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THE EXPECTATION OF SOLUTION OF RANDOM CONTINUITY EQUATION WITH GAUSSIAN VELOCITY FIELD

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Abstract. The continuity equation in d-dimensional space with random velocity field defined by means of a vector-valued Gaussian process is studied. The expectation of corresponding evolution family of operators is explicitly derived, generalizing in a sense the evolution family corresponding to the conventional diffusion equation.

1. Introduction

In the previous paper [3] we studied the random linear differential equation:

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{d} v_i(\omega, t, \mathbf{x}) \frac{\partial u}{\partial x_i} + b(\omega, t, \mathbf{x})u + f(\omega, t, \mathbf{x}). \tag{1}$$

From the physical point of view, equation (1) is the continuity equation that describes the transport of substance in random velocity field $\mathbf{v}(\omega, t, \mathbf{x}) = (v_i(\omega, t, \mathbf{x}))_{i=1}^{d}$, in terms of concentration field $u(\omega, t, \mathbf{x})$. However, it is the mean concentration $\mathbb{E}u$, and not the concentration u itself, that is of primary interest.

From the mathematical point of view, we are required to consider a) existence, uniqueness and measurability of solution, b) the existence of expectation of solution, $\mathbb{E}u$, and c) other properties of $\mathbb{E}u$.

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By using the random evolution family $U(\omega, t, s)$ associated with (1), i.e. generated by

$$A(\omega, t) = \sum_{i=1}^{d} v_i(\omega, t, \mathbf{x}) \frac{\partial}{\partial x_i} + b(\omega, t, \mathbf{x}),$$

solution of (1) can be represented as

$$u(\omega, t) = U(\omega, t, s)u(\omega, s) + \int_s^t dr U(\omega, t, r) f(\omega, r),$$

so the problem of study of u is shifted to the study of the family U. In this way in [3], working in Hilbert (Sobolev) spaces H_k, the existence, uniqueness and measurability of solutions were proved without specific assumptions on the underlying probability space. Next, also in [3], the existence of expectation of solution was proved for the velocity field of the form $v_i(\omega, t, \mathbf{x}) = g_i(\omega, t)\, a_i(\mathbf{x})$, $i = 1, \ldots, d$, where $(g_i(\omega, t))_{i=1}^d$ was d-dimensional Gaussian process. The third problem, namely an explicit calculation of $\mathbb{E}u$, was worked out only for the one-dimensional case:

$$\frac{\partial u}{\partial t} = g(\omega, t)\, a(x) \frac{\partial u}{\partial x},$$

where $g(\omega, t)$ was scalar, Gaussian process. For the zero-mean process g with covariation function $R(t_1, t_2)$, the expectation of corresponding evolution family was given by

$$(\mathbb{E}U)(t, s) = \exp\left\{ \frac{1}{2} h(t, s) \Big(a(x) \frac{d}{dx} \Big)^2 \right\},$$

where $h(t, s) = \int_s^t dr_1 \int_s^t dr_2 R(t_1, t_2)$. In [4] the physically more relevant situation dealing with concept of distributed dispersion was elaborated. Physical consequences were elaborated in [2]. However, regarding the calculation of expectation, both papers [3] and [4] essentially deal with scalar Gaussian processes $g(\omega, t)$.

It is the goal of the present paper to generalize the calculation of $\mathbb{E}u$ given in [3] to the generators of the form

$$A(\omega, t) = \sum_{i=1}^{d} g_i(\omega, t) a_i(t, x_i) \frac{\partial}{\partial x_i}, \tag{2}$$

where $\mathbf{g} = (g_i)_{i=1}^d$ is real, vector-valued Gaussian process, and $\mathbf{a} = (a_i)_{i=1}^d$ is deterministic function. This choice enables an explicit calculation of $\mathbb{E}U$, besides keeping us close to the theory of conventional diffusion equation. First, the problem is solved for bounded generators, i.e. generators of the form $A(\omega, t) = \sum_{i=1}^d g_i(\omega, t) A_i(t)$, where $A_i(\cdot)$ are bounded, linear and commutative operators defined on Banach space. In this case $\mathbb{E}U$ is obtained by summing convergent series of bounded operators, using the commutativity in an essential way. The proof then follows using the Yosida method similarly as in [3], with additional technical difficulties. Namely, Mercer theorem was applied to the covariation function of $\mathbf{g}$ in order to prove that the generators involved are regularly dissipative.

The paper is organized as follows. Section 2 contains all relevant statements, i.e. necessary definitions, assumptions, results from [3] and [4] that are required, and the two new theorems (Theorem 2.7 and the main Theorem 2.11). Section 3 contains the proofs. Nearly all the technical work is shifted into the auxiliary Section 4.

Having in mind that the present paper deals with the exact calculation of mean concentration field, we do not refer to the literature on homogenization and renormalization theories of turbulence. (See, however, a recent review [10], and the literature given therein.) Finally, let us mention that, without commutativity explicit calculation become difficult due to the apparent terms of lower order. Some results for bounded generators, within the framework of Colombeau generalized functions, may be found in [7]. Results presented here comprise a part of author's doctoral dissertation [6].

2. Statements of results

In this section we recall first, the notion of random operator and the notion of evolution family. Than we recall results on existence of random evolution family U and it's expectation $\mathbb{E}U$, and finally state Theorem 2.7 giving the explicit formula for $\mathbb{E}U$ in the case of bounded generators. The same pattern is then followed for the case of differential generators, finishing with the main Theorem 2.11.

Let $\{\Omega, \mathcal{F}, P\}$ be a probability space and X a separable Banach space. We denote by B_X the σ-algebra of Borel subsets of X.

Definition 2.1. a) *A mapping $u : \Omega \to X$ defined on subset $\Omega_u \subseteq \Omega$ is called (X-valued) random element if it is measurable with respect to the σ-algebras $\mathcal{F}$ and B_X, i.e. if for every open set $\mathcal{O} \subset B_X$ it holds $u^{-1}(\mathcal{O}) = \{\omega \in \Omega_u; u(\omega) \in \mathcal{O}\} \subseteq \mathcal{F}$.*

b) *Let Γ be an arbitrary set. A mapping $A : \Omega \times \Gamma \to X$ is said to be a random operator if $y(\cdot) = A(\cdot, \gamma)$ is X-valued random element for every $\gamma \in \Gamma$. In general, domain $D_\omega(A) = \{\gamma \in \Gamma;\ A(\omega, \gamma) \in X\}$ depends on $\omega \in \Omega$.*

c) *If Γ is a linear space then a random operator $A : \Omega \times \Gamma \to X$ is linear if the mapping $A(\omega) : \Gamma \to X$ is a linear operator for a.e. $\omega \in \Omega$.*

Definition 2.2. *Let X be a Banach space, $T > 0$, $A(t) : X \to X$ a family of densely defined linear operators, $0 \leq t \leq T$, and let $U(t, s) \in L(X)$, $0 \leq s \leq t \leq T$, be a family of operators such that:*

(a) *$U(t, s)$ are strongly continuous with respect to $0 \leq s \leq t \leq T$,*

(b) *$U(t, t) = I$, $U(t, s) = U(t, r)U(r, s)$, $0 \leq s \leq r \leq t \leq T$,*

(c) $\|U(t,s)\|_X \le \exp(\beta|t-s|)$, *for some constant* β *that depends only on* T,

(d) $\partial/\partial t\, U(t,s) = A(t)U(t,s)$ *on a dense subset* $Y \subset X$, $s \le t < T$,

(e) $\partial/\partial s\, U(t,s) = -U(t,s)A(s)$ *on a dense subset* $Y \subset X$, $0 < s \le t$.

Then $U(t,s)$ *is called the family of evolution operators generated by the family* $A(t)$.

Next theorem gives the existence of random evolution family in the case of bounded generators. It's proof may be found in [4, Theorem 1.6].

THEOREM 2.3. *Let* X *be a separable Banach space and* $A : \Omega \times [-T,T] \to L(X)$ *a random operator such that* $A(\omega,\cdot) \in C([-T,T], L(X))$ *for a.e.* $\omega \in \Omega$. *Then there exists a mapping* $U : \Omega \times [-T,T] \times [-T,T] \to L(X)$ *that fulfills:*

(a) *For a.e.* $\omega \in \Omega$, $(t,s) \mapsto U(\omega,t,s)$ *is an evolution family with* $Y = X$ *and* $\beta(\omega) = \max_{r\in[-T,T]} \|A(\omega,r)\|_{L(X)}$. *Moreover, all the properties from Definition* 2.1 *remain valid if* $-T \le s \le t \le T$ *is replaced by* $-T \le s,t \le T$. *Thus,* $U(\omega,t,s)$ *is invertible and* $U(\omega,t,s)^{-1} = U(\omega,s,t)$.

(b) *For all* $t,s \in [-T,T]$, *the mapping* $U(\cdot,t,s) : \Omega \to L(X)$ *is a random operator.*

The family $U(\omega,t,s)$ is given by the following series:

$$U(\omega,t,s) = I + \sum_{k=1}^{\infty} \int_s^t dt_1 \cdots \int_s^{t_{k-1}} dt_k A(\omega,t_1)\cdots A(\omega,t_k). \tag{3}$$

DEFINITION 2.4. *Let* X *be a Banach space and let* $u : \Omega \to X$ *be a Bochner integrable random element. The expectation* $\mathbb{E}u \in X$ *of* u *is defined by* $\mathbb{E}u = \int u(\omega)dP(\omega)$. *Random operator* $A : \Omega \times X \to X$ *is said to have expectation in strong sense if for any* $u \in X$ *the random element* Au *has expectation. We write* $(\mathbb{E}A)u = \mathbb{E}(Au)$.

In order to get the existence of expectation of evolution family, $\mathbb{E}U$, we have to specify the probability space more closely.

SUPPOSITION 2.5. *By* $\mathbf{g} : \Omega \times [-T,T] \to \mathbb{R}^d$, $\mathbf{g}(\omega,t) = (g_i(\omega,t))_{i=1}^d$, *we denote a real vector-valued Gaussian process with continuous paths.*

The mean value of $\mathbf{g}$ *is denoted by* $\mathbf{m}(t) = (m_i(t))_{i=1}^d := \mathbb{E}(\mathbf{g}(\cdot,t))$, *and covariation function is denoted by* $\mathsf{R}(t_1,t_2) = (\mu_{i,j}(t_1,t_2))_{i,j=1}^d := \mathbb{E}\big((\mathbf{g}(\cdot,t_1) - \mathbf{m}(t))^{\mathsf{T}}(\mathbf{g}(\cdot,t_2) - \mathbf{m}(t))\big)$.

The functions $\mathbf{m}(t)$ and $\mathsf{R}(t_1,t_2)$ are continuous which follows easily from the continuity of paths.

The following proposition gives the existence of $\mathbb{E}U$, in the case of bounded generators. The proof is analogous to that of [3, Proposition 4.2].

PROPOSITION 2.6. *Let*

$$A(\omega,t)=\sum_{i=1}^{d} g_i(\omega,t)A_i(t), \tag{4}$$

where $\mathbf{g}$ *is a Gaussian process from Supposition* 2.5*, and* $A_i(\cdot)\in C\big([-T,T],L(X)\big)$*,* $i=1,\dots,d$*. Then, the evolution family* $U(\omega,t,s)$ *generated by* $A(\omega,t)$ *has the expectation* $V:=\mathbb{E}U:[-T,T]\times[-T,T]\to L(X)$ *in strong sense and the mapping* $(t,s)\mapsto V(t,s)$ *is continuous.*

Next theorem, using the additional assumption of commutativity of generators, gives the explicit formula for $\mathbb{E}U$ in the case of bounded generators. It's proof is given in Section 3.

THEOREM 2.7. *Let the family* $A(\omega,t)$ *be given by (4) where* $\mathbf{g}$ *is a Gaussian process from Supposition* 2.5*, and* $A_i(\cdot)\in C\big([-T,T],\ L(X)\big)$*,* $i=1,\dots,d$ *commute with each other. Put* $\mathbf{A}(t):=(A_i(t))_{i=1}^d$*,* $M(t)=\mathbf{m}(t)^{\mathsf{T}}\mathbf{A}(t)=\sum_{i=1}^d m_i(t)A_i(t)$*, and* $R(t_1,t_2)=\mathbf{A}(t_2)^{\mathsf{T}}\mathsf{R}(t_1,t_2)\mathbf{A}(t_1)=\sum_{i,j=1}^d \mu_{ij}(t_1,t_2)A_j(t_2)A_i(t_1)$*. Then,* $M(\cdot)$ *and* $R(\cdot,\cdot)$ *are* $L(X)$*-valued continuous functions and the expectation* $V(t,s)$ *of evolution family* $U(\omega,t,s)$ *is given by*

$$V(t,s)=\exp\left\{\int_s^t dt_1 M(t_1)+\frac{1}{2}\int_s^t dt_1\int_s^t dt_2 R(t_1,t_2)\right\},\quad \textit{for } s\le t, \tag{5}$$

$$V(t,s)=\exp\left\{\int_s^t dt_1 M(t_1)+\frac{1}{2}\int_t^s dt_1\int_t^s dt_2 R(t_1,t_2)\right\},\quad \textit{for } t\le s. \tag{6}$$

Finally, we turn to the case of differential operators that is connected with the random linear transport equation. Here, the processes are considered on $\tilde{D}=[-T,T]\times\mathbb{R}^d$. Elements of $\tilde{D}$ are denoted by $\{t,\mathbf{x}\}$, where $t\in[-T,T]$ denotes "time", and $\mathbf{x}=(x_i)_{i=1}^d=(x_1,\dots,x_d)^{\mathsf{T}}$ denotes "space". The differentiation with respect to x_i is denoted by $\partial/\partial x_i$ or by ∂_i. Scalar product in $\mathbb{R}^d$ is denoted by $\mathbf{x}^{\mathsf{T}}\mathbf{y}:=\sum_{i=1}^d x_iy_i$, and in $\mathbb{C}^d$ by $\mathbf{x}^*\mathbf{y}:=\sum_{i=1}^d \overline{x_i}y_i$. By a vector valued function $\mathbf{u}=(u_i)_{i=1}^d$ we mean a sequence of d functions $u_1,\dots,u_d$ on $\tilde{D}$. The gradient of function

$$u\in L_{\infty,1}(\mathbb{R}^d)=\{u\in L_\infty(\mathbb{R}^d);\ \partial_i u\in L_\infty(\mathbb{R}^d)\}$$

is defined by $\nabla u=(\partial_i u)_{i=1}^d\in\big(L_\infty(\mathbb{R}^d)\big)^d$. Then $\mathbf{u}^{\mathsf{T}}\nabla v=\sum u_i\partial v_i$. The divergence of a vector valued function $\mathbf{u}\in\big(L_{\infty,1}(\mathbb{R}^d)\big)^d$ is defined by $\operatorname{div}\mathbf{u}=\sum\partial_i u_i$.

For a Banach space X the norm is denoted by $\|\cdot\|_X$. Only for the Hilbert spaces H_k defined inductively by:

$$H_0=L_2(\mathbb{R}^d),\quad H_k=\{u\in H_{k-1},\partial_i u\in H_{k-1}\},\ \text{for } k=1,2,\dots,$$

the norms and scalar products are denoted by $\|\cdot\|_k$ and $(\cdot,\cdot)_k$, respectively. In particular $(u,v)_1 = (u,v)_0 + \sum(\partial_i u, \partial_i v)_0$. Then we put $H_\infty = \cap_{k\geq 0} H_k$. Furthermore, the Banach algebras of all bounded linear operators defined on H_k, $k = 0,1$ are denoted by $L(H_k)$ and corresponding norms also by $\|\cdot\|_k$, since no ambiguity can occur.

By $\mathcal{D}'(\mathbb{R}^d)$ we denote the space of Schwartz distributions. We shall also need the space $L_{\infty,2}(\mathbb{R}^d) = \{u \in L_{\infty,1}(\mathbb{R}^d);\ \partial_i u \in L_{\infty,1}(\mathbb{R}^d)\}$. The commutator of operators A and B is denoted by $[A,B] := AB - BA$.

Again, we recall the results on existence of random evolution family and it's expectation.

Supposition 2.8. *The functions* $a_i : \Omega \to C\big([-T,T], L_{\infty,1}(\mathbb{R}^d)\big)$, $i = 1,\dots,d$ *and* $b : \Omega \to C\big([-T,T], L_\infty(\mathbb{R}^d)\big)$ *are random elements. In addition,* $\partial_i b \in L_\infty\big((-T,T)\times\mathbb{R}^d\big)$ *for a.e.* $\omega \in \Omega$.

For a.e. $\omega \in \Omega$ we define the quantities:

$$\lambda(\omega,t) = \| -\operatorname{div}\mathbf{a}(\omega,t) + 2b(\omega,t)\|_{L_\infty(\mathbb{R}^d)} \tag{7}$$

$$\beta(\omega) = \sup_{r\in[-T,T]} \lambda(\omega,r). \tag{8}$$

Theorem 2.9. *Let* $X = H_0(\mathbb{R}^d)$ *and let the Supposition* 2.8 *be valid. Then there exists a mapping* $U : \Omega\times[-T,T]\times[-T,T]\times H_0 \to H_0$ *such that:*

(a) *For a.e.* $\omega\in\Omega$, $U(\omega,\cdot,\cdot)$ *is an evolution family with* $Y = H_1(\mathbb{R}^d)$ *and* $\beta(\omega)$ *given by* (8). *Moreover, all the properties from Definition* 2.1 *remain valid if* $-T\leq s\leq t\leq T$ *is replaced by* $-T\leq s,t\leq T$. *Thus* $U(\omega,t,s)$ *is invertible and* $U(\omega,t,s)^{-1} = U(\omega,s,t)$.
(b) *For all* $t,s\in[-T,T]$, *the mapping* $U(\cdot,t,s) : \Omega\to H_0$ *is a random operator.*

Proof may be found in [3, Theorem 3.5].

Proposition 2.10. *Let the family* $A(\omega,t)$ *be given by* (2) *with* $\mathbf{g} = (g_i)_{i=1}^d$ *being a Gaussian process from Supposition* 2.5 *and* $a_i(\cdot)\in C\big([-T,T], L_{\infty,1}(\mathbb{R})\big)$, $i = 1,\dots,d$. *Then, the evolution family* $U(\omega,t,s)$ *generated by* $A(\omega,t)$ *has the expectation* $V : [-T,T]\times[-T,T]\times X\to X$ *in strong sense and* $V(t,s)$ *is strongly continuous* (*with respect to* t,s) *family of bounded, linear operators.*

The proof is the same as in [3, Proposition 4.2]. Before stating the main result, we need some preliminary notions.

Let us denote:

$$\hat{\mathbf{A}}(t) := (\hat{A}_i(t))_{i=1}^d, \qquad \text{where} \qquad \hat{A}_i(t) := a_i(t,\mathbf{x})\frac{\partial}{\partial x_i}. \tag{9}$$

For $u \in H_2$ the operator $B : H_2 \subset H_0 \to H_0$ is defined by (Bochner integrals in H_0)

$$
\begin{aligned}
B(t,s)u &= \int_s^t dt_1 \int_s^t dt_2 \hat{\mathbf{A}}^{\mathsf{T}}(t_1)\mathsf{R}(t_1,t_2)\hat{\mathbf{A}}(t_2)u \\
&= \int_s^t dt_1 \int_s^t dt_2 \sum_{i,j=1}^{d} \mu_{i,j}(t_1,t_2)a_j(t_2,\mathbf{x})\frac{\partial}{\partial x_j}a_i(t_1,\mathbf{x})\frac{\partial}{\partial x_i}u.
\end{aligned}
$$

In the next section it will be shown that B has unique, regularly dissipative extension (still denoted by) $B : D(B) \to H_0$. Hence, the exponential function of B is defined via the Dunford integral.

THEOREM 2.11. *Let*

$$A(\omega,t) = \sum_{i=1}^{d} g_i(\omega,t)a_i(t,x_i)\frac{\partial}{\partial x_i},$$

with $\mathbf{g} = (g_i)_{i=1}^d$ *being a Gaussian process from Supposition* 2.5 *and* $a_i(\cdot) \in C\big([-T,T],\ L_{\infty,2}(\mathbb{R})\big)$, $i = 1,\dots,d$. *Suppose that the process* $\mathbf{g}$ *is centered, i.e.* $\mathbf{m}(t) = 0$. *Then the expectation of the evolution family* $U(\omega,t,s)$ *(whose existence and other properties are given in Theorem 2.9) generated by* $A(\omega,t)$ *is given by* $V(t,s) = \exp\{1/2B(t,s)\}$, *for* $s \le t$ *and* $V(t,s) = V(s,t)$, *for* $t < s$.

3. PROOFS

PROOF OF THEOREM 2.7. Continuity of functions $M(t)$ and $R(t_1,t_2)$ follows easily from the continuity of $\mathbf{m}(t)$, $\mathsf{R}(t_1,t_2)$.

Suppose that $\mathbf{m}(t) = 0$, and take $s \le t$. From (3) we have:

$$
\begin{aligned}
&U(\omega,t,s) = I + \\
&\quad \sum_{k=1}^{\infty} \int_s^t dt_1 \cdots \int_s^{t_{k-1}} dt_k \sum_{i_1,\dots,i_k} g_{i_1}(\omega,t_1)\cdots g_{i_k}(\omega,t_k)A_{i_1}(t_1)\cdots A_{i_k}(t_k).
\end{aligned}
$$

Before applying the expectation operator, note that the moments of odd order of centered Gaussian process vanish, while the moments of even order are given by (see [1])

$$\mathbb{E}(g_{i_1}(\omega,t_1)\cdots g_{i_{2k}}(\omega,t_{2k})) = \sum_{\mathfrak{S}\in\mathfrak{S}_{2k}} \prod_{\{j_1,j_2\}\in\mathfrak{S}} \mu_{i_{j_1} i_{j_2}}(t_{j_1},t_{j_2}). \tag{10}$$

Here, $\mathfrak{S}_{2k}$ denotes the collection of all partitions of the set $\{1,\dots,2k\}$ in two-element subsets. It is easily seen that there are exactly $(2k-1)!! = (2k)!/2^k k!$ such partitions.

A straightforward calculation now gives:

$$\begin{aligned}
V(t,s) &= \mathbb{E}U(\cdot,t,s) \\
&= I + \sum_{k=1}^{\infty}\int_s^t dt_1 \int_s^{t_1} dt_2 \cdots \int_s^{t_{2k-1}} dt_{2k} \\
&\qquad \sum_{i_1,\dots,i_{2k}}^{d} \sum_{\mathfrak{s}\in\mathfrak{S}_{2k}} \prod_{\{j_1,j_2\}\in\mathfrak{s}} \mu_{i_{j_1}i_{j_2}}(t_{j_1},t_{j_2}) A_{i_{j_1}}(t_{j_1}) A_{i_{j_2}}(t_{j_2}) \\
&= I + \sum_{k=1}^{\infty}\frac{1}{(2k)!}\int_s^t dt_1 \int_s^{t} dt_2 \cdots \int_s^{t} dt_{2k} \sum_{\mathfrak{s}\in\mathfrak{S}_{2k}} \prod_{\{j_1,j_2\}\in\mathfrak{s}} R(t_{j_1},t_{j_2}) \\
&= I + \sum_{k=1}^{\infty}\frac{1}{k!}\left[\frac{1}{2}\int_s^t dt_1 \int_s^t dt_2 R(t_1,t_2)\right]^k \\
&= \exp\left\{\frac{1}{2}\int_s^t dt_1 \int_s^t dt_2 R(t_1,t_2)\right\}.
\end{aligned}$$

The second equality is obtained by using Fubini theorem, the above mentioned structure of moments and commutativity of families $A_i(\cdot)$. The same argument gives the symmetry of the function under the integral sign which implies the third equality. Fourth equality follows from Fubini theorem while the last one is obvious.

In the same way, for $t \le s$ we obtain:

$$V(t,s) = \exp\left\{\frac{1}{2}\int_t^s dt_1 \int_t^s dt_2 R(t_1,t_2)\right\}.$$

Now, we consider the case $\mathbf{m}(t) \neq 0$. Generators $A(\omega,t)$ are split in two parts, $A(\omega,t) = A^{(1)}(t) + A^{(2)}(\omega,t)$, where $A^{(1)}(t) = \sum_{i=1}^d m_i(t)A_i(t)$, and $A^{(2)}(\omega,t) = \sum_{i=1}^d (g_i(\omega,t) - m_i(t))A_i(t)$. According to Theorem 2.3 operators $A(\omega,\cdot)$ and $A^{(2)}(\omega,\cdot)$ generate random evolution families $U(\omega,\cdot,\cdot)$, $U^{(2)}(\omega,\cdot,\cdot)$, respectively. Operators $A^{(1)}(t)$ generate deterministic evolution family $U^{(1)}(t,s) = \exp\{\int_s^t dr M(r)\}$, for $-T \le s,t \le T$. Because of commutativity of the operators $A^{(1)}(\cdot)$ and $A^{(2)}(\omega,\cdot)$ we have $U(\omega,t,s) = U^{(1)}(t,s)U^{(2)}(\omega,t,s)$. Applying expectation operator and utilizing linearity and continuity, problem is reduced to the case $\mathbf{m}(t) = 0$. ∎

Proof of Theorem 2.11. Let $\psi \in C_0^\infty(\mathbb{R})$ be an even function such that $\operatorname{supp}\psi = [-1,1]$, $\psi(x) \ge 0$ and $\int dx\,\psi(x) = 1$. For $n \in \mathbb{N}$ and $1 \le i \le d$, let $J_{n,i}$, $J_n : H_0 \to H_0$ be the operators of convolution with functions $\psi_{n,i}(\mathbf{x}) = n\psi(nx_i)$, $\psi_n(\mathbf{x}) = n^d \prod_{i=1}^d \psi(nx_i)$, respectively. J_n is the so called

Friedrichs mollifier. Now, we define a family of bounded operators

$$A_n(t) = \sum_{i=1}^{d} g_i(\omega, t) J_{n,i} a_i(t, x_i) \frac{\partial}{\partial x_i} J_{n,i} \in L(H_0)$$

and also

$$\begin{aligned} B_n(t,s) &= \int_s^t dt_1 \int_s^t dt_2 \hat{\mathbf{A}}_n^{\mathsf{T}}(t_1) \mathsf{R}(t_1, t_2) \hat{\mathbf{A}}_n(t_2) \\ &= \int_s^t dt_1 \int_s^t dt_2 \sum_{i,j=1}^{d} \mu_{i,j}(t_1, t_2) \hat{A}_{n,j}(t_2) \hat{A}_{n,i}(t_1), \end{aligned} \tag{11}$$

where

$$\hat{\mathbf{A}}_n(t) := (\hat{A}_{n,i}(t))_{i=1}^d, \qquad \hat{A}_{n,i}(t) := J_{n,i} a_i(t, x_i) \partial / \partial x_i J_{n,i}.$$

Proof of the Theorem is based on the diagram of Figure 1. The op-

$$\begin{tikzcd} A_n \arrow[r] \arrow[d] & U_n \arrow[r] \arrow[d] & V_n := \mathbb{E}U_n \arrow[r, phantom, "="] \arrow[d] & \exp \frac{1}{2} B_n \arrow[d] \\ A \arrow[r] & U \arrow[r] & V := \mathbb{E}U \arrow[r, phantom, "\leftrightarrow"] & \exp \frac{1}{2} B \end{tikzcd}$$

FIGURE 1. Schematic presentation of the method used to calculate the expectation of evolution family.

erators $A_n(\omega, t)$ generate random evolution families $U_n(\omega, t, s)$, $n \in \mathbb{N}$, with expectation $V_n(t,s) := \mathbb{E}U_n(\omega, t, s) \in L(H_0)$ (Theorem 2.3). Since by the construction, operators $A_n(\omega, t)$ commute with each other, we have $V_n(t,s) = \exp\{1/2 B_n(t,s)\}$ (Theorem 2.7).

On the other side, the family $A(\omega, t)$ generate random evolution family $U(\omega, t, s)$ (Theorem 2.9) with expectation $V(t,s)$ in the strong sense. We know that $U_n(\omega, t, s) \to U(\omega, t, s)$ strongly in H_0 for a.e. $\omega \in \Omega$ (see Theorem 2.5 in [3]). Next, evolution families $U_n(\omega, t, s)$, $n \in \mathbb{N}$ and $U(\omega, t, s)$ have the same constants $\beta(\omega)$ given by (8) (see [3], Lemma 2.4) and the function $\omega \mapsto \exp\{C\beta(\omega)\}$ is integrable for any constant C (see [3, Proposition 4.2]). Lebesgue theorem implies that $V_n(t,s) \to V(t,s)$, strongly in H_0, when $n \to \infty$.

Thus, it is enough to prove that $\exp\{1/2 B_n(t,s)\} \to \exp\{1/2 B(t,s)\}$ strongly in H_0, when $n \to \infty$. Let Σ be the sector from Proposition 4.7 and let $\Gamma : \mathbb{R} \to \mathbb{C}$ be a positively oriented, continuous path in Σ such that for $|s| > 1$, $\Gamma(s) = \tilde{\lambda}' + s \cdot \exp\{\pm i(\pi/2 + \varepsilon'')\}$ for some $\varepsilon'' < \varepsilon'$, where the $-$

sign corresponds to $s \leq -1$, and the $+$ sign corresponds to $s \geq 1$. Utilizing Proposition 4.7 the following representation is valid:

$$V(t,s)u = \exp\left\{\frac{1}{2}B(t,s)\right\}u = \frac{1}{2\pi i}\int_\Gamma dz \exp\left\{\frac{1}{2}z\right\}(B(t,s)-z)^{-1}u.$$

By changing $B(t,s)$ to $B_n(t,s)$, the representation for $V_n(t,s)$ is obtained (Theorem 2.7 and Proposition 4.7). Thus,

$$(V(t,s)-V_n(t,s))u =$$

$$\frac{1}{2\pi i}\int_\Gamma dz \exp\left\{\frac{1}{2}z\right\}\left[(B(t,s)-z)^{-1} - (B_n(t,s)-z)^{-1}\right]u = \tag{12}$$

$$\frac{1}{2\pi i}\int_\Gamma dz \exp\left\{\frac{1}{2}z\right\}\left[(B_n(t,s)-z)^{-1}(B_n-B)(B(t,s)-z)^{-1}\right]u. \tag{13}$$

Now, let $u \in B(H_\infty)$, so we have $v := (B-z)^{-1}u \in H_\infty$. Lemma 4.8 and Proposition 4.7 imply that integrand in (13) tends to zero in H_0 for every $z \in \Gamma$. Proposition 4.7 implies that the sequence $\left[(B(t,s)-z)^{-1} - (B_n(t,s)-z)^{-1}\right]u$ is bounded in H_0, uniformly in $z \in \Gamma$. By Lebesgue theorem $V_n(t,s) \to V(t,s)$ strongly on $B(H_\infty)$. Because the set $B(H_\infty) \subseteq H_0$ is dense (lemma 4.4), we finally obtain that $V_n(t,s) \to V(t,s)$ strongly on H_0. ◻

COROLLARY 3.1. *Let the assumptions of Theorem* 2.11 *be valid, except that the expectation* $\mathbf{m}(t) = \mathbb{E}\mathbf{g}(\omega,t)$ *is allowed to be different from zero. Then the family* $V(t,s)$ *having all the properties from Theorem* 2.11*, is given by:*

$$V(t,s) = \exp\left\{\int_s^t \mathbf{m}^\mathsf{T}(t_1)\hat{\mathbf{A}}(t_1) \;+\; \frac{1}{2}B(t,s)\right\} \textit{ for } -T \leq s \leq t \leq T,$$

$$V(t,s) = \exp\left\{\int_s^t \mathbf{m}^\mathsf{T}(t_1)\hat{\mathbf{A}}(t_1) \;+\; \frac{1}{2}B(s,t)\right\} \textit{ for } -T \leq t \leq s \leq T.$$

PROOF. The proof is analogous to the proof of Corollary 4.4 in [3]. ◻

We may think of $B(t,s)$ as a generalized diffusion operator. However, there is no differential equation governing the expectation of solution of (1), as has been explained in [3, 4]. When the covariation function of $\mathbf{g}$ tends to $\delta(t)I$, which means that $\mathbf{g}$ itself become the Gaussian white noise, formally, the diffusion equation is obtained. Some results in this direction may be found in [6, 7].

4. Auxiliary results

This section contains technical results regarding operators B and B_n from Sections 2 and 3. The two of them are the most important. First result states that $B(t,s)$ may be extended uniquely to the regularly dissipative operator (Proposition 4.3), while the second one states that the image $B(H_\infty)$ is dense in H_0 (Lemma 4.4).

Let Supposition 2.5 be valid. Then the covariation function $\mathsf{R}(t_1,t_2)$ is a positive definite, continuous function of it's arguments. According to the classical Mercer theorem (e.g. [8]), for fixed $s<t$ there is a sequence of continuous, L_2-orthonormal vector functions $\boldsymbol{\phi}^{(j)}=\big(\phi_i^{(j)}\big)_{i=1}^d:[s,t]\to\mathbb{R}^d$ and a sequence of real numbers $\lambda_j>0$, such that:

$$\mathsf{R}(t_1,t_2)=\sum_{j=1}^{\infty}\lambda_j\boldsymbol{\phi}^{(j)}(t_1)\boldsymbol{\phi}^{(j)\mathsf{T}}(t_2),\quad t_1,t_2\in[s,t]. \tag{14}$$

The series (14) converges uniformly in $t_1,t_2\in[s,t]$. Moreover, the series obtained by taking the L_2-norm of each member converges uniformly, too.

From now on, up to Lemma 4.5, we suppose that $a_i(t,\mathbf{x})\in C\big([-T,T],L_{\infty,1}(\mathbb{R}^d)\big)$.

Lemma 4.1. *For $j\in\mathbb{N}$ let $\mathbf{f}^{(j)}:=\big(f_i^{(j)}\big)_{i=1}^d=\Big(\int_s^t dt_1\phi_i^{(j)}(t_1)a_i(t_1,\mathbf{x})\Big)_{i=1}^d$ be a vector function and let us define the operator $F^{(j)}:H_2\subset H_0\to H_0$, by $F^{(j)}=\mathbf{f}^{(j)\mathsf{T}}\nabla$. Then we have $Bu=\sum_{j=1}^\infty\lambda_j{F^{(j)}}^2u$ in H_0, for all $u\in H_2$.*

Proof. Let $u\in H_2$. Then

$$\Big\|\int_s^t dt_2\Big[\mathsf{R}(t_1,t_2)\hat{\mathbf{A}}(t_2)-\sum_{j=1}^N\lambda_j\boldsymbol{\phi}^{(j)}(t_1)\boldsymbol{\phi}^{(j)\mathsf{T}}(t_2)\hat{\mathbf{A}}(t_2)\Big]u\Big\|_1\le$$

$$\le\int_s^t dt_2\max_{t_1,t_2\in[s,t]}\Big\|\mathsf{R}(t_1,t_2)-\sum_{j=1}^N\lambda_j\boldsymbol{\phi}^{(j)}(t_1)\boldsymbol{\phi}^{(j)\mathsf{T}}(t_2)\Big\|_{L(\mathbb{R}^d)}\|\hat{\mathbf{A}}(t_2)u\|_1\to0$$

uniformly in t_1, as $N\to\infty$. Thus

$$\begin{aligned}\sum_{j=1}^N\lambda_j{F^{(j)}}^2u&=\sum_{j=1}^N\lambda_j\left[\int_s^t dt_1\boldsymbol{\phi}^{(j)}(t_1)\hat{\mathbf{A}}(t_1)\right]^2u\\&=\int_s^t dt_1\hat{\mathbf{A}}^{\mathsf{T}}(t_1)\Big[\sum_{j=1}^N\lambda_j\boldsymbol{\phi}^{(j)}(t_1)\int_s^t dt_2\,\boldsymbol{\phi}^{(j)\mathsf{T}}(t_2)\hat{\mathbf{A}}(t_2)\Big]u\to B(t,s)u\end{aligned}$$

in H_0 as $N\to\infty$. ∎

Now, the space $\mathcal{V}\subset H_0$ is defined by the closure of H_∞ in the norm

$$\|u\|_{\mathcal{V}}^2:=\sum_{j=1}^\infty\lambda_j\|F^{(j)}u\|_0^2+\|u\|_0^2.$$

Then bilinear form $\mathfrak{b}:\mathcal{V}\times\mathcal{V}\to\mathbb{C}$ is defined by:

$$\mathfrak{b}(u,v):=-\sum_{j=1}^\infty\lambda_j(F^{(j)}u,F^{(j)}v)_0-\sum_{j=1}^\infty\lambda_j(F^{(j)}u,\operatorname{div}\mathbf{f}^{(j)}\cdot v)_0.$$

Next lemma guarantees that the above definitions are correct and gives much more.

LEMMA 4.2. (a) *For every* $u \in H_\infty$ *(even for* $u \in H_1$*) it holds* $\|u\|_{\mathcal{V}} < \infty$,

(b) *the form* $\mathfrak{b}(\cdot,\cdot)$ *is well defined and continuous,*

(c) *there are constants* $\vartheta > 0$ *and* $\tilde{\lambda} > 0$ *such that for all* $u \in \mathcal{V}$ *it holds* $\operatorname{Re}\mathfrak{b}(u,u) \le -\vartheta\|u\|_{\mathcal{V}}^2 + \tilde{\lambda}\|u\|_0^2$,

(d) *for* $u \in H_2, v \in \mathcal{V}$ *it holds* $(Bu, v)_0 = \mathfrak{b}(u,v)$.

PROOF. (a) For fixed $N \in \mathbb{N}$ and $u \in H_\infty$ (or H_1) it holds:

$$\sum_{j=1}^N \lambda_j \|F^{(j)}u\|_0^2 = \int_s^t dt_2 \Big(\int_s^t dt_1 \sum_{j=1}^N \lambda_j \boldsymbol{\phi}^{(j)}(t_2)\boldsymbol{\phi}^{(j)\mathsf{T}}(t_1)\hat{\mathbf{A}}(t_1)u, \hat{\mathbf{A}}(t_2)u\Big)_0.$$

The assertion follows from Mercer theorem (uniform convergence) and the continuity of function $t \mapsto \hat{\mathbf{A}}(t)u : \mathbb{R} \to (H_0)^d$, for fixed $u \in H_1 \supset H_\infty$.

(b) Let us denote $\partial\mathbf{a}(t) := (\partial_{x_i} a_i(t,\cdot))_{i=1}^d$. Similarly as in (a) we have

$$\sum_{j=1}^N \lambda_j \|\operatorname{div}\mathbf{f}^{(j)} \cdot u\|_0^2 \to \Big(\Big[\int_s^t dt_2 \int_s^t dt_1 \partial\mathbf{a}(t_1)^\mathsf{T} \mathsf{R}(t_1,t_2)\partial\mathbf{a}(t_2)\Big]u, u\Big)_0,$$

as $N \to \infty$. Now, for $u, v \in \mathcal{V}$ it holds

$$\sum_{j=M}^N \lambda_j |(F^{(j)}u, F^{(j)}v)_0| + \sum_{j=M}^N \lambda_j |(F^{(j)}u, \operatorname{div}\mathbf{f}^{(j)} \cdot v)_0| \le$$
$$\Big(\sum_{j=M}^N \lambda_j \|F^{(j)}u\|_0^2\Big)^{1/2} \Big[\Big(\sum_{j=M}^N \lambda_j \|F^{(j)}v\|_0^2\Big)^{1/2} + \Big(\sum_{j=M}^N \lambda_j \|\operatorname{div}\mathbf{f}^{(j)} \cdot v\|_0^2\Big)^{1/2}\Big],$$

whereat the right-hand side tends to zero as $M, N \to \infty$. This shows that the form $\mathfrak{b}$ is well defined.

Put $M = 1$ and let $N \to \infty$. It follows

$$|\mathfrak{b}(u,v)| \le \|u\|_{\mathcal{V}}\big[\|v\|_{\mathcal{V}} + C_1\|v\|_0\big] \le C\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}},$$

where

$$C_1 = \Big\|\int_s^t dt_2 \int_s^t dt_1 \partial\mathbf{a}(t_1)^\mathsf{T}\mathsf{R}(t_1,t_2)\partial\mathbf{a}(t_2)\Big\|_{L_\infty(\mathbb{R}^d)}^{1/2}, \tag{15}$$

so $\mathfrak{b}$ is continuous.

(c) Let $u \in \mathcal{V}$. Then we have

$$\begin{aligned}
\operatorname{Re} \mathfrak{b}(u,u) &= -\sum_{j=1}^{\infty} \lambda_j \|F^{(j)}u\|_0^2 - \operatorname{Re} \sum_{j=1}^{\infty} \lambda_j (F^{(j)}u, \operatorname{div} \mathbf{f}^{(j)} \cdot u)_0 \\
&\le -\sum_{j=1}^{\infty} \lambda_j \|F^{(j)}u\|_0^2 + \Big(\sum_{j=1}^{\infty} \lambda_j \|F^{(j)}u\|_0^2\Big)^{1/2} \cdot \Big(\sum_{j=1}^{\infty} \lambda_j \|\operatorname{div} \mathbf{f}^{(j)} \cdot u\|_0^2\Big)^{1/2} \\
&\le -\sum_{j=1}^{\infty} \lambda_j \|F^{(j)}u\|_0^2 + \frac{\varepsilon}{2} \sum_{j=1}^{\infty} \lambda_j \|F^{(j)}u\|_0^2 + \frac{1}{2\varepsilon} \sum_{j=1}^{\infty} \lambda_j \|\operatorname{div} \mathbf{f}^{(j)} \cdot u\|_0^2 \\
&\le -\big(1 - \frac{\varepsilon}{2}\big)\|u\|_{\mathcal{V}}^2 + \Big(\frac{C_1^2}{2\varepsilon} + \big(1 - \frac{\varepsilon}{2}\big)\Big) \|u\|_0^2,
\end{aligned}$$

whereat C_1 is constant (15), and $\varepsilon \in (0,2)$ is any number.

(d) Let $u \in H_2$ and $v \in \mathcal{V}$. Then we have

$$\begin{aligned}
(Bu,v)_0 &= \lim_{N\to\infty} \Big(\sum_{j=1}^{N} \lambda_j {F^{(j)}}^2 u, v\Big)_0 = \\
&\lim_{N\to\infty} \Big[-\sum_{j=1}^{N} \lambda_j (F^{(j)}u, F^{(j)}v)_0 - \sum_{j=1}^{N} \lambda_j (F^{(j)}u, \operatorname{div} \mathbf{f}^{(j)} \cdot v)\Big] = \mathfrak{b}(u,v).
\end{aligned}$$

The first equality follows from Lemma 4.1, while the second one is obtained by using integration by parts. ∎

Via the form $\mathfrak{b}$, regularly dissipative operator (an extension of B, hence the same letter) $B : H_0 \to H_0$ is defined by: $D(B) := \{u \in H_0$; there is a $f \in H_0$, such that $\mathfrak{b}(u,v) = (f,v)_0$, for every $v \in \mathcal{V}\}$ and $Bu := f$ for such a pair $\{u,f\}$. From the previous lemma and [9, Section 2.2, Remark 3.3.2 and Example 3.6] we have:

Proposition 4.3. *The operator B is well defined, closed operator such that the domain $D(B)$ is dense in $\mathcal{V}$ as well as in H_0. Moreover, $0 \in \mathbb{C}$ is contained in the resolvent set of the operator $B - \tilde{\lambda} I$, i.e. $0 \in \rho(B - \tilde{\lambda} I)$, and $B - \tilde{\lambda} I$ is maximal dissipative.*

Lemma 4.4. *Suppose that $\mathbf{a} = (a_i)_{i=1}^d \in \big(L_{\infty,2}(\mathbb{R}^d)\big)^d$. Then, the image $B(H_\infty)$ is dense in H_0.*

Proof. It is enough to show that H_∞ is dense in $D(B)$ equipped with the graph norm. We have to consider the commutators $[B, J_n]$, where J_n is the Friedrichs mollifier. For $u \in H_\infty$ we have:

$$
\begin{aligned}
[B, J_n] &= \sum_{j=1}^{\infty} \lambda_j [{F^{(j)}}^2, J_n] u \\
&= \sum_{j=1}^{\infty} \lambda_j \big[F^{(j)}, [F^{(j)}, J_n]\big] u + 2 \sum_{j=1}^{\infty} \lambda_j [F^{(j)}, J_n] F^{(j)} u.
\end{aligned}
$$

The proof is now split in three steps.

(i) We claim that the operator $R_n^{(1)} := \sum_{j=1}^{\infty} \lambda_j \big[F^{(j)}, [F^{(j)}, J_n]\big] : H_\infty \subset H_0 \to H_0$ is well defined, bounded uniformly with respect to n and that it converges to zero strongly in H_0, as $n \to \infty$. For $u \in H_\infty$ it holds:

$$
\begin{aligned}
[F^{(j)}, [F^{(j)}, J_n]] u &= \Big[\int_s^t dt_1 \boldsymbol{\phi}^{(j)\mathsf{T}}(t_1) \hat{\mathbf{A}}(t_1), \Big[\int_s^t dt_2 \boldsymbol{\phi}^{(j)\mathsf{T}}(t_2) \hat{\mathbf{A}}(t_2), J_n\Big]\Big] u \\
&= \int_s^t dt_1 \int_s^t dt_2 \sum_{i,i'} \phi_i^{(j)}(t_1) \phi_{i'}^{(j)}(t_1) \Big[\hat{A}_i(t_1), \big[\hat{A}_{i'}(t_1), J_n\big]\Big] u.
\end{aligned}
$$

According to Lemma 4.5, below, there is a constant $C(t_1)$ that depends only on $\|\mathbf{a}(t_1)\|_{L_{\infty,2}(\mathbb{R}^d)}$, such that $\|\big[\hat{A}_i(t_1), [\hat{A}_{i'}(t_1), J_n]\big]\|_{L(H_0)} \le C(t_1)$, and also $\|\big[\hat{A}_i(t_1), [\hat{A}_{i'}(t_1), J_n]\big] u\|_{L(H_0)} \le C(t_1) \eta_{1/n}(u)$, for $u \in H_\infty$. As is well known, $\eta_{1/n}(u) \to 0$ in H_0 for every $u \in H_0$ (see [5]). Thus, by Mercer theorem, the operator $R_n^{(1)}$ satisfies all the three claims.

(ii) Let us prove that $R_n^{(2)} := \sum_{j=1}^{\infty} \lambda_j [F^{(j)}, J_n] F^{(j)} : \mathcal{V} \to H_0$ is well defined, continuous (in the pair of norms $\|\cdot\|_0, \|\cdot\|_{\mathcal{V}}$), and that it converges to zero strongly in H_0, as $n \to \infty$.

First, it is easily seen that there is a constant $C > 0$ such that

$$
\sum_{j=1}^{\infty} \lambda_j \big\|[F^{(j)}, J_n]\big\|_0^2 \le C \tag{16}
$$

for every $n \in \mathbb{N}$. Now, for $u \in \mathcal{V}$ we have:

$$
\begin{aligned}
&\Big\| \sum_{j=1}^{\infty} \lambda_j [F^{(j)}, J_n] F^{(j)} u \Big\|_0 \le \\
&\quad \le \Big(\sum_{j=1}^{\infty} \lambda_j \big\|[F^{(j)}, J_n]\big\|_0^2 \Big)^{1/2} \Big(\sum_{j=1}^{\infty} \lambda_j \big\|F^{(j)} u\big\|_0^2 \Big)^{1/2} \le C \|u\|_{\mathcal{V}},
\end{aligned}
$$

which shows that $R_n^{(2)}$ is well defined and continuous. Furthermore, for any $N \in \mathbb{N}$ it holds:

$$\begin{aligned}
\Big\| \sum_{j=1}^{\infty} \lambda_j [F^{(j)}, J_n] F^{(j)} u \Big\|_0 &\leq \\
&\leq \sum_{j=1}^{N} \lambda_j \big\| [F^{(j)}, J_n] F^{(j)} u \big\|_0 \\
&\quad + \Big(\sum_{j=N+1}^{\infty} \lambda_j \| [F^{(j)}, J_n] \|_0^2 \Big)^{1/2} \Big(\sum_{j=N+1}^{\infty} \lambda_j \| F^{(j)} u \|_0^2 \Big)^{1/2}.
\end{aligned}$$

The second term on the right side tends to zero as $N \to \infty$ (uniformly in n), while the first one tends to zero (for fixed N) as $n \to \infty$ because of the well known Friedrichs lemma [5, Lemma 6.1].

(iii) So far we have shown that the commutator $[B, J_n]$ defined on H_∞ is extended to the continuous operator $R_n := R_n^{(1)} + R_n^{(2)} : \mathcal{V} \to H_0$, and $R_n \to 0$ strongly in H_0. Now we show that the commutator $[B, J_n] = BJ_n - J_n B$, which is defined naturally on $D(B)$, coincide with R_n. For $u \in D(B)$ there is a sequence $u_k \in H_\infty$, such that $u_k \to u$ in $\mathcal{V}$. From

$$[B, J_n] u_k = B J_n u_k - J_n B u_k = R_n u_k$$

it follows that $J_n B u_k$ converges in H_0, as $k \to \infty$. On the other side, for $v \in \mathcal{V}$ we have

$$\begin{aligned}
\lim_{k\to\infty} (J_n B u_k, v)_0 = \lim_{k\to\infty} (B u_k, J_n^* v)_0 &= \lim_{k\to\infty} \mathfrak{b}(u_k, J_n^* v) \\
&= \mathfrak{b}(u, J_n^* v) = (Bu, J_n^* v) = (J_n B u, v),
\end{aligned}$$

hence $J_n B u_k \to J_n B u$ in $\mathcal{D}'$, and consequently also in H_0.

Now, we conclude the proof of lemma. For $u \in D(B)$, it holds $J_n u \to u$ in H_0, and

$$Bu - BJ_n u = (I - J_n) Bu - [B, J_n] u \to 0$$

in H_0 as $n \to \infty$. Hence, $J_n u \to u$ in the graph norm. ∎

LEMMA 4.5. *Let* $\mathbf{a} = (a_i)_{i=1}^d \in \big(L_{\infty,2}(\mathbb{R}^d)\big)^d$, $\hat{A}_i = a_i \partial/\partial x_i$, $A = \sum_{i=1}^d \hat{A}_i$ *and* $R_n := [A, [A, J_n]]$. *Then there is a constant* $C > 0$ *that depends only on* $\|\mathbf{a}\|_{L_{\infty,2}(\mathbb{R}^d)}$, *such that for all* $n \in \mathbb{N}$ *and* $u \in H_\infty$ *it holds:*

(a) $\|R_n u\|_0 \leq C \|u\|_0$,

(b) $\|R_n u\|_0 \leq C \eta_{1/n}(u)^{1/2}$,

with

$$\eta_{1/n}(u) := \sup_{|\mathbf{y}|<1/n} \int_{\mathbb{R}^d} d\mathbf{x} |u(\mathbf{x} - \mathbf{y}) - u(\mathbf{x})|^2.$$

The notion of $\hat{A}_i$ and A used only in this lemma should not be confused with corresponding notions in (2) and (9).

PROOF. Let $u \in H_\infty$. Then we have $[A,[A,J_n]]u = \sum_{i,j=1}^d [\hat{A}_i,[\hat{A}_j,J_n]]u$. A straightforward calculation gives $[\hat{A}_i,[\hat{A}_j,J_n]]u = \sum_{k=1}^4 R_{n,i,j}^{(k)}u$, where the operators $R_{n,i,j}^{(k)}, k=1,2,3,4$ are given by:

$$R_{n,i,j}^{(1)}u = -\int d\mathbf{y}\,\frac{\partial}{\partial y_j}\Big\{\frac{\partial\psi_n}{\partial x_i}(\mathbf{x}-\mathbf{y})(a_j(\mathbf{x})-a_j(\mathbf{y}))\,(a_i(\mathbf{x})-a_i(\mathbf{y}))\Big\}u(\mathbf{y}),$$
$$R_{n,i,j}^{(2)}u = -\int d\mathbf{y}\,\frac{\partial}{\partial y_j}\Big\{\psi_n(\mathbf{x}-\mathbf{y})\Big(\frac{\partial a_j}{\partial x_i}(\mathbf{x})a_i(\mathbf{x}) - \frac{\partial a_j}{\partial y_i}(\mathbf{y})a_i(\mathbf{y})\Big)\Big\}u(\mathbf{y}),$$
$$R_{n,i,j}^{(3)}u = -\int d\mathbf{y}\,\frac{\partial}{\partial y_j}\Big\{\psi_n(\mathbf{x}-\mathbf{y})\big(a_j(\mathbf{x})-a_j(\mathbf{y})\big)\frac{\partial a_i}{\partial y_i}\Big\}u(\mathbf{y}),$$
$$R_{n,i,j}^{(4)}u = \int d\mathbf{y}\,\frac{\partial}{\partial y_i}\Big\{\psi_n(\mathbf{x}-\mathbf{y})\big(a_j(\mathbf{x})-a_j(\mathbf{y})\big)\frac{\partial a_i}{\partial y_j}\Big\}u(\mathbf{y}).$$

Hence, the same method of proof as in [5, Lemma 6.1] may be used. ∎

Previous lemma is the only place where we need the $L_{2,\infty}$-regularity, instead of $L_{1,\infty}$-regularity needed in general theory of evolution equations (e.g. [9]).

In the sequel we suppose that $a_i(t,\mathbf{x}) = a_i(t,x_i)$, $i=1,\dots,d$ and consider operators $B_n(t,s)$ given by (11). Similarly as in Lemma 4.1 we have $B_n u = \sum_{j=1}^\infty \lambda_j {F_n^{(j)}}^2 u$ in $L(H_0)$, where the operators $F_n^{(j)}$ are given by

$$\begin{aligned} F_n^{(j)} &= \int_s^t dt_1 \boldsymbol{\phi}^{(j)\mathsf{T}}(t_1)\hat{\mathbf{A}}_n(t_1) \\ &= \sum_{i=1}^d \Big[\int_s^t dt_1 \phi_i^{(j)}(t_1) J_{n,i} a_i(t_1,x_i)\frac{\partial}{\partial x_i} J_{n,i}\Big] \in L(H_0). \end{aligned}$$

For every $n \in \mathbb{N}$ the space $\mathcal{V}_n$ is defined in analogy with $\mathcal{V}$, replacing operators $F^{(j)}$ by $F_n^{(j)}$. Norm in $\mathcal{V}_n$ is denoted by $\|\cdot\|_{\mathcal{V}_n}$.

LEMMA 4.6. *There are constants $\vartheta > 0$ (the same as in Lemma 4.2(c)) and $\tilde{\lambda}^*$ such that for every $n \in \mathbb{N}$, it holds:*

$$\mathrm{Re}(B_n u,u) \le -\vartheta\|u\|_{\mathcal{V}_n} + \tilde{\lambda}^*\|u\|_0^2.$$

PROOF. Let us denote

$$(\operatorname{div}\mathbf{f}^{(j)})_n := \sum_{i=1}^d J_{n,i}\frac{\partial f_i^{(j)}}{\partial x_i} J_{n,i}.$$

In the same way as in the proof of Lemma 4.2(c), for $u \in H_0$ we obtain

$$\mathrm{Re}(B_n u,u) \le -\big(1-\frac{\varepsilon}{2}\big)\|u\|_{\mathcal{V}_n}^2 + \bigg(\frac{C_{1,n}^2}{2\varepsilon} + \big(1-\frac{\varepsilon}{2}\big)\bigg)\|u\|_0^2,$$

where the constant C_1 reads:

$$C_{1,n} := \left\| \int_s^t dt_2 \int_s^t dt_1 \partial \mathbf{a}_n^{\mathsf{T}}(t_1) \mathsf{R}(t_1, t_2) \partial \mathbf{a}_n(t_2) \right\|_{L_\infty(\mathbb{R}^d)}^{1/2},$$

$\partial \mathbf{a}_n(t) = (J_{n,i} \partial_{x_i} a_i(t, \cdot) J_{n,i})_{i=1}^d$ and $\varepsilon \in (0, 2)$. Since the operators $J_{n,i}$ are bounded in $L_\infty(\mathbb{R}^d)$ uniformly in n, there is a constant C such that $C_{1,n} \leq C$ for all $n \in \mathbb{N}$. Hence, we may take $\vartheta = 1 - \varepsilon/2$, and $\tilde{\lambda}^* = C^2/2\varepsilon + \vartheta$. □

Proposition 4.7. *There are constants $\varepsilon' > 0$ and $\tilde{\lambda}'$ such that the sector*

$$\Sigma = \{\lambda \in \mathbb{C};\ |arg\,(\lambda - \tilde{\lambda}')| \leq \pi/2 + \varepsilon'\} \tag{17}$$

is contained in resolvent sets $\rho(B)$, and $\rho(B_n)$ for all $n \in \mathbb{N}$. Furthermore, there is a constant $\tilde{C} > 0$ such that for every $\lambda \in \Sigma$ it holds:

$$\begin{aligned} &\|(B - \lambda I)^{-1}\|_0 \leq \tilde{C}(\lambda - \tilde{\lambda}')^{-1}, \\ &\|(B_n - \lambda I)^{-1}\|_0 \leq \tilde{C}(\lambda - \tilde{\lambda}')^{-1}, \qquad \textit{for all } n \in \mathbb{N}. \end{aligned}$$

Proof. The proposition follows from Lemma 4.2(c), Lemma 4.6 and [9, Section 2.2, Remark 3.3.2, and Example 3.6]. □

Next lemma explains the relationship between operators B_n and B on a "nice" domain. The proof is obvious.

Lemma 4.8. *$B_n u \to Bu$ in H_0 for every $u \in H_\infty$ (or H_2).*

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