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# APPROXIMATE RESOLUTIONS AND THE FRACTAL CATEGORY

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Abstract. This paper concerns the theory of approximate resolutions and its application to fractal geometry. In this paper, we first characterize a surjective map $f : X \to Y$ between compact metric spaces in terms of a property on any approximate map $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ where $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$ are any choices of approximate resolutions of $X$ and $Y$, respectively. Using this characterization, we construct a category whose objects are approximate sequences so that the box-counting dimension, which was defined for approximate resolutions by the authors, is invariant in this category. To define the morphisms of the category, we introduce an equivalence relation on approximate maps and define the morphisms as the equivalence classes.

## 1. Introduction

The notion of approximate resolution has played important roles in many problems in topology. In particular, it is useful when we wish to study maps between topological spaces even if the spaces are compact metric spaces. Indeed, if we are given a map $f : X \to Y$ between compact metric spaces and polyhedral approximate resolutions $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$, it is not possible in general to obtain a map of systems $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ with strict commutativity which represents $f : X \to Y$. However, this becomes possible if we replace strict commutativity by approximate commutativity [2]. Moreover, the category $\mathsf{CTOP}_{3.5}$ of Tychonoff spaces and continuous maps is in one-to-one correspondence with the category $\mathsf{APRES}_{\mathsf{POL}}$ whose objects are all cofinite polyhedral approximate resolutions and whose morphisms are the

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equivalence classes of approximate maps for a certain equivalence relation [2, Theorem 8.13].

On the other hand, there has been an approach using approximate resolutions to various notions in fractal geometry. In [4], the notion of box-counting dimension was defined for approximate resolutions. This generalizes the traditional notion of box-counting dimension for subsets of Euclidean spaces [1] and gives a useful tool for computations.

Covering dimension is invariant in the category $\mathsf{TOP}$ of topological spaces and continutous maps. However, box-counting dimension is not invariant in that category since for each nonnegative real number $r$ there is a Cantor set $X_r$ with box-counting dimension $r$ [4].

This paper mainly consists of two results. In the first part, we characterize a surjective map $f : X \to Y$ between compact metric spaces in terms of a property on any approximate map $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ where $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$ are any choices of approximate resolutions of $X$ and $Y$, respectively. In the second part, we construct a category whose objects are approximate sequences so that the box-counting dimension is invariant in this category. A category in which the box-counting dimension is invariant was earlier obtained by the authors [5], but the morphisms were based on Lipschitz maps and bi-Lipschitz maps. Our approach follows the one that was taken by [2]. To define the morphisms in the category, we introduce a new equivalence relation on the approximate maps and define the morphism as the equivalence class.

Throughout the paper, *a space means a compact metric space, and a map means a continuous map* unless otherwise stated.

For any space $X$, let $\mathrm{Cov}(X)$ denote the set of all normal open coverings of $X$. For $\mathcal{U}, \mathcal{V} \in \mathrm{Cov}(X)$, $\mathcal{U}$ is said to *refine* $\mathcal{V}$, in notation, $\mathcal{U} < \mathcal{V}$, provided for each $U \in \mathcal{U}$ there is $V \in \mathcal{V}$ such that $U \subseteq V$. For any subset $A$ of $X$ and $\mathcal{U} \in \mathrm{Cov}(X)$, let $\mathrm{st}(A,\mathcal{U}) = \cup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$ and $\mathcal{U}|A = \{U \cap A : U \in \mathcal{U}\}$. If $A = \{x\}$, we write $\mathrm{st}(x,\mathcal{U})$ for $\mathrm{st}(\{x\},\mathcal{U})$. For each $\mathcal{U} \in \mathrm{Cov}(X)$, let $\mathrm{st}\,\mathcal{U} = \{\mathrm{st}(U,\mathcal{U}) : U \in \mathcal{U}\}$. Let $\mathrm{st}^{n+1}\mathcal{U} = \mathrm{st}(\mathrm{st}^n\mathcal{U})$ for each $n = 1, 2, \ldots$ and $\mathrm{st}^1\mathcal{U} = \mathrm{st}\,\mathcal{U}$. For any metric space $(X, \mathrm{d})$ and $r > 0$, let $\mathrm{U_d}(x, r) = \{y \in X : \mathrm{d}(x, y) < r\}$. For any $\mathcal{U} \in \mathrm{Cov}(X)$, two points $x, x' \in X$ are $\mathcal{U}$-*near*, denoted $(x, x') < \mathcal{U}$, provided $x, x' \in U$ for some $U \in \mathcal{U}$. For any $\mathcal{V} \in \mathrm{Cov}(Y)$, two maps $f, g : X \to Y$ between spaces are $\mathcal{V}$-*near*, denoted $(f, g) < \mathcal{V}$, provided $(f(x), g(x)) < \mathcal{V}$ for each $x \in X$. For each $\mathcal{U} \in \mathrm{Cov}(X)$ and $\mathcal{V} \in \mathrm{Cov}(Y)$, let $f\mathcal{U} = \{f(U) : U \in \mathcal{U}\}$ and $f^{-1}\mathcal{V} = \{f^{-1}(V) : V \in \mathcal{V}\}$. For $\mathcal{U} \in \mathrm{Cov}(X)$, let $N_{\mathcal{U}}(X) = \min\{n : X \subseteq \bigcup_{i=1}^{n} U_i,\ U_i \in \mathcal{U}\}$.

## 2. Approximate resolutions and box-counting dimension

In this section we recall the definitions and properties of approximate resolutions and their box-counting dimensions which will be needed in later sections. Although approximate resolutions are defined and useful for arbitrary topological spaces, for our purpose they will be defined only for compact metric spaces. For more details, the reader is referred to [2, 4].

An *approximate inverse sequence* (*approximate sequence*, in short) $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ consists of

i) a sequence of spaces $X_i$ (called *coordinate spaces*), $i \in \mathbb{N}$;
ii) a sequence of $\mathcal{U}_i \in \mathrm{Cov}(X_i)$, $i \in \mathbb{N}$; and
iii) maps $p_{ii'} : X_{i'} \to X_i$ for $i < i'$ where $p_{ii} = 1_{X_i}$ the identity map on $X_i$.

It must satisfy the following three conditions:

(A1) $(p_{ii'}p_{i'i''}, p_{ii''}) < \mathcal{U}_i$ for $i < i' < i''$;
(A2) For each $i \in \mathbb{N}$ and $\mathcal{U} \in \mathrm{Cov}(X_i)$, there exists $i' > i$ such that $(p_{ii_1}p_{i_1i_2}, p_{ii_2}) < \mathcal{U}$ for $i' < i_1 < i_2$; and
(A3) For each $i \in \mathbb{N}$ and $\mathcal{U} \in \mathrm{Cov}(X_i)$, there exists $i' > i$ such that $\mathcal{U}_{i''} < p_{ii''}^{-1}\mathcal{U}$ for $i' < i''$.

An *approximate map* $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X}$ of a space $X$ into an approximate sequence $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ consists of maps $p_i : X \to X_i$ for $i \in \mathbb{N}$ with the following property:

(AS) For each $i \in \mathbb{N}$ and $\mathcal{U} \in \mathrm{Cov}(X_i)$, there exists $i' > i$ such that $(p_{ii''}p_{i''}, p_i) < \mathcal{U}$ for $i'' > i'$.

An *approximate resolution* of a space $X$ is an approximate map $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X}$ of $X$ into an approximate sequence $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ which satisfies the following two conditions:

(R1) For each ANR $P$, $\mathcal{V} \in \mathrm{Cov}(P)$ and map $f : X \to P$, there exist $i \in \mathbb{N}$ and a map $g : X_i \to P$ such that $(gp_i, f) < \mathcal{V}$; and
(R2) For each ANR $P$ and $\mathcal{V} \in \mathrm{Cov}(P)$, there exists $\mathcal{V}' \in \mathrm{Cov}(P)$ such that whenever $i \in \mathbb{N}$ and $g, g' : X_i \to P$ are maps with $(gp_i, g'p_i) < \mathcal{V}'$, then $(gp_{ii'}, g'p_{ii'}) < \mathcal{V}$ for some $i' > i$.

THEOREM 2.1 ([2]). *An approximate map $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ is an approximate resolution of a space $X$ if and only if it satisfies the following two conditions:*

(B1) *For each $\mathcal{U} \in \mathrm{Cov}(X)$, there exists $i_0 \in \mathbb{N}$ such that $p_i^{-1}\mathcal{U}_i < \mathcal{U}$ for $i > i_0$; and*
(B2) *For each $i \in \mathbb{N}$ and $\mathcal{U} \in \mathrm{Cov}(X_i)$, there exists $i_0 > i$ such that $p_{ii'}(X_{i'}) \subseteq \mathrm{st}(p_i(X), \mathcal{U})$ for $i' > i_0$.*

THEOREM 2.2 ([6]). *Every space $X$ admits an approximate resolution $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ such that all $X_i$ are finite polyhedra.*

Throughout the paper, *approximate resolutions are assumed to have the property of Theorem 2.2.*

Let $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ and $\boldsymbol{Y} = \{Y_j, \mathcal{V}_j, q_{jj'}\}$ be approximate sequences of spaces. An *approximate map* $\boldsymbol{f} = \{f_j, f\} : \boldsymbol{X} \to \boldsymbol{Y}$ consists of a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ (i.e., $f(i) < f(j)$ for $i < j$) and maps $f_j : X_{f(j)} \to Y_j$, $j \in \mathbb{N}$, with the following condition:

(AM) For any $j, j' \in \mathbb{N}$ with $j < j'$, there exists $i \in \mathbb{N}$ with $i > f(j')$ such that

$$(q_{jj'} f_{j'} p_{f(j')i'}, f_j p_{f(j)i'}) < \operatorname{st} \mathcal{V}_j \text{ for } i' > i.$$

A map $f : X \to Y$ is a *limit* of $\boldsymbol{f}$ provided the following condition is satisfied:

(LAM) For each $j \in \mathbb{N}$ and $\mathcal{V} \in \operatorname{Cov}(Y_j)$, there exists $j' > j$ such that

$$(q_{jj''} f_{j''} p_{f(j'')}, q_j f) < \mathcal{V} \text{ for } j'' > j'.$$

Remark 2.3. Following the convention from [2], we use the common symbol $f$ for the map $f : X \to Y$ and for the strictly increasing function $f : \mathbb{N} \to \mathbb{N}$.

For each map $f : X \to Y$, an *approximate resolution* of $f$ is a triple $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{f})$ consisting of approximate resolutions $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ of $X$ and $\boldsymbol{q} = \{q_j\} : Y \to \boldsymbol{Y} = \{Y_j, \mathcal{V}_j, q_{jj'}\}$ of $Y$ and of an approximate map $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ with property (LAM).

Theorem 2.4 ([2]). *Let $X$ and $Y$ be spaces. For any approximate resolutions $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$, every map $f : X \to Y$ admits an approximate map $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ such that $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{f})$ is an approximate resolution of $f$.*

For each approximate sequence $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$, let $\operatorname{st} \boldsymbol{X}$ denote the approximate sequence $\{X_i, \operatorname{st} \mathcal{U}_i, p_{ii'}\}$. Then there is a natural approximate map $\boldsymbol{i}_{\boldsymbol{X}} = \{1_{X_i}\} : \boldsymbol{X} \to \operatorname{st} \boldsymbol{X}$, where $1_{X_i} : X_i \to X_i$ is the identity map. For each approximate map $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$, the map $\operatorname{st} \boldsymbol{p} = \{p_i\} : X \to \operatorname{st} \boldsymbol{X} = \{X_i, \operatorname{st} \mathcal{U}_i, p_{ii'}\}$ also satisfies (AS) and hence is an approximate map. Moreover, if $\boldsymbol{p} : X \to \boldsymbol{X}$ is an approximate resolution, so is $\operatorname{st} \boldsymbol{p} : X \to \operatorname{st} \boldsymbol{X}$.

For any approximate sequences $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ and $\boldsymbol{Y} = \{Y_j, \mathcal{V}_j, q_{jj'}\}$ and for each approximate map $\boldsymbol{f} = \{f_j, f\} : \boldsymbol{X} \to \boldsymbol{Y}$, the map $\operatorname{st} \boldsymbol{f} = \{f_j, f\} : \operatorname{st} \boldsymbol{X} \to \operatorname{st} \boldsymbol{Y}$ also satisfies (AM) and hence is an approximate map. Moreover, if $(\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{q})$ is an approximate resolution of a map $f : X \to Y$, then $\operatorname{st} \boldsymbol{f} : \operatorname{st} \boldsymbol{X} \to \operatorname{st} \boldsymbol{Y}$ also satisfies (LAM) and hence $(\operatorname{st} \boldsymbol{f}, \operatorname{st} \boldsymbol{p}, \operatorname{st} \boldsymbol{q})$ is an approximate resolution of $f$.

Iteratively, we define the approximate sequence $\operatorname{st}^k \boldsymbol{X}$ as $\operatorname{st}(\operatorname{st}^{k-1} \boldsymbol{X})$ and similarly for the approximate maps $\operatorname{st}^k \boldsymbol{p}$ and $\operatorname{st}^k \boldsymbol{f}$.

Let $X$ be a compact metric space. For each approximate resolution $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$, consider the following three conditions:

(U) $\operatorname{st}^2 \mathcal{U}_j < p_{ij}^{-1} \mathcal{U}_i$ for $i < j$;

(A) $(p_{ij}p_j, p_i) < \mathcal{U}_i$ for $i < j$; and
(NR) $p_j^{-1}\operatorname{st}\mathcal{U}_j < p_i^{-1}\mathcal{U}_i$ for $i < j$.

An approximate resolution $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ is *admissible* provided it possesses properties (U), (A), (NR).

Proposition 2.5. *Let $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ be an admissible approximate resolution of $X$. Then the following properties hold:*

1) *The family $\mathbb{U}_k = \{p_i^{-1}\operatorname{st}^k\mathcal{U}_i : i \in \mathbb{N}\}$ is a normal sequence on $X$ for $k \geq 0$;*
2) *The approximate resolution $\operatorname{st}^k\boldsymbol{p} = \{p_i\} : X \to \operatorname{st}^k\boldsymbol{X} = \{X_i, \operatorname{st}^k\mathcal{U}_i, p_{ii'}\}$ is admissible for $k \geq 1$.*

For any approximate resolution $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$, we can always find an admissible approximate resolution $\boldsymbol{p'} = \{p_{k_i}\} : X \to \boldsymbol{X'} = \{X_{k_i}, \mathcal{U}_{k_i}, p_{k_ik_j}\}$ by taking a subsequence.

For each approximate resolution $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$, we define the *upper* and *lower box-counting dimensions* of $\boldsymbol{p} : X \to \boldsymbol{X}$ by

$$\overline{\dim}_B(\boldsymbol{p} : X \to \boldsymbol{X}) = \varlimsup_{i\to\infty}\frac{\log_3\beta_i(\boldsymbol{X})}{i}$$

and

$$\underline{\dim}_B(\boldsymbol{p} : X \to \boldsymbol{X}) = \varliminf_{i\to\infty}\frac{\log_3\beta_i(\boldsymbol{X})}{i},$$

where

$$\beta_i(\boldsymbol{X}) = \varlimsup_{j\to\infty} N_{p_{ij}^{-1}\mathcal{U}_i}(X_j) \text{ for } i \in \mathbb{N}.$$

Note here that $\beta_i(\boldsymbol{X}) < \infty$ for each $i$ since each $i$ admits $i_0 \geq i$ such that $N_{p_{ij}^{-1}\mathcal{U}_i}(X_j) \leq N_{p_i^{-1}\mathcal{U}_i}(X)$ for $j \geq i_0$ ([4, Proposition 5.1 part 1)]). If the two values coincide, then we write $\dim_B(\boldsymbol{p} : X \to \boldsymbol{X})$ for the common value and call it the *box-counting dimension* of $\boldsymbol{p} : X \to \boldsymbol{X}$.

Theorem 2.6. 1) ([4, Proposition 5.2]) *If each $p_i$ is onto, the definition is simplified as*

$$\overline{\dim}_B(\boldsymbol{p} : X \to \boldsymbol{X}) = \varlimsup_{i\to\infty}\frac{\log_3 N_{\mathcal{U}_i}(X_i)}{i}$$

*and*

$$\underline{\dim}_B(\boldsymbol{p} : X \to \boldsymbol{X}) = \varliminf_{i\to\infty}\frac{\log_3 N_{\mathcal{U}_i}(X_i)}{i}.$$

2) ([4, Theorem 5.3]) *If $\boldsymbol{p} : X \to \boldsymbol{X}$ is an admissible approximate resolution,*

$$\underline{\dim}_B(\operatorname{st}\boldsymbol{p} : X \to \operatorname{st}\boldsymbol{X}) = \varlimsup_{i\to\infty}\frac{\log_3 N_{p_i^{-1}\operatorname{st}\mathcal{U}_i}(X)}{i}$$

*and*

$$\overline{\dim}_B(\operatorname{st}\boldsymbol{p} : X \to \operatorname{st}\boldsymbol{X}) = \varliminf_{i\to\infty} \frac{\log_3 N_{p_i^{-1}\operatorname{st}\mathcal{U}_i}(X)}{i}.$$

3) ([4, Proposition 5.5]) *If* $\boldsymbol{p} : X \to \boldsymbol{X}$ *is an admissible approximate resolution,*

$$\underline{\dim}_B(\boldsymbol{p} : X \to \boldsymbol{X}) \geq \underline{\dim}_B(\operatorname{st}\boldsymbol{p} : X \to \operatorname{st}\boldsymbol{X}) = \underline{\dim}_B(\operatorname{st}^2\boldsymbol{p} : X \to \operatorname{st}^2\boldsymbol{X})$$

*and*

$$\overline{\dim}_B(\boldsymbol{p} : X \to \boldsymbol{X}) \geq \overline{\dim}_B(\operatorname{st}\boldsymbol{p} : X \to \operatorname{st}\boldsymbol{X}) = \overline{\dim}_B(\operatorname{st}^2\boldsymbol{p} : X \to \operatorname{st}^2\boldsymbol{X}).$$

## 3. Surjective maps

In this section we give a characterization of surjective maps in terms of approximate resolutions, which will be needed in the next section.

For each approximate map $\boldsymbol{f} = \{f_j, f\} : \boldsymbol{X} \to \boldsymbol{Y}$ between approximate sequences $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ and $\boldsymbol{Y} = \{Y_j, \mathcal{V}_j, q_{jj'}\}$, consider the following property:

(APS) $(\forall j \in \mathbb{N})(\forall \mathcal{V} \in \operatorname{Cov}(Y_j))(\exists j_0 > j)(\forall j' > j_0)(\exists j_0' > j')(\forall j'' > j_0')$ $(\exists i_0 > f(j'))(\forall i > i_0)$ :

$$q_{jj''}(Y_{j''}) \subseteq \operatorname{st}(q_{jj'}f_{j'}p_{f(j')i}(X_i), \mathcal{V}).$$

Theorem 3.1. *Let* $f : X \to Y$ *be a map, and* $\boldsymbol{f} = \{f_j\} : \boldsymbol{X} \to \boldsymbol{Y}$ *be an approximate map such that* $(\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{q})$ *is an approximate resolution of* $f$ *where* $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ *and* $\boldsymbol{q} = \{q_j\} : Y \to \boldsymbol{Y} = \{Y_j, \mathcal{V}_j, q_{jj'}\}$ *are approximate resolutions of* $X$ *and* $Y$*, respectively. Then* $f$ *is surjective if and only if* $\boldsymbol{f}$ *satisfies (APS).*

Proof. To show the necessity, let $j \in \mathbb{N}$, and let $\mathcal{V} \in \operatorname{Cov}(Y_j)$. Take $\mathcal{V}' \in \operatorname{Cov}(Y_j)$ such that $\operatorname{st}\mathcal{V}' < \mathcal{V}$. Then, by (A3) and (A2) there exists $j_0 > j$ such that for $j'' > j' > j_0$,

$$\operatorname{st}^2\mathcal{V}_{j'} < q_{jj'}^{-1}\mathcal{V}', \tag{3.1}$$

$$(q_{jj'}q_{j'j''}, q_{jj''}) < \mathcal{V}'. \tag{3.2}$$

Fix $j' > j_0$. Then, by (B2) and (LAM) there exists $j_0' > j'$ such that for $j'' > j_0'$,

$$q_{j'j''}(Y_{j''}) \subseteq \operatorname{st}(q_{j'}(Y), \mathcal{V}_{j'}), \tag{3.3}$$

$$(q_{j'}f, q_{j'j''}f_{j''}p_{f(j'')}) < \mathcal{V}_{j'}. \tag{3.4}$$

Fix $j'' > j_0'$. Then, by (3.4),

$$q_{j'}f(X) \subseteq \operatorname{st}(q_{j'j''}f_{j''}p_{f(j'')}(X), \mathcal{V}_{j'}). \tag{3.5}$$

Since $f$ is surjective, (3.3) implies

$$q_{j'j''}(Y_{j''}) \subseteq \mathrm{st}(q_{j'}f(X), \mathcal{V}_{j'}). \tag{3.6}$$

By (3.5) and (3.6),

$$q_{j'j''}(Y_{j''}) \subseteq \mathrm{st}(\mathrm{st}(q_{j'j''}f_{j''}p_{f(j'')}(X), \mathcal{V}_{j'}), \mathcal{V}_{j'}). \tag{3.7}$$

On the other hand, by (AM) and (AS) there exists $i_0 > f(j'')$ such that for $i > i_0$,

$$(f_{j'}p_{f(j')i}, q_{j'j''}f_{j''}p_{f(j'')i}) < \mathrm{st}\,\mathcal{V}_{j'}, \tag{3.8}$$

$$(p_{f(j'')}, p_{f(j'')i}p_i) < f_{j''}^{-1}q_{j'j''}^{-1}\mathcal{V}_{j'}. \tag{3.9}$$

(3.9) and (3.8) then imply

$$\begin{aligned} q_{j'j''}f_{j''}p_{f(j'')}(X) &\subseteq \mathrm{st}(q_{j'j''}f_{j''}p_{f(j'')i}p_i(X), \mathcal{V}_{j'}) \\ &\subseteq \mathrm{st}(q_{j'j''}f_{j''}p_{f(j'')i}(X_i), \mathcal{V}_{j'}) \\ &\subseteq \mathrm{st}(\mathrm{st}(f_{j'}p_{f(j')i}(X_i), \mathrm{st}\,\mathcal{V}_{j'}), \mathcal{V}_{j'}). \end{aligned} \tag{3.10}$$

By (3.7), (3.10) and (3.1),

$$\begin{aligned} q_{j'j''}(Y_{j''}) &\subseteq \mathrm{st}(\mathrm{st}(\mathrm{st}(\mathrm{st}(f_{j'}p_{f(j')i}(X_i), \mathrm{st}\,\mathcal{V}_{j'}), \mathcal{V}_{j'}), \mathcal{V}_{j'}), \mathcal{V}_{j'}) \\ &\subseteq \mathrm{st}(f_{j'}p_{f(j')i}(X_i), \mathrm{st}^2\,\mathcal{V}_{j'}) \\ &\subseteq \mathrm{st}(f_{j'}p_{f(j')i}(X_i), q_{jj'}^{-1}\mathcal{V}'). \end{aligned}$$

This implies

$$q_{jj'}q_{j'j''}(Y_{j''}) \subseteq \mathrm{st}(q_{jj'}f_{j'}p_{f(j')i}(X_i), \mathcal{V}').$$

This together with (3.2) and the fact that $\mathrm{st}\,\mathcal{V}' < \mathcal{V}$ implies

$$q_{jj''}(Y_{j''}) \subseteq \mathrm{st}(q_{jj'}f_{j'}p_{f(j')i}(X_i), \mathcal{V}),$$

which proves the necessity of the assertion.

To show the sufficiency, let $y \in Y$. We must find $x \in X$ such that $f(x) = y$. Write $y_j = q_j(y)$ for each $j$.

Claim 1. *For each $j$ and $\mathcal{V} \in \mathrm{Cov}(Y_j)$ there exist $j', j''$ with $j < j' < j''$ and a point $z_{f(j'')}$ of $X_{f(j'')}$ such that*

1) $(y_j, q_{jj'}f_{j'}p_{f(j')f(j'')}(z_{f(j'')})) < \mathcal{V}$, *and*
2) $(f_i p_{f(i)f(j'')}, q_{ij'}f_{j'}p_{f(j')f(j'')}) < \mathrm{st}\,\mathcal{V}_i$ *for* $1 \le i < j'$.

Let $j$ and $\mathcal{V} \in \mathrm{Cov}(Y_j)$ be given, and let $\mathcal{V}' \in \mathrm{Cov}(Y_j)$ such that

$$\mathrm{st}^4\,\mathcal{V}' < \mathcal{V}. \tag{3.11}$$

By (A3) and (AS) there is $j' > j$ such that

$$\mathcal{V}_{j'} < q_{jj'}^{-1}\mathcal{V}', \tag{3.12}$$

$$(y_j, q_{jj'}(y_{j'})) < \mathcal{V}'. \tag{3.13}$$

By (AS), (APS), (A2) and (AM), the latter yielding both (3.17) and (3.18), there exist $j'', j_0$ with $j_0 > j'' > j'$ with the following properties: for $k > j_0$,

$$(y_{j'}, q_{j'k}(y_k)) < \mathcal{V}_{j'}, \tag{3.14}$$

(3.15)
$$(q_{j'k}(y_k), q_{j'j''} f_{j''} p_{f(j'')i_k}(w_{i_k})) < \mathcal{V}_{j'} \text{ for some } i_k > f(j'') \text{ and } w_{i_k} \in X_{i_k},$$

$$(p_{f(j')f(j'')} p_{f(j'')i_k}, p_{f(j')i_k}) < f_{j'}^{-1} \mathcal{V}_{j'}, \tag{3.16}$$

$$(f_{j'} p_{f(j')i_k}, q_{j'j''} f_{j''} p_{f(j'')i_k}) < \operatorname{st} \mathcal{V}_{j'}, \tag{3.17}$$

$$(f_i p_{f(i)f(j'')}, q_{ij'} f_{j'} p_{f(j')f(j'')}) < \operatorname{st} \mathcal{V}_i \text{ for } 1 \le i < j'. \tag{3.18}$$

Consider the sequence $\{p_{f(j'')i_k}(w_{i_k})\}$ in $X_{f(j'')}$. Since $X_{f(j'')}$ is compact, there is a subsequence $\{p_{f(j'')m_k}(w_{m_k})\}$ which converges to some point $z_{f(j'')}$ of $X_{f(j'')}$. So there is $k_0 > j_0$ such that

$$(z_{f(j'')}, p_{f(j'')m_k}(w_{m_k})) < p^{-1}_{f(j')f(j'')} f_{j'}^{-1} \mathcal{V}_{j'} \text{ for } k > k_0. \tag{3.19}$$

By (3.14), (3.15) and (3.17),

$$(y_{j'}, f_{j'} p_{f(j')m_k}(w_{m_k})) < \operatorname{st}^2 \mathcal{V}_{j'} \text{ for } k > j_0, \tag{3.20}$$

By (3.20), (3.16) and (3.19),

$$(y_{j'}, f_{j'} p_{f(j')f(j'')}(z_{f(j'')})) < \operatorname{st}^3 \mathcal{V}_{j'}. \tag{3.21}$$

Property 1) now follows from (3.13), (3.21), (3.12) and (3.11). Property 2) is (3.18). Thus Claim 1 has been proven.

Claim 2. *There exists a sequence $\{x_{f(i)}\}$ such that $x_{f(i)} \in X_{f(i)}$ for each $i$ and which satisfies the following properties:*

1) *For each $i$, $x_{f(i)} = \lim_{i'} p_{f(i)f(i')}(x_{f(i')})$,*
2) *For each $i$ and $\mathcal{V} \in \operatorname{Cov}(Y_i)$ there exists $i_0 > i$ such that*

$$(y_i, q_{ii'} f_{i'}(x_{f(i')})) < \mathcal{V} \text{ for } i' > i_0.$$

For each $j$ take $j', j''$ with $j < j' < j''$ and a point $z_{f(j'')}$ of $X_{f(j'')}$ with properties 1) and 2) of Claim 1 with $\mathcal{V}$ being $\mathcal{V}_j$, and write $n_j$ for $j''$.

First, consider the sequence $\{p_{f(1)f(n_j)}(z_{f(n_j)})\}_{j\in\mathbb{N}}$ in $X_{f(1)}$. Since $X_{f(1)}$ is compact, there is a cofinal subset $I_1$ of $\mathbb{N}$ so that the subsequence $\{p_{f(1)f(n_j)}(z_{f(n_j)})\}_{j\in I_1}$ converges to some point $x_{f(1)}$ of $X_{f(1)}$. Inductively, we obtain a cofinal subset $I_i$ of $I_{i-1}$ so that the subsequence $\{p_{f(i)f(n_j)}(z_{f(n_j)})\}_{j\in I_i}$ converges to some point $x_{f(i)}$ of $X_{f(i)}$. We show that the sequence $\{x_{f(i)}\}_{i\in\mathbb{N}}$ has the desired properties.

For 1), let $i \in \mathbb{N}$, and let $\mathcal{U} \in \operatorname{Cov}(X_{f(i)})$. Take $\mathcal{U}' \in \operatorname{Cov}(X_{f(i)})$ such that

$$\operatorname{st}\mathcal{U}' < \mathcal{U}. \tag{3.22}$$

By (A2) there exists $i_0 > i$ such that

$$(p_{f(i)f(i')}p_{f(i')f(i'')}, p_{f(i)f(i'')}) < \mathcal{U}' \text{ for } i'' > i' > i_0. \tag{3.23}$$

Let $i' > i_0$. Then the definitions of $x_{f(i)}$ and $x_{f(i')}$ imply that there is $j$ with $n_j > i'$ such that

$$(x_{f(i)}, p_{f(i)f(n_j)}(z_{f(n_j)})) < \mathcal{U}', \tag{3.24}$$

$$(x_{f(i')}, p_{f(i')f(n_j)}(z_{f(n_j)})) < p^{-1}_{f(i)f(i')}\mathcal{U}'. \tag{3.25}$$

By (3.24), (3.23), (3.25) and (3.22),

$$(x_{f(i)}, p_{f(i)f(i')}(x_{f(i')})) < \mathcal{U},$$

which verifies 1).

To see 2), let $i \in \mathbb{N}$, and let $\mathcal{V} \in \operatorname{Cov}(Y_i)$. Take $\mathcal{V}' \in \operatorname{Cov}(Y_i)$ such that

$$\operatorname{st}^3 \mathcal{V}' < \mathcal{V}. \tag{3.26}$$

By (AS), (A3) and (A2) there is $i' > i$ such that

$$(y_i, q_{ii'}(y_{i'})) < \mathcal{V}', \tag{3.27}$$

$$\mathcal{V}_{i'} < q^{-1}_{ii'}\mathcal{V}', \tag{3.28}$$

$$(q_{ij}, q_{ii'}q_{i'j}) < \mathcal{V}' \text{ for } j > i'. \tag{3.29}$$

By Claim 1, the definition of $x_{f(i')}$, (A2), (AS) and (A3) there is $j > i'$ with the following properties:

$$(y_j, q_{jj'}f_{j'}p_{f(j')f(n_j)}(z_{f(n_j)})) < \mathcal{V}_j, \tag{3.30}$$

$$(f_{i'}p_{f(i')f(n_j)}, q_{i'j'}f_{j'}p_{f(j')f(n_j)}) < \operatorname{st}\mathcal{V}_{i'}, \tag{3.31}$$

$$(x_{f(i')}, p_{f(i')f(n_j)}(z_{f(n_j)})) < f^{-1}_{i'}\mathcal{V}_{i'}, \tag{3.32}$$

$$(q_{i'j'}, q_{i'j}q_{jj'}) < \mathcal{V}_{i'}, \tag{3.33}$$

$$(y_{i'}, q_{i'j}(y_j)) < \mathcal{V}_{i'}, \tag{3.34}$$

$$\mathcal{V}_j < q^{-1}_{i'j}\mathcal{V}_{i'}. \tag{3.35}$$

By (3.32), (3.31) and (3.33),

$$(f_{i'}(x_{f(i')}), q_{i'j}q_{jj'}f_{j'}p_{f(j')f(n_j)}(z_{f(n_j)})) < \operatorname{st}\mathcal{V}_{i'}. \tag{3.36}$$

But by (3.30) and (3.35),

$$(q_{i'j}(y_j), q_{i'j}q_{jj'}f_{j'}p_{f(j')f(n_j)}(z_{f(n_j)})) < \mathcal{V}_{i'}. \tag{3.37}$$

By (3.34), (3.37) and (3.36),

$$(y_{i'}, f_{i'}(x_{f(i')})) < \operatorname{st}^2 \mathcal{V}_{i'}.$$

This together with (3.27), (3.28) and (3.26) imply

$$(y_i, q_{ii'} f_{i'}(x_{f(i')})) < \mathcal{V},$$

as required.

It remains to show that $f(x) = y$ for some $x \in X$. Indeed, Claim 2 1) shows that $\{x_{f(i)}\}$ forms a thread of the subsequence $\boldsymbol{X}' = \{X_{f(i)}, \mathcal{U}_{f(i)}, p_{f(i)f(i+1)}\}$ of $\boldsymbol{X}$, which determines a point $x$ of $X$. We wish to show $f(x) = y$. Let $\mathcal{V} \in \mathrm{Cov}(Y)$. Then by (B1) there exist $i \in \mathbb{N}$ and $\mathcal{V}' \in \mathrm{Cov}(Y_i)$ such that

$$q_i^{-1}\mathcal{V}' < \mathcal{V}. \tag{3.38}$$

Take $\mathcal{V}'' \in \mathrm{Cov}(Y_i)$ such that

$$\mathrm{st}\,\mathcal{V}'' < \mathcal{V}'. \tag{3.39}$$

Then Claim 2 2) and (LAM) imply that there exists $i' > i$ such that

$$(q_{ii'} f_{i'}(x_{f(i')}), y_i) < \mathcal{V}'', \tag{3.40}$$

$$(q_i f, q_{ii'} f_{i'} p_{f(i')}) < \mathcal{V}''. \tag{3.41}$$

By (3.41),

$$(q_i f(x), q_{ii'} f_{i'}(x_{f(i')})) < \mathcal{V}''. \tag{3.42}$$

By (3.40), (3.42) and (3.39),

$$(q_i(y), q_i f(x)) < \mathcal{V}'.$$

This together with (3.38) implies

$$(y, f(x)) < \mathcal{V}.$$

But since $\mathcal{V} \in \mathrm{Cov}(Y)$ is arbitrary, $y = f(x)$. This completes the proof of the theorem. ∎

## 4. The fractal category FRAC

In this section we construct a category, denoted $\mathsf{FRAC}$, in which the box-counting dimension is invariant.

The objects of $\mathsf{FRAC}$ are all admissible approximate resolutions. We wish to define morphisms in $\mathsf{FRAC}$. An approximate map $\boldsymbol{f} = \{f_j, f\} : \boldsymbol{X} \to \boldsymbol{Y}$ is said to be *admissible* provided it satisfies the following three conditions:

(AD1) $\boldsymbol{f}$ is uniform, i.e., $\mathcal{U}_{f(j)} < f_j^{-1}\mathcal{V}_j$ for each $j$;
(AD2) There exists $N \in \mathbb{N}$ such that $|f(j+1) - f(j)| \le N$ for each $j$; and
(AD3) $(f_j p_{f(j)f(j')}, q_{jj'} f_{j'}) < \mathrm{st}\,\mathcal{V}_j$ for $j < j'$.

Remark 4.1. 1) (AD2) means that each $M \in \mathbb{N}$ admits $N \in \mathbb{N}$ such that $|j - j'| \le M$ implies $|f(j) - f(j')| \le N$.

2) Every approximate map $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$, where $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$ are admissible approximate resolutions, admits an admissible approximate map $\boldsymbol{f}' : \boldsymbol{X}' \to \boldsymbol{Y}$ representing the same map $f : X \to Y$ for some admissible approximate resolution $\boldsymbol{p}' : X \to \boldsymbol{X}'$ such that $\boldsymbol{X}'$ is a subsequence of $\boldsymbol{X}$.
3) If a map of systems $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ satisfies (AD3) and (AD1), then $\operatorname{st}\boldsymbol{f} : \operatorname{st}\boldsymbol{X} \to \operatorname{st}\boldsymbol{Y}$ satisfies (AM), i.e., is an approximate map.

Proof. By (AD3), for $j < j'$,

$$(f_j p_{f(j)f(j')}, q_{jj'} f_{j'}) < \operatorname{st}\mathcal{V}_j.$$

Let $i > f(j')$. Then this implies

$$(f_j p_{f(j)f(j')} p_{f(j')i}, q_{jj'} f_{j'} p_{f(j')i}) < \operatorname{st}\mathcal{V}_j. \tag{4.1}$$

By (A1),

$$(p_{f(j)f(j')} p_{f(j')i}, p_{f(j)i}) < \mathcal{U}_{f(j)}.$$

This together with (AD1) implies

$$(f_j p_{f(j)f(j')} p_{f(j')i}, f_j p_{f(j)i}) < \mathcal{V}_j. \tag{4.2}$$

By (4.1) and (4.2),

$$(f_j p_{f(j)i}, q_{jj'} f_{j'} p_{f(j')i}) < \operatorname{st}^2 \mathcal{V}_j,$$

which proves (AM) for $\operatorname{st}\boldsymbol{f} : \operatorname{st}\boldsymbol{X} \to \operatorname{st}\boldsymbol{Y}$. ∎

For any admissible approximate resolutions $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$, let $\mathsf{AP}(\boldsymbol{X}, \boldsymbol{Y})$ denote the set of all admissible approximate maps from $\boldsymbol{X}$ to $\boldsymbol{Y}$. For any $\boldsymbol{f} = \{f_j, f\}, \boldsymbol{f}' = \{f'_j, f'\} \in \mathsf{AP}(\boldsymbol{X}, \boldsymbol{Y})$, we write $\boldsymbol{f} \sim \boldsymbol{f}'$ provided there exist $m, N \in \mathbb{N}$ with the property that each $j$ admits $i_0 > f(j), f'(j)$ such that

$$i_0 - f(j) \le m,$$
$$i_0 - f'(j) \le m,$$
$$(f_j p_{f(j)i}, f'_j p_{f'(j)i}) < \operatorname{st}^N \mathcal{V}_j \text{ for each } i \ge i_0.$$

Proposition 4.2. *For any admissible approximate resolutions $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$, $\sim$ is an equivalence relation on $\mathsf{AP}(\boldsymbol{X}, \boldsymbol{Y})$.*

Proof. It suffices to show the transitivity. Suppose $\boldsymbol{f} = \{f_j, f\}, \boldsymbol{f}' = \{f'_j, f'\}, \boldsymbol{f}'' = \{f''_j, f''\} \in \mathsf{AP}(\boldsymbol{X}, \boldsymbol{Y})$ and $\boldsymbol{f} \sim \boldsymbol{f}'$ and $\boldsymbol{f}' \sim \boldsymbol{f}''$. Then there exist $m', m'', N', N'' \in \mathbb{N}$ such that each $j$ admits $i'_0 > f(j), f'(j)$ and $i''_0 > f'(j), f''(j)$ such that $i'_0 - f(j) \le m'$, $i'_0 - f'(j) \le m'$, $i''_0 - f'(j) \le m''$, $i''_0 - f''(j) \le m''$ and

$$(f_j p_{f(j)i}, f'_j p_{f'(j)i}) < \operatorname{st}^{N'} \mathcal{V}_j \text{ for } i \ge i'_0, \tag{4.3}$$

$$(f'_j p_{f'(j)i}, f''_j p_{f''(j)i}) < \operatorname{st}^{N''} \mathcal{V}_j \text{ for } i \ge i''_0. \tag{4.4}$$

Now let $i_0 = \max\{i'_0, i''_0\}$ and $m = m' + m''$. Then $i_0 - f(j) \le m$ and $i_0 - f''(j) \le m$. By (4.3) and (4.4), for $i \ge i_0$,

$$(f_j p_{f(j)i}, f''_j p_{f''(j)i}) < \operatorname{st}^N \mathcal{V}_j$$

where $N = \max\{N', N''\} + 1$. $\square$

Now let $\widetilde{\mathsf{AP}}(\boldsymbol{X}, \boldsymbol{Y}) = \mathsf{AP}(\boldsymbol{X}, \boldsymbol{Y})/\sim$. Since $\boldsymbol{f} \sim \boldsymbol{f}'$ implies $\operatorname{st}\boldsymbol{f} \sim \operatorname{st}\boldsymbol{f}'$, there is a well-defined direct sequence

$$\widetilde{\mathsf{AP}}(\boldsymbol{X}, \boldsymbol{Y}) \to \widetilde{\mathsf{AP}}(\operatorname{st}\boldsymbol{X}, \operatorname{st}\boldsymbol{Y}) \to \cdots \to \widetilde{\mathsf{AP}}(\operatorname{st}^n \boldsymbol{X}, \operatorname{st}^n \boldsymbol{Y}) \to \cdots.$$

For any admissible approximate resolutions $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$, let the set $\mathsf{FRAC}(\boldsymbol{p}, \boldsymbol{q})$ of morphisms in $\mathsf{FRAC}$ be the limit of this sequence.

Now we wish to define the composition. For $\boldsymbol{\varphi} \in \mathsf{FRAC}(\boldsymbol{p}, \boldsymbol{q})$ and $\boldsymbol{\psi} \in \mathsf{FRAC}(\boldsymbol{q}, \boldsymbol{r})$ where $\boldsymbol{p} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$, $\boldsymbol{q} : Y \to \boldsymbol{Y} = \{Y_j, \mathcal{V}_j, q_{jj'}\}$ and $\boldsymbol{r} : Z \to \boldsymbol{Z} = \{Z_k, \mathcal{W}_k, r_{kk'}\}$ are admissible approximate resolutions, the composition $\boldsymbol{\psi} \circ \boldsymbol{\varphi} \in \mathsf{FRAC}(\boldsymbol{p}, \boldsymbol{r})$ is defined as follows: let $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ be represented by $\boldsymbol{f} = \{f_j, f\} \in \mathsf{AP}(\operatorname{st}^n \boldsymbol{X}, \operatorname{st}^n \boldsymbol{Y})$ and $\boldsymbol{g} = \{g_k, g\} \in \mathsf{AP}(\operatorname{st}^n \boldsymbol{Y}, \operatorname{st}^n \boldsymbol{Z})$, respectively. Then, let $h = fg$ and for each $k$ let $h_k = g_k f_{g(k)} : X_{fg(k)} \to Z_k$, and we have a map of systems $\boldsymbol{h} = \{h_k, h\} : \boldsymbol{X} \to \boldsymbol{Z}$.

PROPOSITION 4.3. *The map of systems $\boldsymbol{h} = \{h_k, h\}$ defines an admissible approximate map $\boldsymbol{h} = \{h_k, h\} : \operatorname{st}^{n+2} \boldsymbol{X} \to \operatorname{st}^{n+2} \boldsymbol{Z}$.*

PROOF. We must verify (AM), (AD1), (AD2) and (AD3) for $\boldsymbol{h} : \operatorname{st}^{n+2} \boldsymbol{X} \to \operatorname{st}^{n+2} \boldsymbol{Y}$. For simplicity, we may assume $n = 0$. (AD1) holds since by (AD1) for $\boldsymbol{f}$ and $\boldsymbol{g}$, $\mathcal{U}_{fg(k)} < f_{g(k)}^{-1} g_k^{-1} \mathcal{W}_k$ for each $k$, which implies $\operatorname{st}^2 \mathcal{U}_{fg(k)} < f_{g(k)}^{-1} g_k^{-1} \operatorname{st}^2 \mathcal{W}_k$ for each $k$. By (AD2) for $\boldsymbol{g}$, there exists $M \in \mathbb{N}$ such that $|g(k+1) - g(k)| \le M$ for each $k$, and hence by (AD2) for $\boldsymbol{f}$ there exists $N \in \mathbb{N}$ such that $|fg(k+1) - fg(k)| \le N$ for each $k$, verifying (AD2) for $\boldsymbol{h}$. It remains to verify (AD3) and (AM). For $k < k'$, (AD3) for $\boldsymbol{g}$ implies

$$(g_k q_{g(k)g(k')}, r_{kk'} g_{k'}) < \operatorname{st}\mathcal{W}_k, \tag{4.5}$$

and (AD3) for $\boldsymbol{f}$ implies

$$(f_{g(k)} p_{fg(k)fg(k')}, q_{g(k)g(k')} f_{g(k')}) < \operatorname{st}\mathcal{V}_{g(k)}. \tag{4.6}$$

But (AD1) for $\boldsymbol{g}$ implies

$$\operatorname{st}\mathcal{V}_{g(k)} < g_k^{-1} \operatorname{st}\mathcal{W}_k.$$

This together with (4.6) implies

$$(g_k f_{g(k)} p_{fg(k)fg(k')}, g_k q_{g(k)g(k')} f_{g(k')}) < \operatorname{st}\mathcal{W}_k. \tag{4.7}$$

By (4.5) and (4.7),

$$(g_k f_{g(k)} p_{fg(k)fg(k')}, r_{kk'} g_{k'} f_{g(k')}) < \operatorname{st}^2 \mathcal{W}_k.$$

This means (AD3) for $\boldsymbol{h} : \operatorname{st} \boldsymbol{X} \to \operatorname{st} \boldsymbol{Z}$, which together with (AD1) for $\boldsymbol{h} : \operatorname{st} \boldsymbol{X} \to \operatorname{st} \boldsymbol{Z}$ implies (AD3) and (AM) for $\boldsymbol{h} : \operatorname{st}^2 \boldsymbol{X} \to \operatorname{st}^2 \boldsymbol{Z}$ by Remark 4.1 3). ∎

The admissible approximate resolution $\boldsymbol{h} = \{h_k, h\} : \operatorname{st}^{n+2} \boldsymbol{X} \to \operatorname{st}^{n+2} \boldsymbol{Y}$ is denoted by $\boldsymbol{gf}$.

PROPOSITION 4.4. *Let $\boldsymbol{f}, \boldsymbol{f}' \in \mathsf{AP}(\boldsymbol{X}, \boldsymbol{Y})$ and $\boldsymbol{g}, \boldsymbol{g}' \in \mathsf{AP}(\boldsymbol{Y}, \boldsymbol{Z})$. Then*

1) *if $\boldsymbol{f} \sim \boldsymbol{f}'$, then $\boldsymbol{gf} \sim \boldsymbol{gf}'$, and*
2) *if $\boldsymbol{g} \sim \boldsymbol{g}'$, then $\boldsymbol{gf} \sim \boldsymbol{g}'\boldsymbol{f}$.*

PROOF. For 1), let $\boldsymbol{f} = \{f_j, f\}$, $\boldsymbol{f}' = \{f'_j, f'\} : \boldsymbol{X} \to \boldsymbol{Y}$, and suppose $\boldsymbol{f} \sim \boldsymbol{f}'$. Then there exist $m, M \in \mathbb{N}$ such that each $j$ admits $i_0 > f(j), f'(j)$ such that $i_0 - f(j) \le m$, $i_0 - f'(j) \le m$ and

$$(f_j p_{f(j)i}, f'_j p_{f'(j)i}) < \operatorname{st}^M \mathcal{V}_j,$$

and (AD1) for $\boldsymbol{g}$ implies $\operatorname{st}^M \mathcal{V}_{g(k)} < g_k^{-1} \operatorname{st}^M \mathcal{W}_k$. So each $k$ admits $i_0 > fg(k), f'g(k)$ such that $i_0 - fg(k) \le m$, $i_0 - f'g(k) \le m$ and

$$(g_k f_{g(k)} p_{fg(k)i}, g_k f'_{g(k)} p_{f'g(k)i}) < \operatorname{st}^M \mathcal{W}_k \text{ for } i > i_0.$$

This shows $\boldsymbol{gf} \sim \boldsymbol{gf}'$.

For 2), let $\boldsymbol{g} = \{g_k, g\}, \boldsymbol{g}' = \{g'_k, g'\} : \boldsymbol{Y} \to \boldsymbol{Z}$, and suppose $\boldsymbol{g} \sim \boldsymbol{g}'$. Then there exist $n, N \in \mathbb{N}$ with the property that each $k$ admits $j_0 > g(k), g'(k)$ such that $j_0 - g(k) \le n$, $j_0 - g'(k) \le n$ and

$$(g_k q_{g(k)j}, g'_k q_{g'(k)j}) < \operatorname{st}^N \mathcal{W}_k \text{ for } j \ge j_0. \tag{4.8}$$

Fix $k$, and let $i_0 = f(j_0)$. By (AD3) for $\boldsymbol{f}$ and (AD1) for $\boldsymbol{g}$,

$$(g_k f_{g(k)} p_{fg(k)i_0}, g_k q_{g(k)j_0} f_{j_0}) < \operatorname{st} \mathcal{W}_k. \tag{4.9}$$

Similarly,

$$(g'_k f_{g'(k)} p_{fg'(k)i_0}, g'_k q_{g'(k)j_0} f_{j_0}) < \operatorname{st} \mathcal{W}_k. \tag{4.10}$$

By (4.9), (4.8) and (4.10),

$$(g_k f_{g(k)} p_{fg(k)i_0}, g'_k f_{g'(k)} p_{fg'(k)i_0}) < \operatorname{st}^{N+1} \mathcal{W}_k. \tag{4.11}$$

(AD2) for $\boldsymbol{f}$ implies that there exists $m \in \mathbb{N}$ such that $i_0 - fg(k) \le m$ and $i_0 - fg'(k) \le m$ (see Remark 4.1 1)). Let $i \ge i_0$. Then by (A1) for $\boldsymbol{p} : X \to \boldsymbol{X}$,

$$(p_{fg(k)i}, p_{fg(k)i_0} p_{i_0 i}) < \mathcal{U}_{fg(k)}, \tag{4.12}$$

$$(p_{fg'(k)i}, p_{fg'(k)i_0} p_{i_0 i}) < \mathcal{U}_{fg'(k)}. \tag{4.13}$$

By (4.12), (4.11), (4.13) and (AD1) for $\boldsymbol{g}$,

$$(g_k f_{g(k)} p_{fg(k)i}, g'_k f_{g'(k)} p_{fg'(k)i}) < \operatorname{st}^{N+2} \mathcal{W}_k,$$

which proves $\boldsymbol{gf} \sim \boldsymbol{g}'\boldsymbol{f}$. ∎

Proposition 4.5. 1) *Let* $\boldsymbol{\varphi} \in \mathsf{FRAC}(\boldsymbol{p},\boldsymbol{q})$*,* $\boldsymbol{\psi} \in \mathsf{FRAC}(\boldsymbol{q},\boldsymbol{r})$*,* $\boldsymbol{\rho} \in \mathsf{FRAC}(\boldsymbol{r},\boldsymbol{s})$*. Then* $(\boldsymbol{\varphi}\circ\boldsymbol{\psi})\circ\boldsymbol{\rho} = \boldsymbol{\varphi}\circ(\boldsymbol{\psi}\circ\boldsymbol{\rho})$*.*

2) *For each* $\boldsymbol{\varphi} \in \mathsf{FRAC}(\boldsymbol{p},\boldsymbol{q})$*,* $\boldsymbol{\varphi}\circ\mathbf{1}_{\boldsymbol{p}} = \boldsymbol{\varphi}$ *and* $\mathbf{1}_{\boldsymbol{q}}\circ\boldsymbol{\varphi} = \boldsymbol{\varphi}$*, where* $\mathbf{1}_{\boldsymbol{p}} \in \mathsf{FRAC}(\boldsymbol{p},\boldsymbol{q})$ *is the morphism represented by* $1_{\boldsymbol{X}} \in \mathsf{AP}(\boldsymbol{X},\boldsymbol{Y})$*.*

Proof. For 1), suppose $\boldsymbol{\varphi},\boldsymbol{\psi},\boldsymbol{\rho}$ are represented by

$$\begin{aligned}\boldsymbol{f} &= \{f_j, f\} \in \mathsf{AP}(\mathrm{st}^n \boldsymbol{X}, \mathrm{st}^n \boldsymbol{Y}),\\ \boldsymbol{g} &= \{g_k, g\} \in \mathsf{AP}(\mathrm{st}^n \boldsymbol{Y}, \mathrm{st}^n \boldsymbol{Z}),\\ \boldsymbol{h} &= \{h_l, h\} \in \mathsf{AP}(\mathrm{st}^n \boldsymbol{Z}, \mathrm{st}^n \boldsymbol{W}),\end{aligned}$$

respectively. Here $\boldsymbol{p} : X \to \boldsymbol{X}$, $\boldsymbol{q} : Y \to \boldsymbol{Y}$, $\boldsymbol{r} : Z \to \boldsymbol{Z}$ and $\boldsymbol{s} : W \to \boldsymbol{W}$ are admissible approximate resolutions. Then $(\boldsymbol{\varphi}\circ\boldsymbol{\psi})\circ\boldsymbol{\rho}$ and $\boldsymbol{\varphi}\circ(\boldsymbol{\psi}\circ\boldsymbol{\rho})$ are both represented by the approximate map of the form $\{h_l g_{h(l)} f_{gh(l)}, fgh\}$, and hence these are the same morphism. 2) is proven similarly to Proposition 4.4. ∎

Theorem 4.6. $\mathsf{FRAC}$ *is a category.*

Proof. Propositions 4.3 and 4.4 show that the composition of morphisms is well-defined, and Proposition 4.5 shows the associativity and the existence of the identity morphism. Hence $\mathsf{FRAC}$ forms a category. ∎

Lemma 4.7. *Let* $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{i,i+1}\}$ *and* $\boldsymbol{q} = \{q_j\} : Y \to \boldsymbol{Y} = \{Y_j, \mathcal{V}_j, q_{j,j+1}\}$ *be admissible approximate resolutions of* $X$ *and* $Y$*, respectively. Let* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *be an admissible approximate map such that* $\boldsymbol{f}$ *satisfies (APS) and there is* $m \in \mathbb{N}$ *with* $f(j) \le j+m$ *for any* $j$*. Then*

$$\underline{\dim}_B(\mathrm{st}\,\boldsymbol{q} : Y \to \mathrm{st}\,\boldsymbol{Y}) \le \underline{\dim}_B(\mathrm{st}\,\boldsymbol{p} : X \to \mathrm{st}\,\boldsymbol{X})$$

*and*

$$\overline{\dim}_B(\mathrm{st}\,\boldsymbol{q} : Y \to \mathrm{st}\,\boldsymbol{Y}) \le \overline{\dim}_B(\mathrm{st}\,\boldsymbol{p} : X \to \mathrm{st}\,\boldsymbol{X}).$$

Proof. Let $i \in \mathbb{N}$. Take $j_0 > i$ as in (APS), and fix $j > j_0$. Then there exists $j_0' > j$ with the property that each $j' > j_0'$ admits $i_0 > i+m, f(j)$ such that

$$q_{jj'}(Y_{j'}) \subseteq \mathrm{st}(q_{ij} f_j p_{f(j)i'}(X_{i'}), \mathcal{V}_i) \text{ for } i' > i_0. \tag{4.14}$$

Fix $j' > j_0'$ and $i' > i_0$. By hypothesis, $f(i) \le i+m$, and by (A1),

$$(p_{f(i),i+m} p_{i+m,i'}, p_{f(i)i'}) < \mathcal{U}_{f(i)}. \tag{4.15}$$

By (4.15) and (AD1),

$$p_{i+m,i'}^{-1} p_{f(i),i+m}^{-1} \mathcal{U}_{f(i)} < p_{f(i)i'}^{-1}\,\mathrm{st}\,\mathcal{U}_{f(i)} < p_{f(i)i'}^{-1} f_i^{-1}\,\mathrm{st}\,\mathcal{V}_i. \tag{4.16}$$

By (AD3),

$$(f_i p_{f(i)f(j)}, q_{ij} f_j) < \mathrm{st}\,\mathcal{V}_i, \tag{4.17}$$

and by (A1) for $\boldsymbol{p}$,

$$(p_{f(i)f(j)}p_{f(j)i'}, p_{f(i)i'}) < \mathcal{U}_{f(i)}. \tag{4.18}$$

So, by (4.18), (AD1) and (4.17),

$$(f_i p_{f(i)i'}, q_{ij} f_j p_{f(j)i'}) < \mathrm{st}^2\, \mathcal{V}_i. \tag{4.19}$$

Also by (U) for $\boldsymbol{p}$,

$$\mathrm{st}^2\, \mathcal{U}_{i+m} < p^{-1}_{f(i),i+m} \mathcal{U}_{f(i)}. \tag{4.20}$$

Hence by (4.20), (4.16) and (4.19),

$$p^{-1}_{i+m,i'}\, \mathrm{st}^2\, \mathcal{U}_{i+m} < p^{-1}_{f(i)i'} f_j^{-1} q_{ij}^{-1}\, \mathrm{st}^3\, \mathcal{V}_i.$$

This implies that

$$N_{p^{-1}_{i+m,i'}\, \mathrm{st}^2\, \mathcal{U}_{i+m}}(X_{i'}) \geq N_{p^{-1}_{f(i)i'} f_j^{-1} q_{ij}^{-1}\, \mathrm{st}^3\, \mathcal{V}_i}(X_{i'}) = N_{\mathrm{st}^3\, \mathcal{V}_i}(q_{ij} f_j p_{f(i)i'}(X_{i'})). \tag{4.21}$$

Since $q_{ij} f_j p_{f(i)i'}(X_{i'}) \subseteq V_1 \cup \cdots \cup V_n$ for some open subsets $V_1, \ldots, V_n$ implies that

$$\mathrm{st}(q_{ij} f_j p_{f(i)i'}(X_{i'}), \mathcal{V}_i) \subseteq \mathrm{st}(V_1, \mathcal{V}_i) \cup \cdots \mathrm{st}(V_n, \mathcal{V}_i),$$

then

$$N_{\mathrm{st}^3\, \mathcal{V}_i}(q_{ij} f_j p_{f(i)i'}(X_{i'})) \geq N_{\mathrm{st}^4\, \mathcal{V}_i}(\mathrm{st}(q_{ij} f_j p_{f(i)i'}(X_{i'}), \mathcal{V}_i)). \tag{4.22}$$

But by (4.14),

$$N_{\mathrm{st}^4\, \mathcal{V}_i}(\mathrm{st}(q_{ij} f_j p_{f(i)i'}(X_{i'}), \mathcal{V}_i)) \geq N_{\mathrm{st}^4\, \mathcal{V}_i}(q_{ij'}(Y_{j'})) = N_{q^{-1}_{ij'}\, \mathrm{st}^4\, \mathcal{V}_i}(Y_{j'}). \tag{4.23}$$

(4.21),(4.22) and (4.23) show that each $j' > j'_0$ admits $i_0$ such that for each $i' > i_0$,

$$N_{p^{-1}_{i+m,i'}\, \mathrm{st}^2\, \mathcal{U}_{i+m}}(X_{i'}) \geq N_{q^{-1}_{ij'}\, \mathrm{st}^4\, \mathcal{V}_i}(Y_{j'}),$$

and hence $\beta_{i+m}(\mathrm{st}^2 \boldsymbol{X}) \geq \beta_i(\mathrm{st}^4 \boldsymbol{Y})$ for each $i$. This implies $\underline{\dim}_B(\mathrm{st}^2 \boldsymbol{p} : X \to \mathrm{st}^2 \boldsymbol{X}) \geq \underline{\dim}_B(\mathrm{st}^4 \boldsymbol{q} : Y \to \mathrm{st}^4 \boldsymbol{Y})$. But $\underline{\dim}_B(\mathrm{st}^2 \boldsymbol{p} : X \to \mathrm{st}^2 \boldsymbol{X}) = \underline{\dim}_B(\mathrm{st}\, \boldsymbol{p} : X \to \mathrm{st}\, \boldsymbol{X})$ and $\underline{\dim}_B(\mathrm{st}^4 \boldsymbol{q} : Y \to \mathrm{st}^4 \boldsymbol{Y}) = \underline{\dim}_B(\mathrm{st}\, \boldsymbol{q} : Y \to \mathrm{st}\, \boldsymbol{Y})$ by Theorem 2.6 3), and hence we have the first assertion. The second assertion is similarly obtained. $\square$

Now, for any admissible approximate resolution $\boldsymbol{p} : X \to \boldsymbol{X}$, we define the *upper* and *lower star box-counting dimensions* $\underline{\mathrm{Dim}}_B$ and $\overline{\mathrm{Dim}}_B$ as

$$\underline{\mathrm{Dim}}_B(\boldsymbol{p} : X \to \boldsymbol{X}) = \underline{\dim}_B(\mathrm{st}\, \boldsymbol{p} : X \to \mathrm{st}\, \boldsymbol{X})$$

and

$$\overline{\mathrm{Dim}}_B(\boldsymbol{p} : X \to \boldsymbol{X}) = \overline{\dim}_B(\mathrm{st}\, \boldsymbol{p} : X \to \mathrm{st}\, \boldsymbol{X}).$$

If these values coincide, the common value is called the *star box-counting dimension* of $\boldsymbol{p} : X \to \boldsymbol{X}$ and denoted by $\mathrm{Dim}_B(\boldsymbol{p} : X \to \boldsymbol{X})$. Then we have

THEOREM 4.8. *$\underline{\mathrm{Dim}}_B$ and $\overline{\mathrm{Dim}}_B$ are invariant in the category* $\mathsf{FRAC}$.

Proof. It suffices to show that if $\boldsymbol{\varphi} \in \mathsf{FRAC}(\boldsymbol{p}, \boldsymbol{q})$ and $\boldsymbol{\psi} \in \mathsf{FRAC}(\boldsymbol{q}, \boldsymbol{p})$ satisfy $\boldsymbol{\psi} \circ \boldsymbol{\varphi} = \mathbf{1}_{\boldsymbol{p}}$, then

$$\underline{\dim}_B(\mathrm{st}\, \boldsymbol{q} : Y \to \mathrm{st}\, \boldsymbol{Y}) \leq \underline{\dim}_B(\mathrm{st}\, \boldsymbol{p} : X \to \mathrm{st}\, \boldsymbol{X}) \tag{4.24}$$

and

$$\overline{\dim}_B(\mathrm{st}\, \boldsymbol{q} : Y \to \mathrm{st}\, \boldsymbol{Y}) \leq \overline{\dim}_B(\mathrm{st}\, \boldsymbol{p} : X \to \mathrm{st}\, \boldsymbol{X}). \tag{4.25}$$

Let $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ be represented by $\boldsymbol{f} \in \mathsf{AP}(\mathrm{st}^n \boldsymbol{X}, \mathrm{st}^n \boldsymbol{Y})$ and $\boldsymbol{g} \in \mathsf{AP}(\mathrm{st}^n \boldsymbol{Y}, \mathrm{st}^n \boldsymbol{Z})$, respectively, and $\boldsymbol{gf} \sim 1_{\mathrm{st}^{n+2} \boldsymbol{X}}$. So there exists $m \in \mathbb{N}$ such that $fg(i) \leq i+m$ for each $i$. Then $g(i) \leq i+m$ for each $i$. Moreover, our equivalence relation $\sim$ implies the equivalence relation $\sim$ in the sense of [2, §. 7]. So, $\boldsymbol{gf}$ represents the identity map $1_X : X \to X$, and by [2, Lemma 8.8] $\boldsymbol{gf}$ also represents the map $gf$ where $f : X \to Y$ and $g : Y \to X$ are the maps represented by $\boldsymbol{f} : \mathrm{st}^n \boldsymbol{X} \to \mathrm{st}^n \boldsymbol{Y}$ and $\boldsymbol{g} : \mathrm{st}^n \boldsymbol{Y} \to \mathrm{st}^n \boldsymbol{X}$, respectively. Thus $gf = 1_X$. So $g$ is onto, which implies by Theorem 3.1 that $\boldsymbol{g}$ satisfies (APS). Now Lemma 4.7 implies that

$$\underline{\dim}_B(\mathrm{st}^{n+1} \boldsymbol{q} : Y \to \mathrm{st}^{n+1} \boldsymbol{Y}) \leq \underline{\dim}_B(\mathrm{st}^{n+1} \boldsymbol{p} : X \to \mathrm{st}^{n+1} \boldsymbol{X}).$$

This together with Theorem 2.6 part 3) implies (4.24). Similarly we obtain (4.25). ∎

Remark 4.9. There is an obvious functor from $\mathsf{FRAC}$ to the category $\mathsf{APRES}_{\mathsf{POL}}$ of approximate resolutions which was introduced by Mardešić and Watanabe [2]. The latter category is equivalent to the category $\mathsf{CTOP}_{3.5}$ of Tychonoff spaces and maps. $\mathsf{FRAC}$ is strictly finer than $\mathsf{CTOP}_{3.5}$. Indeed, for each $r > 0$ there exist a Cantor set $X_r$ and an admissible approximate resolution $\boldsymbol{p}_r : X_r \to \boldsymbol{X}_r = \{X_i, \mathcal{U}_i, p_{i,i+1}\}$ such that $\dim_B(\boldsymbol{p}_r : X_r \to \boldsymbol{X}_r) = r$ where for each $i$ the coordinate space $X_i$ is the discrete space consisting of a finite number of points, and the open covering $\mathcal{U}_i$ consists of the discrete points. So, $\mathrm{st}\, \boldsymbol{p}_r : X_r \to \mathrm{st}\, \boldsymbol{X}_r$ is nothing but $\boldsymbol{p}_r : X_r \to \boldsymbol{X}_r$, and hence $\mathrm{Dim}_B(\boldsymbol{p}_r : X_r \to \boldsymbol{X}_r) = \dim_B(\boldsymbol{p}_r : X_r \to \boldsymbol{X}_r) = r$. For $r \neq s$, $\boldsymbol{p}_r : X_r \to \boldsymbol{X}_r$ and $\boldsymbol{p}_s : X_s \to \boldsymbol{X}_s$ are distinct objects in $\mathsf{FRAC}$ in view of Theorem 4.8 but the same object in $\mathsf{APRES}_{\mathsf{POL}}$ since $X_r$ and $X_s$ are Cantor sets ([4, §. 8]).

## References


[1] K. Falconer, Fractal geometry: mathematical foundations and applications, John Wiley & Sons, Chichester, 1990.

[2] S. Mardešić and T. Watanabe, *Approximate resolutions of spaces and maps*, Glas. Mat. **24** (1989), 587–637.

[3] T. Miyata and T. Watanabe, *Approximate resolutions and Lipschitz functions*, Topology Appl. **122** (2002), 353–375.

[4] T. Miyata and T. Watanabe, *Approximate resolutions and box-counting dimension*, Topology Appl. **132** (2003), 49–69.

[5] T. Miyata and T. Watanabe, *Bi-Lipschitz maps and the category of approximate resolutions*, Glas. Mat. **38(58)** (2003), 129–156.

[6] T. Watanabe, *Approximative shape I*, Tsukuba J. Math. **11** (1987), 17–59.

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