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A FAN X ADMITS A WHITNEY MAP FOR $C(X)$ IFF IT IS METRIZABLE

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Abstract. Let X be a non-metric continuum, and $C(X)$ be the hyperspace of subcontinua of X. It is known that there is no Whitney map on the hyperspace 2^X for non-metrizable Hausdorff compact spaces X. On the other hand, there exist non-metrizable continua which admit and ones which do not admit a Whitney map for $C(X)$. In this paper we will show that a generalized fan X admits a Whitney map for $C(X)$ if and only if it is metrizable.

1. Introduction

Introduction contains some basic definitions, results and notations. An external characterization of non-metric continua which admit a Whitney map is given in Section 2 (Theorem 2.3). In Section 3 we study hereditarily irreducible mappings onto a fan. The main theorem of this paper is Theorem 4.20.

All spaces in this paper are compact Hausdorff and all mappings are continuous. The weight of a space X is denoted by $w(X)$. The cardinality of a set A is denoted by card(A). We shall use the notion of inverse system as in [3, pp. 135-142]. An inverse system is denoted by $\mathbf{X} = \{X_a, p_{ab}, A\}$.

A *generalized arc* is a Hausdorff continuum with exactly two non-separating points. Each separable arc is homeomorphic to the closed interval $I = [0, 1]$.

For a compact space X we denote by 2^X the hyperspace of all nonempty closed subsets of X equipped with the Vietoris topology. $C(X)$ and $X(n)$, where n is a positive integer, stand for the sets of all connected members of 2^X and of all nonempty subsets consisting of at most n points, respectively, both considered as subspaces of 2^X, see [6].

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For a mapping $f : X \to Y$ define $2^f : 2^X \to 2^Y$ by $2^f(F) = f(F)$ for $F \in 2^X$. By [14, 5.10] 2^f is continuous, $2^f(C(X)) \subset C(Y)$ and $2^f(X(n)) \subset Y(n)$. The restriction $2^f|C(X)$ is denoted by $C(f)$.

An element $\{x_a\}$ of the Cartesian product $\prod\{X_a : a \in A\}$ is called a *thread* of $\mathbf{X}$ if $p_{ab}(x_b) = x_a$ for any $a, b \in A$ satisfying $a \leq b$. The subspace of $\prod\{X_a : a \in A\}$ consisting of all threads of $\mathbf{X}$ is called the limit of the inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ and is denoted by $\lim \mathbf{X}$ or by $\lim\{X_a, p_{ab}, A\}$ [3, p. 135].

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces with the natural projections $p_a : \lim \mathbf{X} \to X_a$, for $a \in A$. Then $2^{\mathbf{X}} = \{2^{X_a}, 2^{p_{ab}}, A\}$, $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ and $\mathbf{X}(n) = \{X_a(n), 2^{p_{ab}}|\, X_b(n), A\}$ form inverse systems. For each $F \in 2^{\lim \mathbf{X}}$, i.e., for each closed $F \subseteq \lim \mathbf{X}$ the set $p_a(F) \subseteq X_a$ is closed and compact. Thus, we have a mapping $2^{p_a} : 2^{\lim \mathbf{X}} \to 2^{X_a}$ induced by p_a for each $a \in A$. Define a mapping $M : 2^{\lim \mathbf{X}} \to \lim 2^{\mathbf{X}}$ by $M(F) = \{p_a(F) : a \in A\}$. Since $\{p_a(F) : a \in A\}$ is a thread of the system $2^{\mathbf{X}}$, the mapping M is continuous and one-to-one. It is also onto since for each thread $\{F_a : a \in A\}$ of the system $2^{\mathbf{X}}$ the set $F' = \bigcap\{p_a^{-1}(F_a) : a \in A\}$ is non-empty and $p_a(F') = F_a$. Thus, M is a homeomorphism. If $P_a : \lim 2^{\mathbf{X}} \to 2^{X_a}, a \in A$, are the projections, then $P_a M = 2^{p_a}$. Identifying F with $M(F)$ we have $P_a = 2^{p_a}$.

LEMMA 1.1 ([6, Lemma 2.]). *Let $X = \lim \mathbf{X}$. Then $2^X = \lim 2^{\mathbf{X}}$, $C(X) = \lim C(\mathbf{X})$ and $X(n) = \lim \mathbf{X}(n)$.*

An *arboroid* is an hereditarily unicoherent continuum which is arcwise connected by generalized arcs. A metrizable arboroid is a *dendroid.* If X is an arboroid and $x, y \in X$, then there exists a unique arc $[x, y]$ in X with endpoints x and y. If $[x, y]$ is an arc, then $[x, y] \setminus \{x, y\}$ is denoted by (x, y).

A point t of an arboroid X is said to be a *ramification point* of X if t is the only common point of some three arcs such that it is the only common point of any two, and an end point of each of them.

A point e of an arboroid X is said to be *end point* of X if there exists no arc $[a, b]$ in X such that $x \in [a, b] \setminus \{a, b\}$.

If an arboroid X has only one ramification point t, it is called a *generalized fan* with the top t. A metrizable generalized fan is called a *fan.*

We say that an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is *σ-directed* if for each sequence $a_1, a_2, ..., a_k, ...$ of the members of A there is an $a \in A$ such that $a \geq a_k$ for each $k \in \mathbb{N}$.

In the sequel we shall use the following theorem.

THEOREM 1.2 ([7, Lemma 2.2]). *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a σ-directed inverse system of compact spaces with surjective bonding mappings and the limit X. Let Y be a metric compact space. Then for each surjective mapping $f : X \to Y$ there exists an $a \in A$ such that for each $b \geq a$ there exists a mapping $g_b : X_b \to Y$ such that $f = g_b p_b$.*

If the bonding mappings are not surjective, then we consider the inverse system $\{p_a(X),\, p_{ab}|p_b(X),\, A\}$ which has surjective bonding mappings. Moreover, $p_a(X) = \cap\{p_{ab}(X_b) : b \geq \ a\}$. Applying Theorem 1.2 we obtain the following theorem.

THEOREM 1.3. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be a* σ*-directed inverse system of compact spaces with the limit* X*. Let* Y *be a metric compact space. Then for each surjective mapping* $f : X \to Y$ *there exists an* $a \in A$ *such that for each* $b \geq a$ *there exists a mapping* $g_b : p_b(X) \to Y$ *such that* $f = g_b p_b$*.*

In the sequel we shall use the following results.

LEMMA 1.4 ([3, Corollary 2.5.7]). *Any closed subspace* Y *of the limit* X *of an inverse system* $\mathbf{X} = \{X_a,\, p_{ab},\, A\}$ *is the limit of the inverse system* $\mathbf{X}_Y = \{Cl(p_a(Y)),\, p_{ab}|Cl(p_b(Y)),\, A\}$.

LEMMA 1.5 ([3, Corollary 2.5.11]). *Let* $\mathbf{X} = \{X_a,\, p_{ab},\, A\}$ *be an inverse system and* B *a subset cofinal in* A*. The mapping consisting in restricting all threads from* $X = \lim \mathbf{X}$ *to* B *is a homeomorphism of* X *onto the space* $\lim\{X_b,\, p_{bc},\, B\}$.

Now we will prove some expanding theorems of non-metric compact spaces into σ-directed inverse systems of compact metric spaces.

THEOREM 1.6. *If* X *is the Cartesian product* $X = \prod\{X_s : s \in S\}$*, where* $\mathrm{card}(S)\ > \aleph_0$ *and each* X_s *is compact, then there exists a* σ*-directed inverse system* $\mathbf{X} = \{Y_a, P_{ab}, A\}$ *of the countable products* $Y_a = \prod\{X_\mu : \mu \in a\}$*,* $\mathrm{card}(a) = \aleph_0$*, such that* X *is homeomorphic to* $\lim \mathbf{X}$.

PROOF. Let A be the set of all subsets of S of the cardinality $\aleph_0$ ordered by inclusion. If $a \subseteq b$, then we write $a \leq b$. It is clear that A is σ-directed. For each $a \in A$ there exists the product $Y_a = \prod\{X_\mu : \mu \in a\}$. If $a, b \in A$ and $a \leq b$, then there exists the projection $P_{ab} : Y_b \to Y_a$. Finally, we have the system $\mathbf{X} = \{Y_a, P_{ab}, A\}$. Let us prove that X is homeomorphic to $\lim \mathbf{X}$. Let $x \in X$. It is clear that $P_a(x) = x_a$ is a point of Y_a and that $P_{ab}(x_b) = x_a$ if $a \leq b$. This means that (x_a) is a thread in $\mathbf{X} = \{Y_a, P_{ab}, A\}$. Set $H(x) = (x_a)$. We have the mapping $H : X \to \lim \mathbf{X}$. It is clear that H is continuous, 1-1 and onto. Hence, H is a homeomorphism. □

COROLLARY 1.7. *For each Tychonoff cube* I^m*,* $m \geq \aleph_1$*, there exists a* σ*-directed inverse system* $\mathbf{I} = \{I^a, P_{ab}, A\}$ *of the Hilbert cubes* I^a *such that* I^m *is homeomorphic to* $\lim \mathbf{I}$.

THEOREM 1.8. *Let* X *be a compact Hausdorff space such that* $w(X) \geq \aleph_1$*. There exists a* σ*-directed inverse system* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *of metric compacta* X_a *such that* X *is homeomorphic to* $\lim \mathbf{X}$.

Proof. By [3, Theorem 2.3.23.] the space X is embeddable in $I^{w(X)}$. From Corollary 1.7 it follows that $I^{w(X)}$ is a limit of $\mathbf{I} = \{I^a, P_{ab}, A\}$, where every I^a is the Hilbert cube. Now, X is a closed subspace of $\lim \mathbf{I}$. Let $X_a = P_m(X)$, where $P_m : I^m \to I^a$ is a projection of the Tychonoff cube I^m onto the Hilbert cube I^a. Let p_{ab} be the restriction of P_{ab} on X_b. We have the inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that $w(X_a) \leq \aleph_0$. By virtue of Lemma 1.4 X is homeomorphic to $\lim \mathbf{X}$. Moreover, $\mathbf{X}$ is a σ-directed inverse system since $\mathbf{I} = \{I^a, P_{ab}, A\}$ is a σ-directed inverse system. □

2. Whitney map and hereditarily irreducible mappings

The notion of an irreducible mapping was introduced by Whyburn [21, p. 162]. If X is a continuum, a surjection $f : X \to Y$ is *irreducible* provided no proper subcontinuum of X maps onto all of Y under f. Some theorems for the case when X is semi-locally-connected are given in [21, p. 163].

A mapping $f : X \to Y$ is said to be *hereditarily irreducible* [15, p. 204, (1.212.3)] provided that for any given subcontinuum Z of X, no proper subcontinuum of Z maps onto $f(Z)$.

A mapping $f : X \to Y$ is *light* (*zero-dimensional*) if all fibers $f^{-1}(y)$ are hereditarily disconnected (zero-dimensional or empty) [3, p. 450], i.e., if $f^{-1}(y)$ does not contain any connected subsets of cardinality larger that one ($\dim f^{-1}(y) \leq 0$). Every zero-dimensional mapping is light, and in the realm of mappings with compact fibers the two classes of mappings coincide.

Lemma 2.1. *Every hereditarily irreducible mapping is light.*

Lemma 2.2. *If $f : X \to Y$ is monotone and hereditarily irreducible, then f is 1-1.*

Let Λ be a subspace of 2^X. By a *Whitney map* for Λ [15, p. 24, (0.50)] we will mean any mapping $g : \Lambda \to [0, +\infty)$ satisfying

a) if $\{A\}, \{B\} \in \Lambda$ such that $A \subset B, A \neq B$, then $g(\{A\}) < g(\{B\})$ and

b) $g(\{x\}) = 0$ for each $x \in X$ such that $\{x\} \in \Lambda$.

If X is a metric continuum, then there exists a Whitney map for 2^X and $C(X)$ ([15, pp. 24-26], [5, p. 106]). On the other hand, if X is non-metrizable, then it admits no Whitney map for 2^X [2]. It is known that there exist non-metrizable continua which admit and ones which do not admit a Whitney map for $C(X)$ [2]. Moreover, if X is a non-metrizable locally connected or a rim-metrizable continuum, then X admits no Whitney map for $C(X)$ [9, Theorem 8, Theorem 11]. In what follows we shall show that a generalized fan X does not admit any Whitney map for $C(X)$.

The first step in proving this statement is an external characterization of non-metric continua which admit a Whitney map.

Theorem 2.3. *Let X be a non-metric continuum. Then X admits a Whitney map for $C(X)$ if and only if for each σ-directed inverse system $\mathbf{X} =$*

$\{X_a,\ p_{ab},\ A\}$ of continua which admit Whitney maps for $C(X_a)$ and $X = \lim \mathbf{X}$ there exists a cofinal subset $B \subset A$ such that for every $b \in B$ the projection $p_b : \lim \mathbf{X} \to X_b$ is hereditarily irreducible.

PROOF. *Necessity.* Consider inverse system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ whose limit is $C(X)$ (Lemma 1.1). If $\mu : C(X) \to [0, \infty)$ is a Whitney map for $C(X)$, then, by Theorem 1.3, there exists a cofinal subset B of A such that for every $b \in B$ there is a mapping $\mu_b : C(p_b)(X) \to [0, \infty)$ with $\mu = \mu_b C(p_b)$. Suppose that p_b is not hereditarily irreducible. Then there exists a pair F, G of subcontinua of X with $F \subseteq G$, $F \neq G$, (i.e., F is a proper subcontinuum of G) such that $p_b(F) = p_b(G)$. It is clear that $C(p_b)(\{F\}) = C(p_b)(\{G\})$. This means that $\mu_b C(p_b)(\{F\}) = \mu_b C(p_b)(\{G\})$. From $\mu = \mu_b C(p_b)$ it follows that $\mu(\{F\}) = \mu(\{G\})$. This is impossible since μ is a Whitney map for $C(X)$ and from $F \subseteq G$, $F \neq G$ it follows $\mu(\{F\}) < \mu(\{G\})$.

Sufficiency. Suppose that there exists a cofinal subset $B \subset A$ such that for every $b \in B$ the projection $p_b : \lim \mathbf{X} \to X_b$ is hereditarily irreducible. Consider inverse system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ whose limit is $C(X)$ (Lemma 1.1). Let $\mu_b : C(X_b) \to [0, \infty)$ be a Whitney map for $C(X_b)$, where $b \in B$ is fixed. We shall prove that $\mu = \mu_b C(p_b) : C(X) \to [0, \infty)$ is a Whitney map for $C(X)$. Let F, G be a pair of subcontinua of X with $F \subseteq G$, $F \neq G$. We must prove that $\mu(\{F\}) < \mu(\{G\})$. Now, $p_b(F) \subset p_b(G)$ and $p_b(F) \neq p_b(G)$ since p_b is hereditarily irreducible. We infer that $\mu_b(\{p_b(F)\}) < \mu_b(\{p_b(G)\})$ since μ_b is a Whitney map for $C(X_b)$. Moreover, $\{p_b(F)\} = C(p_b)(\{F\})$ and $\{p_b(G)\} = C(p_b)(\{G\})M$. From $\mu_b(\{p_b(F)\}) < \mu_b(\{p_b(G)\})$ we have $\mu_b(C(p_b)(\{F\})) < \mu_b(C(p_b)(\{G\}))$, i.e., $\mu_b C(p_b)(\{F\}) < \mu_b C(p_b)(\{G\})$. Finally, $\mu(\{F\}) < \mu(\{G\})$ since $\mu = \mu_b C(p_b)$. □

REMARK 2.4. It follows from Theorem 2.3 and Lemma 2.1 that the projections p_b are light for every $b \in B$. It is a question are the bonding mappings p_{ab} light mappings. The following theorem shows that it is possible to find such inverse system which has the light bonding mappings.

THEOREM 2.5. *If X is a non-metric continuum which admits a Whitney map for $C(X)$, then there exists a σ-directed inverse system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ of metric continua X_a such that the bonding mappings p_{ab} are light and $X = \lim \mathbf{X}$.*

PROOF. By virtue of Theorem 1.8 there exists a σ-directed inverse system $\mathbf{Y} = \{Y_a,\ q_{ab},\ B\}$ of metric compact spaces Y_a such that $X = \lim \mathbf{Y}$. From Remark 2.4 it follows that there exists a metric space Y_b such that the projection $q_b : X \to Y_b$ is light. Using [18, p. 204, Theorem 7.10] we obtain an inverse system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ of metric compact spaces and zero-dimensional bonding mappings such that $X = \lim \mathbf{X}$. Since every zero-dimensional mapping is light, and in the realm of mappings with compact fibers the two classes of mappings coincide [3, p. 450], we infer that p_{ab} are light. Applying Theorem 2.3 we conclude that there exists a $B \subset A$ which is cofinal in A and such that the projections p_b are light for every $b \in B$. ☐

We close this section with the following theorem.

THEOREM 2.6. *If X is the Cartesian product $X = \prod\{X_s : s \in S\}$, where* $\mathrm{card}(S) > \aleph_0$ *and each X_s is a continuum, then there is no Whitney map for $C(X)$.*

PROOF. By virtue of Theorem 1.6 it follows that for the Cartesian product $X = \prod\{X_s : s \in S\}$, $\mathrm{card}(S) > \aleph_0$, there exists a σ-directed inverse system $\mathbf{X} = \{Y_a, P_{ab}, A\}$ of the products $Y_a = \prod\{X_\mu : \mu \in a\}$, $\mathrm{card}(a) = \aleph_0$, such that X is homeomorphic to $\lim \mathbf{X}$. If every $X_s : s \in S$, is a continuum, then every bonding mapping P_{ab} in $\mathbf{X} = \{Y_a, P_{ab}, A\}$ is monotone since $P_{ab}^{-1}(x)$ is the product of all X_s which are factors in Y_b but not factors in Y_a. The statement of Theorem follows from Theorem 2.3. ☐

3. HEREDITARILY IRREDUCIBLE MAPPINGS ONTO ARBOROIDS

Theorem 2.3 suggests the study of hereditarily irreducible mappings. In this section we will consider hereditarily irreducible mappings onto arboroids.

A continuum X is said to be *arcwise connected* provided for every two points $x, y \in X, x \neq y$, there is a generalized or a metrizable arc $[x, y] \subset X$.

LEMMA 3.1. *If X is an arboroid and if Y is an arboroid which contains finitely many ramification points, then every hereditarily irreducible mapping $f : X \to Y$ is a homeomorphism.*

PROOF. Suppose that f is not a homeomorphism. Then there exists a point $y \in Y$ such that $f^{-1}(y)$ is not a single point. This means that there exist points $x_1, x_2 \in X$ such that $f(x_1) = f(x_2) = y$. Since X is an arboroid there exists the unique generalized arc Z in X such that x_1, x_2 are end points of Z.

CLAIM 1. *There exists a segment $[a, b]$ of Z such that $f^{-1}(y) \cap (a, b) = \emptyset$ and $f^{-1}(y) \cap [a, b] = \{a, b\}$.*

It is clear that $f^{-1}(y)$ is not dense in Z. In the opposite case we have that Z is a proper subcontinuum of $f^{-1}(y)$. This is impossible since $f^{-1}(y)$ contains no continuum. It follows that there exists a segment $[c, d] \subset Z$ such that $f^{-1}(y) \cap Z \subset [c, d]$ and $\{c, d\} \subset f^{-1}(y) \cap Z$. It is again clear that there exists a subinterval (a_1, b_1) of $[c, d]$ such that $f^{-1}(y) \cap (a_1, b_1) = \emptyset$. Let $\mathcal{A}$ be a family of all segments (a_α, b_α) which contains (a_1, b_1) and $f^{-1}(y) \cap (a_\alpha, b_\alpha) = \emptyset$. It is clear that the union of all elements of $\mathcal{A}$ is a subsegment (a, b) of $[c, d]$. Let us prove that $a, b \in f^{-1}(y)$. Suppose that $a \notin f^{-1}(y)$. Then $f(a) \neq y$. There exists an open set U containing a such that $f(U)$ does not contain the point y. It is clear that there exists a segment (e, h) contained in

U. Then $(a,\, b) \cup (e,\, h)$ is a segment which contains $(a_1,\, b_1)$. It is clear that $(a,\, b) \cup (e,\, h)$ is not in $\mathcal{A}$, a contradiction. Hence, $a \in f^{-1}(y)$. Similarly, one can prove that $b \in f^{-1}(y)$.

In the remaining part of the proof we shall consider the restriction $g = f|[a,b]$. Let us recall that g is hereditarily irreducible and that $W = f([a,b])$, as a subcontinuum of Y, is an arboroid. Thus we have a hereditarily irreducible surjection g of the arc $[a,b]$ onto a dendroid W such that $g^{-1}(y) = \{a,b\}$.

CLAIM 2. *There exist subarcs* $[a,x]$ *and* $[z,b]$ *such that* $g([a,x]) \subset g([z,b])$ *or* $g([a,x]) \supseteq g([z,b])$.

Let U_y be a neighborhood of y such that $U_y \diagdown \{y\}$ does not contain ramification points. There exist segments $[a,x]$ and $[z,b]$ such that $g([a,x]) \subset U_y$ and $g([z,b]) \subset U_y$. It follows that $g([a,x])$ and $g([z,b])$ are arcs since $g((a,x])$ and $g([z,b))$ do not contain ramification points. Suppose that $g([a,x]) \cap g([z,b]) = \{y\}$. Then $C = g([a,x]) \cup g([z,b])$ is a continuum. Because of Claim 1, $g([x,z])$ is a continuum not containing the point y. It follows that $C \cap g([x,z])$ is not a continuum since $C \cap g([x,z])$ contains $\{y\}$ and two disjoint subsets $g([a,x]) \cap g([x,z] \supseteq \{g(x)\}$ and $g([x,z]) \cap g([z,b] \supseteq \{g(z)\}$ not containing $\{y\}$. This is impossible since is W is hereditarily unicoherent. Hence, $D = g([a,x]) \cap g([z,b])$ is a non-degenerate continuum containing the point $\{y\}$. It is clear that D does not contain ramification points. It follows that $g([a,x]) \subset g([z,b])$ or $g([a,x]) \supseteq g([z,b])$ since in the opposite case we obtain a triod in U_y.

CLAIM 3. *We may assume that* $g([a,x]) \supseteq g([z,b])$.

Now, $g([a,z]) = g([a,b])$ since $g([a,x]) \supseteq g([z,b])$. This is impossible since g is hereditarily irreducible. ☐

COROLLARY 3.2. *If* X *is an arboroid and if* Y *is a generalized fan, then every hereditarily irreducible mapping* $f : X \to Y$ *is a homeomorphism.*

Now we are ready to prove the following theorem.

THEOREM 3.3. *Let* $\mathbf{X} = \{X_a,\, p_{ab},\, A\}$ *be a* σ*-directed inverse system of fans. If* $X = \lim \mathbf{X}$ *is arcwise connected, then* X *admits a Whitney map for* $C(X)$ *if and only if* X *is metrizable.*

PROOF. If X is metrizable, then it admits a Whitney map for $C(X)$ [15, pp. 24-26]. Suppose now that X admits a Whitney map for $C(X)$. From Theorem 2.3 it follows that there exists a cofinal subset B of A such that for every $b \in B$ the projection p_b is hereditarily irreducible. By Corollary 3.2 we infer that p_b is a homeomorphism. Hence, X is metrizable since each X_b is metrizable. ☐

4. AM-fans

We say that an arboroid X is an *AM-arboroid* if each arc in X is metrizable. Now we shall prove that every arboroid is a limit of a σ-directed inverse systems of AM-arboroids.

Theorem 4.1. *Let X be an arboroid. There exists an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is an AM-arboroid, every p_{ab} is monotone and X is homeomorphic to* $\lim \mathbf{X}$.

Proof. If X is an AM-dendroid, then it has metrizable arcs and Theorem is obvious. If X is not an AM-dendroid, then there exists an inverse σ-system $\mathbf{Y} = \{Y_a, q_{ab}, A\}$ of metric continua X_a such that X is homeomorphic to $\lim \mathbf{Y}$ (Theorem 1.8). It is clear that the projections q_a are not light since then the restrictions $q_a|L$ are light for every arc L in X. Then from [10, Theorem 1] it follows that L is metrizable. Hence, q_a is not light. Let q_a be the natural projection of X onto Y_a. Applying the monotone-light factorization [3, pp. 450-451] to q_a, we get compact spaces X_a, monotone surjections $m_a : X \to X_a$ and light surjections $l_a : X_a \to Y_a$ such that $q_a = l_a m_a$. By [10, Lemma 8] there exist monotone surjections $p_{ab} : X_b \to X_a$ such that $p_{ab} m_b = m_a$, $a \leq b$. It follows that $\mathbf{X} = \{X_a, p_{ab}, A\}$ is an inverse system such that X is homeomorphic to $\lim \mathbf{X}$. Let us prove that X_a is an AM-arboroid. The space X_a is hereditarily unicoherent since m_a is monotone. Moreover, X_a is arcwise connected. Namely, if x_a, y_a are distinct points of X_a, then there exists a pair x, y of points of X such that $x_a = m_a(x)$ and $y_a = m_a(y)$. Let L be the arc with end points x and y. Now, $m_a(L)$ is a continuous image of an arc and, consequently, arcwise connected [19]. Hence, X_a is an arboroid. Since every map l_a is light, we infer that each arc in X_a is metrizable (by [20, Theorem 1.2, p. 464] saying that if X is rim-metrizable and a surjective mapping $l : X \to Y$ is light, then $w(X) = w(Y)$; compare also [10, Theorem 1]). Hence, every X_a is an AM-arboroid. □

Corollary 4.2. *Let X be a generalized fan. There exists an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is an AM-fan, every p_{ab} is monotone and X is homeomorphic to* $\lim \mathbf{X}$.

Proof. By Theorem 4.1 there exists an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is an AM-arboroid, every p_{ab} is monotone and X is homeomorphic to $\lim \mathbf{X}$. Let us observe that the projections $p_a : X \to X_a$ are monotone [3, 6.3.16.(a), pp. 462-463]. It remains to prove that each X_a is an AM-fan. Suppose that some X_a is not AM-fan. This means that X_a has two different ramification points. It follows that X_a contains two different triods T_1 and T_2. Hence, there is a triod, say T_2, such that $p_a^{-1}(T_2)$ is a subset of some arc L in X since X is a generalized fan. It is clear that this impossible since $p_a^{-1}(T_2)$ is a continuum. □

THEOREM 4.3. *If a generalized fan X admits a Whitney map for $C(X)$, then X is an AM-fan.*

PROOF. It follows from Corollary 4.2 that there exists an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is an AM-fan, every p_{ab} is monotone and X is homeomorphic to $\lim \mathbf{X}$. If X admits a Whitney map for $C(X)$, then there exists a cofinal subset B of A such that p_b is hereditarily irreducible for every $b \in B$ (Theorem 2.3). From Lemma 2.2 we infer that p_b is 1-1. Hence, p_b is a homeomorphism. This means that X is an AM-fan. □

Now we shall expand every non-metric AM-fan into inverse system of a metric finite fan. This is done in Theorem 4.19. The proof of this Theorem requires some preliminary definitions and results which are straightforward modifications of [4].

A *chain,* in a topological space, is a collection $\mathcal{E} = \{E_1, ...E_m\}$ of open sets E_i such that $E_i \cap E_j \neq \emptyset$ if and only if $|i-j| \leq 1$. The elements of $\mathcal{E}$ are *links*. Let $\mathcal{U}$ be an open cover of a space X. We say that a chain $\mathcal{E} = \{E_1, ...E_m\}$ is a $\mathcal{U}$-chain if each link E_i of $\mathcal{E}$ is contained in some member U of $\mathcal{U}$.

Let $\mathcal{E} = \{E_1, ...E_m\}$ be a chain; frequently we denote $\mathcal{E}$ by $E(1,m)$ and denote $\cup\{E_i : 1 \leq i \leq m\}$ by $E^*(1,m)$ or by $\mathcal{E}^*$.

DEFINITION 4.4. *If $[a,b]$ is an arc and $\mathcal{E} = E(1,m)$ is a chain covering $[a,b]$ then $[a,b]$ is straight in $\mathcal{E}$ provided:*

1. *$\mathcal{E}$ is a chain from a to b i.e. $a \in E_1 \setminus Cl\ E_2$, $b \in E_m \setminus Cl\ E_{m-1}$,*
2. *$(\partial E_i \cap [a,b])$ is a one point set if $i = 1$ or $i = m$ and a two point set otherwise.*

LEMMA 4.5. *Suppose that X is an AM-arboroid, Y is a finite tree, $Y \subset X$ and $p \in Y$. Let $\mathcal{K} = \{K : K$ is a component of $Y \setminus \{p\}\}$. Then for each open set U such that $p \in U$ there exists an open set V such that $p \in V \subset U$ and* $\mathrm{card}(Y \cap \partial V) = \mathrm{card}(\mathcal{K})$.

PROOF. The proof is the same as the proof of Lemma 1 of [4] since X has metrizable arcs and Y is metrizable. Namely, $\mathcal{K}$ is a finite set, since each component of $Y \setminus \{p\}$ contains an end point of Y. This follows from the fact that if $K \in \mathcal{K}$, then K is arcwise connected, because Y is locally connected. The end points of Y are precisely the end points of maximal arcs in Y. Since $K \cup \{p\}$ is a tree and K is arcwise connected, then if A is a maximal arc in $K \cup \{p\}$, at least one end point of A is an end point of Y. Suppose $\mathcal{K} = \{K_1, ..., K_n\}$. According to [21, p. 88] there is a set V', open in Y such that $p \in V' \subset U$, and $\partial_Y V'$, the boundary of V' relative to Y, contains exactly n points. Now V' must be connected, since if V'' is the component of V' containing p, then V'' is open in Y and $\partial_Y V'' \subset \partial_Y V'$. Since we may assume that for each i, $K_i \subsetneqq Cl\ U$, $\partial_Y V''$ contains a point from each K_i.

Since $\partial_Y V'$ contains only n points, $V' = V''$. Thus $Y \setminus \partial_Y V'$ is the union of two separated sets, one of which is V' and the other contains $Y \setminus U$. There are disjoint sets S and T, open in X, such that $V' \subset S$ and $Y \setminus U \subset T$. Now let $V = U \setminus Cl\ T$. Then $(\partial V) \cap Y = (\partial T) \cap Y = \partial_Y V'$, an n-point set. □

LEMMA 4.6. *Suppose* $[a, b]$ *is straight in* $\mathcal{E} = E(1, m)$ *and* W *is an open set containing* $[a, b]$. *Then* $[a, b]$ *is straight in* $\{E_1 \cap W, E_2 \cap W, ..., E_m \cap W\}$.

PROOF. This is Lemma 2 of [4]. It is clear from the definition of straightness that for each i, $\partial(E_i \cap W)$ contains at least as many points of $[a, b]$ as ∂E_i does. Conversely, since $\partial(E_i \cap W) \subset (\partial E_i) \cap (\partial W)$ and $[a, b] \subset W$, $(\partial(E_i \cap W)) \cap [a, b] \subset (\partial E_i) \cap [a, b]$. Thus $\partial(E_i \cap W)$ contains exactly as many points of $[a, b]$ as ∂E_i does. That is, $[a, b]$ is straight in $\{E_1 \cap W, E_2 \cap W, ..., E_m \cap W\}$. □

We now show that each arc in AM-arboroid can be covered by chains in which that arc is straight.

LEMMA 4.7. *If* $[a, b]$ *is an arc in an AM-dendroid* X *and* $\mathcal{U}$ *an open covering of* X, *then there an chain* $\mathcal{E} = E(1, m)$ *of sets open in* X *such that* $\mathcal{E} = E(1, m)$ *refines* $\mathcal{U}$ *and* $[a, b]$ *is straight in* $\mathcal{E}$.

PROOF. The proof is a straightforward modification of the proof of [4, Proposition 1]. Suppose, to the contrary, that there is an arc $[a, b]$ in X such that $[a, b]$ is not straight in any chain which refines $\mathcal{U}$. For fixed $\mathcal{U}$ and fixed arc $[a, b]$, we say that a subarc $[a', b']$ of $[a, b]$ has property $\mathcal{P}$ iff $[a', b']$ is not straight in any chain which refines $\mathcal{U}$. Clearly $[a, b]$ has property $\mathcal{P}$. We now show that property $\mathcal{P}$ is inductive. Let $\mathcal{L} = \{L_\alpha : \alpha < \omega_\tau\}$ be a transfinite sequence such that, for each ordinal $\alpha < \omega_\tau$, L_α has property $\mathcal{P}$ and $L_\beta \subset L_\alpha$ if $\alpha < \beta < \omega_\tau$. We must show that $L = \cap\{L_\alpha : \alpha < \omega_\tau\}$ has property $\mathcal{P}$. If it does not, then L is not degenerate, hence it is a subarc $[c, d]$ of $[a, b]$. Since $[a, b]$ has property $\mathcal{P}$, $[c, d] \neq [a, b]$. Without loss of generality, we may assume that $a < c < d \leq b$, $<$ denoting the usual order from a to b on $[a, b]$. Since $[c, d]$ does not have property $\mathcal{P}$, there is a chain $\mathcal{F} = F(1, n)$ of open sets in X such that $[c, d]$ is straight in $\mathcal{F}$ and $\mathcal{F}$ refines $\mathcal{U}$. Let U be an open set such that $c \in U$ and $Cl\ U \subset F_1 \setminus Cl\ F_2$. According to Lemma 4.5, there is an open set V such that $c \in V \subset U$ and $(\partial V) \cap [c, d]$ is degenerate. Similarly, there is an open set R such that $d \in R \subset Cl\ R \subset F_n \setminus Cl\ F_{n-1}$ and $(\partial R) \cap [c, d]$ is degenerate. Now $(V \cup [c, d] \cup R) \cap [a, b]$ is open in $[a, b]$ and contains L. Hence there is an $\alpha < \omega_\tau$ such that $L_\alpha \subset (V \cup [c, d] \cup R) \cap [a, b]$. If $L_\alpha = [a_\alpha, b_\alpha]$, then we may assume that $a_\alpha \in V$ and $b_\alpha \in R$, since $L_\alpha \setminus [c, d] \subset V \cup R$. Since $V \subset F_1 \setminus Cl\ F_2$ and $R \subset F_n \setminus Cl\ F_{n-1}$, $\mathcal{F}$ is a chain from a_α to a_β covering $[a_\alpha, b_\alpha]$. Since, for each α, $(\partial F_\alpha) \cap (V \cup R) = \emptyset$, $\partial F_\alpha \cap [a_\alpha, b_\alpha] = \partial F_\alpha \cap [c, d]$, which is degenerate if F_α is an end link of $\mathcal{F}$ and a two point set otherwise. Thus $L_\alpha = [a_\alpha, b_\alpha]$ is straight in $\mathcal{F}$. This is impossible, for L_α was assumed to have property $\mathcal{P}$. It follows that $[c,d]$ must have property $\mathcal{P}$, hence that property $\mathcal{P}$ is inductive.

Since $[a,b]$ has property $\mathcal{P}$, there is a subcontinuum of $[a,b]$ which is irreducible with respect to having property $\mathcal{P}$. This subcontinuum must be non-degenerate; we shall simply assume that $[a,b]$ is irreducible with respect to having property $\mathcal{P}$. Let x be a non-end point of $[a,b]$. Since $[a,x]$ and $[x,b]$ are proper subarcs of $[a,b]$, neither has property $\mathcal{P}$. Hence there are $\mathcal{U}$-chains $\mathcal{G} = G(1,j)$ and $\mathcal{H} = H(1,k)$ of open sets in X such $[a,x]$ is straight in $\mathcal{G}$ and $[x,b]$ is straight in $\mathcal{H}$.

Using regularity and Lemma 4.5, we obtain an open set Q such that $x \in Q \subset Cl\ Q \subset (G_j \setminus Cl\ G_{j-1}) \cap (H_1 \setminus Cl\ H_2)$ and $(\partial Q) \cap [a,b]$ contains exactly two points, one in $[a,x]$, the other in $[x,b]$. Clearly, $[a,x] \setminus Q$ and $[x,b] \setminus Q$ are disjoint closed sets. It follows that $X \setminus Q$ is the union of two disjoint closed sets A and B, with $[a,x] \setminus Q \subset A$ and $[x,b] \setminus Q \subset B$. From the normality of X we infer that there exist open sets S and T such that $A \subset S$, $B \subset T$ and $Cl\ S \cap Cl\ T = \emptyset$. We now define chains $\mathcal{G}' = G'(1,j)$ and $\mathcal{H}' = H'(1,k)$, one-to-one refinements of $\mathcal{G}$ and $\mathcal{H}$, respectively, by $G'_i = G_i \cap (S \cup Q)$, $H'_i = H_i \cap (T \cup Q)$. Lemma 4.6 shows that $[a,x]$ is straight in $\mathcal{G}'$ and $[x,b]$ is straight in $\mathcal{H}'$. Since the only points in a link of $\mathcal{G}'$ and a link of $\mathcal{H}'$ are those in Q, we may define a chain $\mathcal{E} = E(1,m)$ by $E_i = G'_i$, if $1 \le i \le j$; $G_i = H'_{i-j}$, if $j+1 \le i \le j+k$. One can prove (see [4, p. 116]) that $[a,b]$ is straight in $\mathcal{E}$. □

Lemma 4.7 shows that one can cover each arc from the top of an AM-fan to an end point by a chain in which the arc is straight and a finite collection of these chains cover the AM-fan. However, different chains may intersect very badly. In order to cut them apart, we will need some control over boundaries of the links. Hence we establish

Lemma 4.8. *Suppose X is an AM-fan, t is the top of X and W is the set of end points of X. For each cover $\mathcal{U}$ of X and each $w \in W$ there is a U-chain $\mathcal{E} = E(1,m)$ of sets open in X such that $[t,w]$ is straight in $\mathcal{E} = E(1,m)$ and $\partial E^*(2,m) \subset E_1$.*

Given an AM-fan X and an open cover $\mathcal{U}$ of X we want to cover X with a $\mathcal{U}$-tree chain whose nerve is a triangulation of a finite fan as does Figure 3 in [4]. The following Lemma shows that we can do this for a finite subfan Y of X.

Lemma 4.9 ([4, Proposition 3]). *Suppose X is an AM-fan, Y is a finite subfan of X, the top of X, t, is the top of Y and each end point w of Y, $w \neq t$, is an end point of X. If $Y = \cup\{[t,w_i] : i \in \{1,2,...,n\}\}$ and $\mathcal{U}$ is a cover of X, then there exists a finite collection $\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_n$ such that:*

(i) *each $\mathcal{F}_j = F_j(1,r_j) = \{F_{j1}, F_{j2}, ..., F_{jr_j}\}$ is a $\mathcal{U}$-chain consisting of at least 3 links,*

(ii) *for each* j, $[t, w_j]$ *is straight in* $\mathcal{F}_j$,
(iii) *for each* j, $\partial F^*(2, r_j) \subset F_{j1}$,
(iv) *for each* j, $F_{j1} = F_{11}$,
(v) *if* $i \neq j$ *then* $([t, w_j] \cup F^*(2, r_j)) \cap$ *Cl* $F^*(2, r_i) = \emptyset$.

Let $\mathcal{U}$ be a cover of a space X. We shall write $(x, y) < \mathcal{U}$ if there is an element $U \in \mathcal{U}$ such that $x, y \in U$.

Once we have covered the AM-fan X as in Figure 3 of [4], we use the cover to construct the retraction. To do this, we will piece together the retractions of chains onto straight arcs. We therefore prove

LEMMA 4.10. *Let a compact space* X *contain an arc* $[a, b]$ *that is straight in a* $\mathcal{U}$*-chain* $\mathcal{E} = E(1, m)$, $\mathcal{E}^* \subset X$, $\partial \mathcal{E}^*(2, m) \subset E_1$ *and* $p = (\partial E_1) \cap [a, b]$. *Then there is a continuous function* $f : (\mathcal{E}^* \setminus E_1) \to [p, b]$ *such that* f *is a retraction onto* $[p, b]$, $f[(\partial E_1) \cap E_2] = p$ *and for each* $x \in \mathcal{E}^* \setminus E_1, (x, f(x)) < \mathcal{U}$.

PROOF. This is actually Proposition 4 of [4] whose proof is valid in the case of AM-fans.

Since $\partial \mathcal{E}^*(2, m) \subset E_1$, $\mathcal{E}^* \setminus E_1$ is compact and for each $i \in \{2, ..., m-1\}$, ∂E_i is the union of two disjoint closed sets, $(\partial E_i) \cap E_{i-1}$ and $(\partial E_i) \cap E_{i+1}$. Since $[a, b]$ is straight in $\mathcal{E}$, for each $i \in \{1, ..., m-1\}$, $(\partial E_i) \cap E_{i+1} \cap [a, b]$ is a single point, r_i. Then $p = r_1$. Let $b = r_m$. Again, straightness guarantees that $p = r_1 < r_2 < ... < r_m = b$, where $<$ denotes the usual order from a to b on $[a, b]$. For each $i \in \{1, ..., m-2\}$, we define a function

$$g_i : ((\partial E_i) \cap E_{i+1}) \cup [r_i, r_{i+1}] \cup ((\partial E_{i+1}) \cap E_{i+2}) \to [r_i, r_{i+1}]$$

by

$$g_i(x) = \begin{cases} r_i & \text{if} \quad x \in (\partial E_i) \cap E_{i+1}, \\ x & \text{if} \quad x \in [r_i, r_{i+1}], \\ r_{i+1} & \text{if} \quad x \in (\partial E_{i+1}) \cap E_{i+2}. \end{cases}$$

Clearly, each g_i is a continuous retraction onto $[r_i, r_{i+1}]$. Since metric arcs are absolute retracts, for each i there is a continuous extension h_i of g_i, $h_i : Cl\ E_{i+1} \setminus E_i \to [r_i, r_{i+1}]$. The function $f = h_1 \cup h_2 \cup ... \cup h_{m-1}$ is a continuous retraction of $\mathcal{E}^* \setminus E_1$ onto $[p, b]$ such that, for each $x \in \mathcal{E}^* \setminus E_1, (x, f(x)) < \mathcal{U}$. □

The final step is the following theorem.

THEOREM 4.11. *Suppose* X *is an* AM*-fan and* $\mathcal{U}$ *is a cover of* X. *Then there is a finite fan* $Y \subset X$ *and a retraction* $r : X \to Y$ *such that if* $x \in X$, *then* $(x, f(x)) < \mathcal{U}$.

PROOF. Let t denote the top of X and let W denote the set of end points of X. Then $X = \cup\{[t, w] : w \in W\}$. For each $w \in W$, we apply Lemma 4.8 to obtain a chain $\mathcal{E}_w$ such that $[t, w]$ is straight in $\mathcal{E}_w$ and $\partial(\mathcal{E}_w \setminus E_{w1})^* \subset E_{w1}$.

There is a finite subset $W' \subset W$ such that $\{\mathcal{E}^*_w : w \in W'\}$ covers X. If $W' = \{w_1, ..., w_n\}$, let us relabel the corresponding chains $\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n$. For each $j \in \{1, 2, ..., n\}$ let $\mathcal{E}_j = \{E_{j1}, E_{j2}, ..., E_{jm_j}\} = E_j(1, m_j)$. As in Step III of the proof of Theorem 1 of [4] one can construct the new $\mathcal{U}$-chains $\mathcal{K}_1 = \{K_{11}, K_{12}, ..., K_{1p_1}\}$, $\mathcal{K}_2 = \{K_{21}, K_{22}, ..., K_{2p_2}\}, ...,$ $\mathcal{K}_n = \{K_{n1}, K_{n2}, ..., K_{np_n}\}$ such that:

(1) $\bigcup\{\mathcal{K}_j : j \in \{1, 2, ..., n\}\}$ covers X,
(2) For each $j \in \{1, 2, ..., n\}$ the arc $[t, w_j]$ is straight in $\mathcal{K}_j$,
(3) If $j \in \{1, 2, ..., n\}$, then $K_{j1} = K_{11}$, and
(4) If $j \neq i$, then $K^*_i(2, p_i) \cap K^*_j(2, p_j) = \emptyset$.

See Figure 3 in [4, p. 124]. We now construct a retraction r of X onto $Y = \cup\{[t, w_j] : j \in (1, ..., n)\}$. We will assume that each $\mathcal{K}_j$ has more that one link; if this is not true, the needed modifications in the definition of r are obvious.

For each $j \in \{1, 2, ..., n\}$, there exists a point $s_j \in [t, w_j]$ such that $(\partial K_{j1}) \cap [t, w_j] = (\partial K_{j1}) \cap Y \cap K_{j2} = \{s_j\}$. Since each $[t, w_j]$ is straight in the $\mathcal{U}$-chain $\mathcal{K}_j$, we apply Lemma 4.10 to obtain a retraction $f_j : (K^*_j \setminus K_{j1}) \to [s_j, w_j]$ such that $f_j[(\partial K_{j1}) \cap K_{j2}] = \{s_j\}$ and f_j moves each point less than $\mathcal{U}$. If $i \neq j$, then (domain f_i) $\cap$ (domain f_j) $\subset K^*_i(2, p_i) \cap K^*_j(2, p_j) = \emptyset$. Hence we may define $f = \cup\{f_i : i \in \{1, 2, ..., n\}\}$. Clearly, f is a retraction of $X \setminus K_{11}$ onto $\cup\{[s_i, w_i] : i \in \{1, 2, ..., n\}\}$ moving each point less than $\mathcal{U}$.

Now $Y \cap K_{11} = \cup\{[t, s_i) : i \in \{1, 2, ..., n\}\}$ and $(Cl K_{11}) \cap Y = \cup\{[t, s_i] : i \in \{1, 2, ..., n\}\}$ since each $[t, w_i]$ is straight in $\mathcal{K}_i$. We define $g : (\partial K_{11}) \cup ((Cl\ K_{11}) \cap Y) \to (Cl\ K_{11}) \cap Y$ by $g(x) = x$ if $x \in Y$ and $g(x) = s_i$ if $x \in (\partial K_{11}) \cap K_{i2}$. Since $(Cl\ K_{11}) \cap Y$ is a metric tree, it is an absolute retract. Hence g can be extended to a map $h : Cl\ K_{11} \to (Cl\ K_{11}) \cap Y$. Since $Cl\ K_{11}$ is contained in some member of $\mathcal{U}$, f moves each point less than $\mathcal{U}$. Finally, let $r = h \cup f$. Since f and h agree on the intersection of their domains, ∂K_{11}, r is well-defined and continuous. Obviously, r is a retraction of X onto Y. Since neither f nor h moves any point as much as $\mathcal{U}$, neither does r. ∎

Remark 4.12. Theorem 4.11 is a modification of Theorem 1 of [4]. The proof is valid for AM-fans. Let us observe that from the proof of this Theorem it follows that if t is the top of X, then $r(t) = t$.

In the case that X is a fan, we obtain [4, Theorem 2].

Theorem 4.13. *Each fan is an inverse limit of a sequence of finite fans.*

Given an open covering $\mathcal{U}$ of a compact space X, we say that a mapping $f : X \to Y$ is a *$\mathcal{U}$-mapping* provided there is an open covering $\mathcal{V}$ of Y such that $f^{-1}(\mathcal{V})$ refines $\mathcal{U}$, written as $f^{-1}(\mathcal{V}) \geq \mathcal{U}$.

Let $\mathcal{P}$ be a class of compact polyhedra. We say that a compact space X is *$\mathcal{P}$-like* provided for every open covering $\mathcal{U}$ of X there is a polyhedron $P \in \mathcal{P}$ and a $\mathcal{U}$-mapping $f : X \to P$ which is surjective.

Now let $\mathcal{F}$ be a class of finite metrizable fans. From Lemmas 4.5-4.10 and Theorems 4.11-4.13 it follows the following theorem.

THEOREM 4.14. *Every AM-fan is $\mathcal{F}$-like.*

In what follows we shall use the notion of approximate inverse systems in the sense of S. Mardešić [11]. Cov(X) is the set of all normal coverings of a topological space X. If $\mathcal{U}$, $\mathcal{V} \in$ Cov(X) and $\mathcal{V}$ refines $\mathcal{U}$, we write $\mathcal{V} < \mathcal{U}$.

An *approximate inverse system* is a collection $\mathbf{X} = \{X_a, p_{ab}, A\}$, where $(A, \leq)$ is a directed preordered set, $X_a, a \in A$, is a topological space and $p_{ab} : X_b \to X_a, a \leq b$, are mappings such that $p_{aa} = id$ and the following condition (A2) is satisfied:

(A2) For each $a \in A$ and each normal cover $\mathcal{U} \in \mathrm{Cov}(X_a)$ there is an index $b \geq a$ such that

$$(p_{ac}p_{cd}, p_{ad}) < \mathcal{U}, \text{ whenever } a \leq b \leq c \leq d.$$

An *approximate map* [13, Definition (1.9), p. 592] $\mathbf{p} = \{p_a : a \in A\} : X \to \mathbf{X}$ into an approximate system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is a collection of maps $p_a : X \to X_a, a \in A$, such that the following condition holds

(AS) For any $a \in A$ and any $\mathcal{U} \in \mathrm{Cov}(X_a)$ there is $b \geq a$ such that $(p_{ac}p_c, p_a) < \mathcal{U}$, for each $c \geq b$.

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system and let $\mathbf{p} =\{p_a : a \in A\} : X \to \mathbf{X}$ be an approximate map. We say that $\mathbf{p}$ is a *limit* of $\mathbf{X}$, written as $\lim \mathbf{X}$, provided it has the following universal property:

(UL) For any approximate map $\mathbf{q} =\{q_a : a \in A\} : Y \to \mathbf{X}$ of a space Y there exists a unique map $g : Y \to X$ such that $p_a g = q_a$.

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate system. A point $x = (x_a) \in \prod\{X_a : a \in A\}$ is called an *approximate thread* of $\mathbf{X}$ provided it satisfies the following condition:

(L) $(\forall a \in A)(\forall \mathcal{U} \in \mathrm{Cov}(X_a))(\exists b \geq a)(\forall c \geq b)\ \ p_{ac}(x_c) \in \mathrm{st}(x_a, \mathcal{U})$.

If X_a is a $\mathrm{T}_{3.5}$-space, then the sets $\mathrm{st}(x_a, \mathcal{U})$, $\mathcal{U} \in \mathrm{Cov}(X_a)$, form a basis of the topology at the point x_a. Therefore, for an approximate system of Tychonoff spaces condition **(L)** is equivalent to the following condition:

(L)* $(\forall a \in A) \lim\{p_{ac}(x_c) : c \geq a\} = x_a$.

The existence of the limit of any approximate system was proved in [13, (1.14) Theorem].

THEOREM 4.15. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system. Let $X \subseteq \prod\{X_a : a \in A\}$ be the set of all threads of $\mathbf{X}$ and let $p_a : X \to X_a$ be the restriction $p_a = \pi_a|X$ of the projection $\pi_a : \prod\{X_a : a \in A\} \to X_a, a \in A$. Then $\mathbf{p} = \{p_a : a \in A\} : X \to \mathbf{X}\}$ is a limit of $\mathbf{X}$.*

We call this limit the *canonical limit* of $\mathbf{X} = \{X_a, p_{ab}, A\}$. In the sequel limit means the canonical limit.

A preordered set $(A, \leq)$ is *cofinite* provided each $a \in A$ the set of all predecessors of a is a finite set.

We shall use the following theorem from [12, Theorem 3].

THEOREM 4.16. *Let $\mathcal{P}$ be a class of polyhedra with no isolated points. Let X be a compact Hausdorff space which is $\mathcal{P}$ - like. Then there exists an approximate inverse system of compact polyhedra $\mathbf{P} = \{P_a, \varepsilon_a, p_{ab}, A\}$ such that $P_a \in \mathcal{P}$, all the bonding mappings p_{ab} are surjective and the limit $\lim \mathbf{P}$ is homeomorphic to X. Moreover, A is cofinite and $\operatorname{card}(A) \leq w(X)$.*

THEOREM 4.17. *For every AM-fan X there exists an approximate inverse system $\mathbf{F} = \{F_a, \varepsilon_a, p_{ab}, B\}$ of finite metric fans such that $F_a \in \mathcal{F}$, all the bonding mappings p_{ab} are surjective and the limit $\lim \mathbf{P}$ is homeomorphic to X.*

PROOF. Theorem follows from Theorems 4.14 and 4.16. ☐

REMARK 4.18. Let us observe that from the proof of [12, Theorem 3], in particular, from the proof [12, Lemma 2] it follows that $p_{ab} : P_b \to P_a$ is a simplicial map such that $p_{ab}(r_b(t)) = r_a(t)$, where t is the top of the fan X and $r_a : X \to P_a$ is a retraction from Theorem 4.11.

Now we shall expand each non-metrizable AM-fan into usual inverse systems of metric fans.

THEOREM 4.19. *For every AM-fan X there exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric fans such that all the bonding mappings p_{ab} are surjective and the limit $\lim \mathbf{X}$ is homeomorphic to X.*

PROOF. By Theorem 4.17 there exists an approximate inverse system $\mathbf{F} = \{F_a, \varepsilon_a, q_{ab}, B\}$ of finite metric fans such that $F_a \in \mathcal{F}$, all the bonding mappings q_{ab} are surjective and the limit $\lim \mathbf{F}$ is homeomorphic to X. By forgetting the meshes ε_a [13, (1.7) Definition] and using Corollary 1 of [8] we obtain a usual σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$, where each X_a is the limit of an approximate inverse subsystem $\{F_\alpha, q_{\alpha\beta}, \Phi\}$, $\operatorname{card}(\Phi) = \aleph_0$, of the system $\mathbf{F}^* = \{F_a, q_{ab}, B\}$. Let us prove that every X_a is a metric fan. Firstly, each X_a is arcwise connected since there exists the projection $p_a : X \to X_a$ and X is arcwise connected. Now we shall prove that X is hereditarily unicoherent. From Lemma 3 of [8] it follows that we may assume that Φ is order isomorphic to the set of natural numbers $\mathbb{N}$. Then from Proposition 8 of [1] it follows that there exists an inverse sequence $\{F_n, q^*_{nm}, \mathbb{N}\}$ such that $\lim\{F_\alpha, q_{\alpha\beta}, \Phi\}$ is homeomorphic to $\lim\{F_n, q^*_{nm}, \mathbb{N}\}$. It is known that $\lim\{F_n, q^*_{nm}, \mathbb{N}\}$ is hereditarily unicoherent [16, Corollary 1, p. 228] since each F_n is hereditarily unicoherent. It remains to prove that $\lim\{F_n, q^*_{nm}, \mathbb{N}\}$ is a fan. For each $n \in \mathbb{N}$ let t_n be the top of F_n. From Remark 4.12 it follows that $t = (t_n)$ is a point of $\lim\{F_n, q^*_{nm}, \mathbb{N}\}$. It is clear that t is a ramification point of $\lim\{F_n, q^*_{nm}, \mathbb{N}\}$. Suppose that there exists a ramification point u of $\lim\{F_n, q^*_{nm}, \mathbb{N}\}$ such that $u \neq t$. Then there exists a triod T in $\lim\{F_n, q^*_{nm}, \mathbb{N}\}$ which contains u and $t \notin T$. There exists an $n \in \mathbb{N}$ such that $T_n = q^*_n(T)$ contains no $t_n = q^*_n(t)$. This means that T_n is an arc since F_n is a fan. Now, $\lim\{T_n, q^*_n|T_m, m > n\}$ is chainable. Hence, $\lim\{T_n, q^*_n|T_m, m > n\}$ is atriodic [17, Theorem 12.4]. This is impossible since $T = \lim\{T_n, q^*_n|T_m, m > n\}$. Thus, $\lim\{F_n, q^*_{nm}, \mathbb{N}\}$ contains only one ramification point t. Hence, $\lim\{F_n, q^*_{nm}, \mathbb{N}\}$ is a fan. □

Now we are ready to prove the main result of this paper.

THEOREM 4.20. *A generalized fan X admits a Whitney map for $C(X)$ if and only if it is metrizable.*

PROOF. If X is metrizable, then X admits a Whitney map for $C(X)$. Conversely, if X admits a Whitney map for $C(X)$, then, by Theorem 4.3 X is an AM-fan. From Theorem 4.19 it follows that there exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric fans such that all the bonding mappings p_{ab} are surjective and the limit $\lim \mathbf{X}$ is homeomorphic to X. Theorem 3.3 completes the proof. □

Let $\mathcal{AM}$ be a class of AM-arboroids. From Theorem 4.1 it follows that each arboroid is $\mathcal{AM}$-like. Using Theorem 4.14 we obtain the following result.

COROLLARY 4.21. *Each generalized fan is $\mathcal{F}$-like.*

By a similar method of proof as in the proof of Theorem 4.19 we obtain the following theorem.

THEOREM 4.22. *For every generalized fan X there exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric fans such that all the bonding mappings p_{ab} are surjective and the limit $\lim \mathbf{X}$ is homeomorphic to X.*

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