usefully in a study of various notions of projectivity. It is a Hilbert C^*-module version of a result which states that a C^*-algebra is projective if and only if it is corona projective ([7], Theorem 10.1.9). Again, the proof below follows the proof for C^*-algebras without changes.

THEOREM 3.15. *Let $q : V \to Z$ be a surjective morphism of full Hilbert C^*-modules, denote* $\operatorname{Ker} q$ *by X. Suppose that a full Hilbert C^*-module P has the property that for each morphism $\Psi : P \to Q(X)$ there exists a morphism $\tilde{\Psi} : P \to X_d$ such that $\Psi = q_d \tilde{\Psi}$. Then for each diagram of the form*

$$\begin{array}{ccc} & & P \\ & & \downarrow \Phi \\ V & \xrightarrow{q} & Z \end{array} \tag{3.25}$$

there exists a morphism $\tilde{\Phi}$ making the diagram

$$\begin{array}{ccc} & & P \\ & \overset{\tilde{\Phi}}{\swarrow} & \downarrow \Phi \\ V & \xrightarrow{q} & Z \end{array} \tag{3.26}$$

commutative.

PROOF. Consider diagram (3.25) and the extension $0 \to X \xrightarrow{\iota} V \xrightarrow{q} Z \longrightarrow 0$ with the corresponding Busby invariant $\Delta : Z \to Q(V)$ to obtain the following diagram

$$\begin{array}{ccccc} & & & & P \\ & & & & \downarrow \Phi \\ X & \xrightarrow{\iota} & V & \xrightarrow{q} & Z \\ \downarrow \mathrm{id} & & \downarrow \Lambda & & \downarrow \Delta \\ X & \xrightarrow{\Gamma} & X_d & \xrightarrow{q_d} & Q(X) \end{array} \tag{3.27}$$

By assumption applied to the map $\Delta\Phi$, there exists a morphism $\tilde{\Psi} : P \to X_d$ such that $\Delta\Phi = q_d \tilde{\Psi}$. By Theorem 3.6 V is (unitarily equivalent to) the pullback for the triple $(X_d, Z, Q(x))$. Applying the pullback property to the coherent pair of morphisms Φ and $\tilde{\Psi}$ we get the map $\tilde{\Phi}$ which serves as a lift for Φ. ∎

ACKNOWLEDGEMENTS.

The authors wish to thank to the referee for several valuable suggestions improving the original manuscript. In particular, the proof of Proposition 3.10 is suggested by the referee. In the original version of the paper Proposition 3.10 was proved in a weaker form with the additional assumption that the image of Φ should contain an essential ideal submodule.

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Received: 26.11.2002.
Revised: 09.01.2003.