

ON A GENERALIZATION OF THE SINE FUNCTION

TOMASZ SZOSTOK
Silesian University, Poland

ABSTRACT. A function $s(x, y) = \inf_{\lambda \in \mathbb{R}} \frac{\|x + \lambda y\|}{\|x\|}$ strictly connected with Birkhoff-James orthogonality is considered. This function may be used in functional equations theory, to provide unconditional equations in place of orthogonal equations in the sense of Birkhoff-James. Moreover, we deal with another generalization of the sine function which, in particular leads to a characterization of inner product spaces.

1. INTRODUCTION

The main idea of the present paper comes from the theory of conditional functional equations. More precisely we talk about the so called orthogonal equations. Consider, for example, the well known Cauchy equation

$$(1.1) \quad f(x + y) = f(x) + f(y).$$

Any solution of this equation is called an additive function. Let us now consider the related conditional equation

$$(1.2) \quad x \perp y \Rightarrow f(x + y) = f(x) + f(y).$$

The condition $x \perp y$ may be understood in various ways. For instance, if we deal with functions defined on inner product spaces then we can use the orthogonality defined by an inner product. In an abstract normed linear space the orthogonality has to be defined in another way. The most widely used kinds of orthogonalities are:

James orthogonality

$$x \perp_J y \Leftrightarrow \|x + y\| = \|x - y\|;$$

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Birkhoff-James orthogonality

$$x \perp_{BJ} y \Leftrightarrow (\|x + \lambda y\| \geq \|x\| \quad \text{for all } \lambda \in \mathbb{R});$$

and Pythagorean orthogonality

$$x \perp_P y \Leftrightarrow \|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

Clearly, every solution of equation (1.1) is a solution of (1.2). The converse implication is false, in general. For example if X is an inner product space and $f : X \rightarrow \mathbb{R}$ is defined by the formula $f(x) := \|x\|^2$, then f is a solution of (1.2) that is not additive. Have a look at the equation

$$(1.3) \quad f(x + y) = g\left(\frac{\|x - y\|}{\|x + y\|}\right) [f(x) + f(y)].$$

It is an unconditional equation and it may be assumed for almost all values of x and y . Every additive function f is a solution of this equation (with function $g = 1$) and $f(x) = \|x\|^2$ is also a solution of this equation (with $g(a) = \frac{2}{1+a^2}$). That means that equation (1.3) preserves the most important solutions of (1.2). Further taking here $x, y \neq 0, x \perp_J y$ we obtain

$$x, y \neq 0, x \perp_J y \Rightarrow f(x + y) = g(1)[f(x) + f(y)],$$

which means that we get a modified version of James orthogonal additivity as a special case of equation (1.3). Consequently, this equation can be easily solved with help of results concerning orthogonally additive functions (see [2]). In the case of normed spaces which are not inner product spaces Gy. Szabó papers devoted to James orthogonal additivity [3] and [4] are useful. Such problems were considered in papers paper [5] and [6]. A natural question arises whether a similar procedure may be applied with respect to another orthogonalities. In order to deal with the Birkhoff-James orthogonality we may consider a function similar to the quotient $\frac{\|x-y\|}{\|x+y\|}$ connected with the James orthogonality, namely

$$s(x, y) := \inf_{\lambda \in \mathbb{R}} \frac{\|x + \lambda y\|}{\|x\|}.$$

Then the following equation is to be considered

$$f(x + y) = g(s(x, y))[f(x) + f(y)].$$

However, our present goal is only to examine some properties of the function s just introduced.

2. RESULTS

We begin with the following

DEFINITION 2.1. Let $(X, \|\cdot\|)$ be a real normed linear space. Define a function $s : X \times X \rightarrow \mathbb{R}$ by the following formula:

$$s(x, y) := \begin{cases} \inf_{\lambda \in \mathbb{R}} \frac{\|x + \lambda y\|}{\|x\|}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Note that, actually, the infimum in definition of s is achieved and consequently may be replaced by the minimum. The following properties of function s are obvious.

PROPOSITION 2.2. Let $(X, \|\cdot\|)$ be a real normed linear space and let α, β be some nonzero real numbers. Then $s(\alpha x, \beta y) = s(x, y)$ for all $x, y \in X$.

Further function s takes all its values in the interval $[0, 1]$, and $s(x, y) = 0$ if and only if x, y are linearly dependent.

Function s is strictly connected with the Birkhoff-James orthogonality. Namely we have the following simple

REMARK 2.3. Let x, y be given elements of a normed linear space $X(\|\cdot\|)$. Then

$$x \perp_{BJ} y \Leftrightarrow s(x, y) = 1.$$

REMARK 2.4. Let $(X, (\cdot|\cdot))$ be an inner product space. Then

$$(2.1) \quad s(x, y) = \sqrt{1 - \frac{(x|y)^2}{\|x\|^2 \|y\|^2}} \quad \text{for all } x, y \in X \setminus \{0\};$$

in particular $s(x, y) = s(y, x)$. On the other hand if $(X, \|\cdot\|)$ is a normed space of dimension at least 3 and we have $s(x, y) = s(y, x)$ for all $x, y \in X$, then X has to be an inner product space.

PROOF. Fix $x, y \in X$ $x, y \neq 0$ and put $a := \|x\|^2, b := \|y\|^2, c := (x|y)$. We shall determine the real number λ_0 such that for this number the expression $h(\lambda) := \frac{\|x + \lambda y\|}{\|x\|}$ is minimal. Since h is nonnegative it suffices to consider its square:

$$\frac{\|x + \lambda y\|^2}{\|x\|^2} = \frac{a + 2\lambda c + \lambda^2 b}{a}.$$

Obviously, the latter expression (and hence also h) is minimal for $\lambda_0 = -\frac{c}{b}$. Using this value of λ we get

$$\begin{aligned} s(x, y) &= \sqrt{\frac{a + 2\lambda_0 c + \lambda_0^2 b}{a}} = \sqrt{\frac{a - 2\frac{c}{b}c + \frac{c^2}{b^2}b}{a}} \\ &= \sqrt{1 - \frac{c^2}{b}} = \sqrt{1 - \frac{c^2}{ab}} = \sqrt{1 - \frac{(x|y)^2}{\|x\|^2 \|y\|^2}}. \end{aligned}$$

Now, assume that X is a normed space of dimension at least 3, such that we have $s(x, y) = s(y, x)$ for all $x, y \in X$. From Remark 2.3 we infer that

$x \perp_{BJ} y \Leftrightarrow y \perp_{BJ} x$. The symmetry of the Birkhoff-James orthogonality in a space with dimension greater than 2 forces this space to be an inner product space (see [1]). \square

What about continuity of function s ? If we interpret s as the absolute value of the sinus of the angle between the vectors x and y , then we cannot expect the continuity of s at a point (x, y) such that one of the vectors x, y is equal to zero. However, except for these points, the function s is continuous.

THEOREM 2.5. *Let $(X, \|\cdot\|)$ be a real normed linear space of dimension at least 2. Then the function*

$$s_{|(X \setminus \{0\}) \times (X \setminus \{0\})}$$

is continuous.

PROOF. Since $s(X \times X) \subset [0, 1]$, we infer that

$$\limsup_{n \rightarrow \infty} s(x_n, y_n)$$

is finite. Now, take $(x, y) \in X \times X$, $x, y \neq 0$ and assume that there exist sequences $(x_n, y_n) \rightarrow (x, y) \leftarrow (x_n^1, y_n^1)$ such that

$$s(x_n, y_n) \rightarrow \alpha \quad \text{and} \quad s(x_n^1, y_n^1) \rightarrow \alpha_1$$

for some real numbers $\alpha \neq \alpha_1$. Assume also that $x_n, y_n, x_n^1, y_n^1 \neq 0$. We are going to show that this assumption leads to a contradiction. Assume that $\alpha > \alpha_1$ and denote $\varepsilon := \alpha - \alpha_1$. Since $x \neq 0$, we can replace (x, y) by $\left(\frac{x}{\|x\|}, y\right)$ and do the same with the coordinates of (x_n, y_n) and (x_n^1, y_n^1) . The resulting points will still have all the properties assumed. Consequently there is no loss of generality in assuming that $\|x\| = \|x_n\| = \|x_n^1\| = 1$ for all $n \in \mathbb{N}$ whence

$$\min_{\lambda} \|x_n + \lambda y_n\| \rightarrow \alpha \quad \text{and} \quad \min_{\lambda} \|x_n^1 + \lambda y_n^1\| \rightarrow \alpha_1.$$

Thus

$$\|x_n + \lambda_n y_n\| \rightarrow \alpha \quad \text{and} \quad \|x_n^1 + \lambda_n^1 y_n^1\| \rightarrow \alpha_1$$

where λ_n and λ_n^1 are such real numbers that the considered expressions are minimal; moreover,

$$(2.2) \quad \|x_n + \lambda_n y_n\| \leq \|x_n + \lambda y_n\| \quad \text{for all } n \in \mathbb{N} \text{ and all } \lambda \in \mathbb{R}.$$

We have also $\|x_n^1 + \lambda_n^1 y_n^1\| = s(x_n^1, y_n^1) \leq 1, n \in \mathbb{N}$. On the other hand

$$\|x_n^1 + \lambda_n^1 y_n^1\| \geq |\lambda_n^1| \|y_n^1\| - \|x_n^1\|, \quad n \in \mathbb{N}.$$

These conditions together with $y_n^1 \rightarrow y \neq 0$ mean that the sequence (λ_n^1) is bounded. Consequently, without loss of generality, we may assume that there is a real number λ_1 such that $\lambda_n^1 \rightarrow \lambda_1$.

Note that

$$| \|x_n + \lambda_1 y_n\| - \|x_n^1 + \lambda_1 y_n^1\| | < \frac{\varepsilon}{9},$$

$$| \|x_n^1 + \lambda_1 y_n^1\| - \|x_n^1 + \lambda_n^1 y_n^1\| | < \frac{\varepsilon}{9}$$

as well as

$$| \alpha_1 - \|x_n^1 + \lambda_n^1 y_n^1\| | < \frac{\varepsilon}{9}$$

for almost all $n \in \mathbb{N}$ and, therefore,

$$(2.3) \quad | \alpha_1 - \|x_n + \lambda_1 y_n\| | < \frac{\varepsilon}{9} + \frac{\varepsilon}{9} + \frac{\varepsilon}{9} = \frac{\varepsilon}{3}.$$

for almost all $n \in \mathbb{N}$ say $n \geq n_0$. On the other hand

$$(2.4) \quad \|x_n + \lambda_n y_n\| > \alpha - \frac{\varepsilon}{3},$$

say for $n \geq n_1$. Finally, using (2.3), and (2.4), we conclude that for all $n \geq \max(n_0, n_1)$ one has

$$\|x_n + \lambda_n y_n\| > \alpha - \frac{\varepsilon}{3} > \alpha_1 + \frac{\varepsilon}{3} > \|x_n + \lambda_1 y_n\|,$$

which contradicts (2.2). This contradiction shows that for every pair $(x, y) \in X \times X$, $x, y \neq 0$ the function s has a limit at this point. To finish the proof it is enough to observe that we can take (x_n, y_n) equal to (x, y) in the first part of the proof. \square

As we already mentioned the function s can be viewed as the absolute value of a sinus of the angle between the vectors x and y . Several further facts about function s can also be observed. Especially, in inner product spaces this function has many interesting properties. For example, our next remark states that in an inner product space in every isosceles triangle the absolute values of the sinus of the angle between the median and the both equal sides are the same.

REMARK 2.6. If $(X, (\cdot|\cdot))$ is a real inner product space, then for all $x, y \in X$, with $\|x\| = \|y\|$ one has

$$(2.5) \quad s(x, x + y) = s(y, x + y).$$

With the aid of function s we are also able to define a real function which stands for an analogue of sinus in a given normed space. To do this we need earlier the following

REMARK 2.7. Let $(X, \|\cdot\|)$ be a normed linear space and let $x, y \in X$. If a, b, c, d are such real numbers that $\angle((1, 0), (a, b)) = \angle((1, 0), (c, d))$, then $s(x, ax + by) = s(x, cx + dy)$.

This is a direct consequence of Proposition 2.2. Now we are able to formulate the following definition.

DEFINITION 2.8. Let $(X, \|\cdot\|)$ be a real normed linear space of dimension at least 2. Take $x, y \in X$, with $\|x\| = \|y\| = 1$ and such that $x \perp_{BJ} y$. Define a function $\phi_{x,y} : \mathbb{R} \rightarrow \mathbb{R}$ by the following formula

$$(2.6) \quad \phi_{x,y}(t) := \operatorname{sgn} b \cdot s(x, ax + by), \quad t \in \mathbb{R},$$

where $a, b \in \mathbb{R}$ are such that the angle between the vector (a, b) and the vector $(1, 0)$ is equal to t .

REMARK 2.9. If $(X, (\cdot|\cdot))$ is a real inner product space and $t \in (0, \frac{\pi}{2})$, then for all $x, y \in X$, with $x \perp y$, $\|x\| = \|y\| = 1$ we have

$$(2.7) \quad \phi_{x,y}\left(\frac{\pi}{2} + t\right) = \phi_{x,y}\left(\frac{\pi}{2} - t\right).$$

PROOF. Fix $x, y \in X$ satisfying the assumptions of the theorem and put $z := ax + by$ where a, b are any real numbers satisfying the equality $\frac{b}{a} = \tan\left(\frac{\pi}{2} + t\right)$. Then

$$\begin{aligned} \phi_{x,y}\left(\frac{\pi}{2} + t\right) &= s(x, z) = s(z, x) = s(-z, x) \\ &= s(-ax - by, x) = s(-ax - by, -2ax). \end{aligned}$$

Now, using Remark 2.6, we get

$$s(-ax - by, -2ax) = s(-ax + by, -2ax)$$

and further

$$s(-ax + by, -2ax) = s(-ax + by, x) = s(x, -ax + by) = \phi_{x,y}\left(\frac{\pi}{2} - t\right).$$

Thus the remark has been proved. \square

EXAMPLE 2.10. If $(X, (\cdot|\cdot))$ is a real inner product space, then for all unit vectors $x, y \in X$ such that $x \perp y$ and for all $t \in \mathbb{R}$ we have $\phi_{x,y}(t) = \operatorname{sint}$.

PROOF. Take $x \in X$, $\|x\| = 1$ and $y \in X$ such that $y \perp x$, $\|y\| = 1$. Then for every z of the form $z = ax + by$ we have $\|z\| = \sqrt{a^2 + b^2}$. Take a $t \in (0, \frac{\pi}{2})$ and write

$$\begin{aligned} \phi_{x,y}(t) &= s(x, x + \tan t y) = \min_{\lambda \in \mathbb{R}} \frac{\|x + \lambda(x + \tan t y)\|}{\|x\|} \\ &= \sqrt{\min_{\lambda \in \mathbb{R}} ((1 + \lambda)^2 + \lambda^2 \tan^2 t)}. \end{aligned}$$

Obviously the above formula as a function of λ attains its minimum at $\lambda = -\cos^2 t$ whence

$$\phi_{x,y}(t) = \sqrt{(1 - \cos^2 t)^2 + \cos^4 t \frac{\sin^2 t}{\cos^2 t}} = \operatorname{sint}.$$

If now $t \in (\frac{\pi}{2}, \pi)$ then the desired equality results from Remark 2.9. For $t \in (\pi, 2\pi)$ we also have $\phi_{x,y}(t) = \operatorname{sint}$ because $s(x_1, -x_2) = s(x_1, x_2)$. For

bigger values of t we only need to note that $\phi_{x,y}$ is periodic with period equal to 2π . For $t = \pi$ we have $\phi_{x,y}(t) = 0$, if $t = \frac{\pi}{2}$ then we have $\phi_{x,y}(t) = 1$. \square

It is easy to check that in normed spaces which are not inner product spaces functions $\phi_{x,y}$ fail to be the sine function, in general. Have a look at the following example.

EXAMPLE 2.11. Consider space \mathbb{R}^2 with the norm $\|(a, b)\| = \max\{|a|, |b|\}$. Then

$$\phi_{(1,0),(0,1)}(t) = \begin{cases} \frac{\tan t}{1+\tan t}, & t \in [0, \frac{\pi}{2}), \\ 1, & t = \frac{\pi}{2}. \end{cases}$$

Natural is the question whether the converse of Example 2.10 holds true. The answer is positive and we shall give it later on in this paper. Namely, among others it will be proved that if in a 2-dimensional space one can find vectors $x, y, \|x\| = \|y\| = 1, x \perp_{BJ} y$ such that $\phi_{x,y} = \sin$, then this space must be an inner product one. However we have to start with the following lemma.

LEMMA 2.12. *Let $f, g : (a, b) \rightarrow \mathbb{R}$ with $-\infty < a < b < \infty$ be given concave functions. If there exists a point $x_0 \in (a, b)$ such that $g(x_0) > f(x_0)$, then there exists a straight line $l := \{(x, cx + d) : x \in \mathbb{R}\}$ which is not horizontal, has a common point with the graph of g and satisfies the following condition: $cx + d > f(x) + \varepsilon$ for all $x \in (a, b)$ and some $\varepsilon > 0$.*

PROOF. Put $\varepsilon := \frac{g(x_0) - f(x_0)}{2}$ and let us distinguish two cases:

1^o either $f'_+ \neq 0$ or $f'_- \neq 0$.

2^o $f'(x_0) = 0$.

If 1^o occurs, then there exists a nontrivial straight line $\tilde{l} = \{(x, \alpha x + \tilde{\beta}) : x \in \mathbb{R}\}$ with $\alpha \neq 0$ that supports the graph of f at the point $(x_0, f(x_0))$ and it suffices to take $l := \{(x, \alpha x + \beta) : x \in \mathbb{R}\}$ where $\beta := \tilde{\beta} + 2\varepsilon$.

In the case 2^o we have $f'(x_0) = 0$. Then function f has a local maximum at x_0 , further from the concavity of this function we infer that

$$f(x) \leq f(x_0) < g(x_0)$$

for all $x \in (a, b)$. Define $c := \frac{\varepsilon}{x_0 - a}$ and $d := g(x_0) - \frac{\varepsilon x_0}{x_0 - a}$. Then the function $h(x) = cx + d$ is increasing and

$$h(a) = ca + d = g(x_0) - \varepsilon > \max_{x \in (a, b)} f(x) + \varepsilon,$$

which means that the line defined by the formula $y = cx + d$ satisfies the latter of the conditions desired. To finish the proof it is enough to note that

$$cx_0 + d = g(x_0)$$

i.e. this line has a common point with the graph of g . \square

LEMMA 2.13. *Let $(X, \|\cdot\|)$ be a normed space. If $x, y \in X$ are unit vectors, then the set*

$$S := \{(a, b) \in \mathbb{R}^2 : \|ax + by\| = 1, |a| < 1, b > 0\}$$

yields a graph of some concave function $f : (-1, 1) \rightarrow (0, \infty)$.

PROOF. For the sake of brevity we shall be writing $\|(a, b)\|$ instead of $\|ax + by\|$. Note that for every $t \in (-1, 1)$ there exists $f(t) > 0$ such that the point $(t, f(t))$ is an element of S . Indeed, $\|(t, 0)\| = |t| < 1$ and, on the other hand, for every $t \in (-1, 1)$ one has

$$\lim_{s \rightarrow \infty} \|(t, s)\| = \infty.$$

Now we are going to show that for every $t \in (-1, 1)$ there exists exactly one corresponding point $f(t)$. Suppose that

$$(2.8) \quad \|(t, s_1)\| = \|(t, s_2)\| = 1$$

for some real numbers s_1, s_2 and some $t \in (-1, 1)$. Assume that $s_1 \geq s_2$, then $(t, s_2) = \lambda(t, s_1) + (1 - \lambda)(t, 0)$ and $s_2 = \lambda s_1$ for some $\lambda \in [0, 1]$. We shall show that $\lambda = 1$. Indeed, consider the case of $t > 0$ and define

$$u := \alpha(t, s_1) + (1 - \alpha)(1, 0)$$

for $\alpha = \frac{\lambda}{t + \lambda - \lambda t}$. Directly from the definition of u we infer that $\|u\| \leq 1$. Put $\beta := \frac{1}{t + \lambda - \lambda t}$, then $u = \beta(t, s_2)$ whence $1 \geq \|u\| = \beta \geq 1$, i.e. $\beta = 1$, which implies that $\lambda = 1$, and consequently, $s_1 = s_2$. If we had $t < 0$, it would be sufficient to repeat the above calculations taking

$$u := \alpha(t, s_1) + (1 - \alpha)(-1, 0)$$

with $\alpha = \frac{\lambda}{\lambda t + \lambda - t}$ and $\beta := \frac{1}{\lambda t + \lambda - t}$. For $t = 0$ the equality results directly from (2.8).

Till now we have shown that there exists a function f such that the set S is a graph of this function. To finish the proof we have to show that this function is concave. To this end take $t_1, t_2 \in (-1, 1)$ and $\alpha \in (0, 1)$. Then

$$(2.9) \quad \|\alpha(t_1, f(t_1)) + (1 - \alpha)(t_2, f(t_2))\| \leq 1,$$

on the other hand from the definition of f we have

$$(2.10) \quad \|(\alpha t_1 + (1 - \alpha)t_2, f(\alpha t_1 + (1 - \alpha)t_2))\| = 1.$$

Further, equation (2.9) implies the existence of exactly one $s \geq \alpha f(t_1) + (1 - \alpha)f(t_2)$ such that $\|\alpha t_1 + (1 - \alpha)t_2, s\| = 1$. Therefore, from equation (2.10) we infer that $s = f(\alpha t_1 + (1 - \alpha)t_2)$, which shows the concavity of f . \square

REMARK 2.14. Let $(X, \|\cdot\|)$ be a real normed linear space. Take $x, y \in X$ such that, $\|x\| = \|y\| = 1$ and $x \perp_{BJ} y$. If $a, b \in \mathbb{R}$ are such that $\|ax + by\| = \alpha$, then $|a| \leq \alpha$.

REMARK 2.15. Let $(X, \|\cdot\|)$ be a real normed linear space. Take $x, y \in X$ such that, $\|x\| = \|y\| = 1$ and $x \perp_{BJ} y$. Let $f : (-1, 1) \rightarrow \mathbb{R}$ be such a function that

$$\{(a, b) \in \mathbb{R}^2 : \|ax + by\| = 1, |a| < 1, b > 0\} = \text{Gr } f.$$

Let further α be a given positive number. Define the function $\psi_\alpha : (1 - \alpha, 1 + \alpha) \rightarrow \mathbb{R}$ by the formula $\psi_\alpha(a) := \alpha f\left(\frac{1}{\alpha}(a - 1)\right)$. Then the set

$$S_\alpha := \{(a, b) \in \mathbb{R}^2 : \|x - (ax + by)\| = \alpha, |a - 1| < \alpha, b > 0\}.$$

coincides with the graph of ψ_α . Moreover the (finite) limits of ψ_α at the endpoints of the domain exist and the following equalities hold:

$$\begin{aligned} \lim_{a \rightarrow 1 - \alpha} \psi_\alpha(a) &= \sup\{b : \|(1 - \alpha, b)\| = \alpha\}, \\ \lim_{a \rightarrow 1 + \alpha} \psi_\alpha(a) &= \sup\{b : \|(1 + \alpha, b)\| = \alpha\}. \end{aligned}$$

PROOF. We shall present only a sketch of the proof. One has to show that for every $(a, b) \in S_\alpha$ we get $b = \psi_\alpha(a)$ (the converse implication can be proved similarly). From the definition of S_α we have

$$\|(ax + by) - x\| = \alpha,$$

whence

$$\left\| \frac{1}{\alpha}((a - 1)x + by) \right\| = 1.$$

We have also $|a - 1| < \alpha$ i.e. $\left|\frac{1}{\alpha}(a - 1)\right| < 1$ whence, $\frac{1}{\alpha}b = f\left(\frac{1}{\alpha}(a - 1)\right)$ and, consequently, $b = \psi_\alpha(a)$.

The second statement results from the following facts: every concave function defined on an interval is monotone in some neighbourhoods of the ends of this interval, and the 2-dimensional unit sphere is a compact set. \square

LEMMA 2.16. Let $(X, \|\cdot\|)$ be a real normed linear space. Take $x, y \in X$, such that $\|x\| = \|y\| = 1$ and $x \perp_{BJ} y$. Let $f_0 : (-1, 1) \rightarrow \mathbb{R}$ such a function that

$$\text{Gr } f_0 = \{(a, b) \in \mathbb{R}^2 : \|ax + by\| = 1, |a| < 1, b > 0\}.$$

Take also $t \in (0, \frac{\pi}{2})$ and put $l := \{(a, (\tan t)a) : a \in \mathbb{R}\}$. Extend f_0 to a continuous function, $f : [-1, 1] \rightarrow \mathbb{R}$. Let further ψ_α be defined in the same way as in Remark 2.15. Then $\phi_{x,y}(t) = \alpha_0$ if and only if

$$l \cap \text{Gr } \psi_{\alpha_0} \neq \emptyset \text{ and } l \cap \text{Gr } \psi_\alpha = \emptyset \text{ for every } \alpha \in (0, \alpha_0)$$

(in the case of $t \in (\frac{\pi}{2}, \pi)$ we would have to define and use an analogue of function ψ_α with $(a - 1)$ replaced by $(a + 1)$).

PROOF. If $\phi_{x,y}(t) = \alpha_0$, then

$$(2.11) \quad \min_{\lambda \in \mathbb{R}} \|x + \lambda(x + \tan t y)\| = \alpha_0.$$

Now we can find a $\lambda_0 \in \mathbb{R}$ such that the above minimum is achieved for λ equal to $-\lambda_0$. Then we have $\|x - \lambda_0(x + \tan t y)\| = \alpha_0$. Since $\alpha_0 \leq 1$, we have $\|(1 - \lambda_0)x - \lambda_0 \tan t y\| \leq 1$. From Remark 2.14 we infer that $\lambda_0 \geq 0$ and, moreover, $\lambda_0(x + \tan t y) \in \partial K(x, \alpha_0)$, where $\partial K(x, \alpha_0)$ stands for a sphere centered at x and having radius equal to α_0 . Let us write these facts in the following way

$$\lambda_0(x + \tan t y) \in \partial K^+(x, \alpha_0) := \{ax + by : \|x - (ax + by)\| = \alpha_0, b \geq 0\}.$$

On the other hand equation (2.11) implies that

$$(2.12) \quad \lambda(x + \tan t y) \notin \text{Int } K(x, \alpha_0).$$

for all $\lambda \in \mathbb{R}$. Have a closer look at the set $\partial K^+((1, 0), \alpha_0)$ (the point $((1, 0)$ is here being identified with the vector x). From Remark 2.14 it follows that this set can be written in the following way.

$$\partial K^+((1, 0), \alpha_0) = \{(a, b) \in \mathbb{R}^2 : \|(1, 0) - (a, b)\| = \alpha_0, b \geq 0\} = K_1 \cup K_2$$

where

$$K_1 := \{(a, b) \in \mathbb{R}^2 : \|(1, 0) - (a, b)\| = \alpha_0, a \in (1 - \alpha, 1 + \alpha), b \geq 0\}$$

and

$$K_2 := \{(a, b) \in \mathbb{R}^2 : \|(1, 0) - (a, b)\| = \alpha_0, a \in \{1 - \alpha, 1 + \alpha\}, b \geq 0\}.$$

Let us consider the case when $(\lambda_0, \lambda_0 \tan t) \in K_1$. We have then

$$K_1 \subset \text{Gr} \psi_{\alpha_0}$$

whence $l \cap \text{Gr} \psi_{\alpha_0} \neq \emptyset$, which means that the first assertion is true. Moreover, from condition (2.12) we infer that l not only has a common point with the graph of considered function but also

$$\|(1, 0) - (\lambda, \lambda \tan t)\| \geq \alpha_0$$

for all $\lambda \in \mathbb{R}$, which means that

$$l \cap \text{Gr} \psi_{\alpha} = \emptyset \quad \text{for every } \alpha \in (0, \alpha_0).$$

Let us now consider the case of $l \cap K_1 = \emptyset$. Then $(\lambda_0, \lambda_0 \tan t) \in K_2$, and we have $\lambda_0 \in \{1 - \alpha, 1 + \alpha\}$ getting

$$\lambda_0 \tan t \in \{\sup\{a : \|(1 - \alpha, a)\| = \alpha\}, \sup\{a : \|(1 + \alpha, a)\| = \alpha\}\}.$$

The first one of the above two statements is obvious. Suppose that the latter does not hold. In such a case we obtain a contradiction with condition (2.12). Indeed, the line l contains a point (a_1, b_1) such that $a_1 \in (1 - \alpha, 1 + \alpha)$ and $b_1 < f(a_1)$. It is a consequence of the continuity of the function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $\text{Gr} g = l$ and of Remark 2.15. We have $\|(a_1, b_1)\| < 1$, which means that the point (a_1, b_1) is an element of $\text{Int} K((1, 0), \alpha_0)$.

The converse implication may be proved similarly. \square

Now we are able to prove the main result of the present paper. Namely, our next theorem states that if in two normed spaces the generalizations of sinus are the same, then these spaces are essentially the same.

THEOREM 2.17. *Let $(X_1, \|\cdot\|_1), (X_2, \|\cdot\|_2)$ be two normed real linear spaces. Further, let $x_1, y_1 \in X_1$ and $x_2, y_2 \in X_2$ be such that $\|x_1\|_1 = \|y_1\|_1 = \|x_2\|_2 = \|y_2\|_2 = 1$ and $x_1 \perp_{BJ} y_1$ $x_2 \perp_{BJ} y_2$. Then the equality*

$$\phi_{x_1, y_1}^1 = \phi_{x_2, y_2}^2$$

implies that

$$\|ax_1 + by_1\|_1 = \|ax_2 + by_2\|_2$$

for all real numbers a, b .

PROOF. Take unit vectors $x_1, x_2, y_1, y_2 \in X$ which are respectively Birkhoff-James orthogonal. Denote, for brevity, $\|ax_1 + by_1\|_1$ by $\|(a, b)\|_1$ and $\|ax_2 + by_2\|_2$ by $\|(a, b)\|_2$. Suppose that there exist $a_0, b_0 \in \mathbb{R}$ such that

$$(2.13) \quad \|(a_0, b_0)\|_1 \neq \|(a_0, b_0)\|_2.$$

We are going to show that there exists a $\theta \in [0, 2\pi]$, such that

$$\phi_{x_1, y_1}^1(\theta) \neq \phi_{x_2, y_2}^2(\theta).$$

Note that there is no loss of generality in assuming that $b_0 > 0$; in fact, if $\|(a_0, b_0)\|_1 \neq \|(a_0, b_0)\|_2$, then also $\|(-a_0, -b_0)\|_1 \neq \|(-a_0, -b_0)\|_2$. Define the sets

$$S_1 := \{(a, b) : b > 0, |a| < 1, \|(a, b)\|_1 = 1\}$$

and

$$S_2 := \{(a, b) : b > 0, |a| < 1, \|(a, b)\|_2 = 1\}$$

From Lemma 2.13 we know that these sets yield graphs of some concave functions, $f_0, g_0 : (-1, 1) \rightarrow \mathbb{R}$, respectively. The first step of the proof is to show that condition (2.13) implies that $S_1 \neq S_2$ and, in consequence, there exists a $t \in (-1, 1)$ such that $f_0(t) \neq g_0(t)$. To this end let us note that

$$(2.14) \quad 1 = \left\| \left(\frac{a_0}{\|(a_0, b_0)\|_1}, \frac{b_0}{\|(a_0, b_0)\|_1} \right) \right\|_1 \neq \left\| \left(\frac{a_0}{\|(a_0, b_0)\|_1}, \frac{b_0}{\|(a_0, b_0)\|_1} \right) \right\|_2.$$

If now $\left| \frac{a_0}{\|(a_0, b_0)\|_1} \right| < 1$, then obviously $S_1 \neq S_2$. Note that the case $\left| \frac{a_0}{\|(a_0, b_0)\|_1} \right| > 1$ is impossible since, otherwise, using Remark 2.14, we would obtain

$$\left\| \left(\frac{a_0}{\|(a_0, b_0)\|_1}, \frac{b_0}{\|(a_0, b_0)\|_1} \right) \right\|_1 > 1$$

which contradicts condition (2.14). Assume that $\left| \frac{a_0}{\|(a_0, b_0)\|_1} \right| = 1$. In this case from (2.14), using the Birkhoff-James orthogonality of x_2 and y_2 , we obtain

$$\alpha := \left\| \left(\frac{a_0}{\|(a_0, b_0)\|_1}, \frac{b_0}{\|(a_0, b_0)\|_1} \right) \right\|_2 > 1.$$

Consider the vector

$$z := \frac{1}{\alpha} \left(\frac{a_0}{\|(a_0, b_0)\|_1}, \frac{b_0}{\|(a_0, b_0)\|_1} \right) = (z_1, z_2).$$

Thus we have found $z_1, z_2 \in \mathbb{R}$ such that $|z_1| < 1, \|z\|_2 = 1$ and $z_2 > 0$. That means $z \in S_2$ or, equivalently, $g_0(z_1) = z_2$. On the other hand $\|z\|_1 < 1$ i.e. $z \notin S_1$ and, finally, $f_0(z_1) \neq z_2$ thus $f_0(z_1) \neq g_0(z_1)$ for some $z_1 < 1$. Let f and g stand for the continuous extensions of f_0 and g_0 , respectively, onto the closed interval $[-1, 1]$. From Lemma 2.12 we get the existence of a line $b = ca + d_0$, $c \neq 0$ such that this line has a common point with the graph of f and lies above the graph of g . We can assume that this function is a supporting line for f (it is enough to replace the original number d_0 by another one). Denote the abscissa of the contact point by t_0 . Then

$$f(t_0) = ct_0 + d_0 \quad \text{and} \quad ct + d_0 > g(t)$$

for all $t \in [-1, 1]$. Now take a d_1 such that the line $b = ca + d_1$ is a supporting line for g at a point t_1 ; note that $d_1 < d_0$.

Assume that $c > 0$; then we have $d_0 > c$. Indeed, $g(-1) \geq 0$ and

$$d_1 + ct \geq g(t)$$

for all $t \in [a, b]$. In particular $d_0 - c > d_1 - c \geq 0$. In the case where $c < 0$ one can similarly show that $d_0 > -c$.

Take an $\alpha \in (0, 1]$ and consider the function $\psi_\alpha : (1 - \alpha, 1 + \alpha) \rightarrow \mathbb{R}$ defined by the formula

$$\psi_\alpha(a) := \alpha f \left(\frac{1}{\alpha}(a - 1) \right)$$

(if $c < 0$ then we have to consider the similar function with $(a - 1)$ replaced by $(a + 1)$ and the remaining part of the proof is unchanged). We have

$$\psi_\alpha(1 + \alpha t_0) = \alpha f(t_0).$$

Let us also note that the function ψ_α has a supporting line at the point $1 + \alpha t_0$ with the slope equal to c . If ψ_α were differentiable at this point we would have

$$\psi'_\alpha(a) = \left[\alpha f \left(\frac{1}{\alpha}(a - 1) \right) \right]' = \alpha \frac{1}{\alpha} f' \left(\frac{1}{\alpha}(a - 1) \right) = f' \left(\frac{1}{\alpha}(a - 1) \right),$$

whence

$$\psi'_\alpha(1 + \alpha t_0) = f' \left(\frac{1}{\alpha}(1 + \alpha t_0 - 1) \right) = f' \left(\frac{1}{\alpha} \alpha t_0 \right) = f'(t_0).$$

If ψ_α were not differentiable at $1 + \alpha t_0$, then the similar evaluations could be made for the one-sided derivatives and the line with the slope equal to c would be a supporting line for function ψ_α if some line with the same slope would be a supporting line for the function f at the point t_0 .

Summarizing, we have two lines $l := \{(a, ca + d_0) : a \in \mathbb{R}\}$ and $l_1 := \{(a, ca + d_1) : a \in \mathbb{R}\}$, $d_0 \neq d_1$ which are supporting lines for f and g , respectively, with the corresponding points of contact: $(t_0, f(t_0))$ and $(t_1, g(t_1))$ for some $t_0, t_1 \in [-1, 1]$, i.e.

$$ct_0 + d_0 = f(t_0) \quad \text{and} \quad ct_1 + d_1 = g(t_1).$$

Since $d_0 \neq 0$ we are able to define $\alpha_1 := \frac{c}{d_0}$; then

$$\begin{aligned} \psi_{\alpha_1}(1 + \alpha_1 t_0) &= \alpha_1 f\left(\frac{1}{\alpha_1}(1 + \alpha_1 t_0 - 1)\right) = \alpha_1 f(t_0) \\ &= \frac{c}{d_0} f(t_0) = \frac{c}{d_0}(ct_0 + d_0) = \frac{c^2 t_0}{d_0} + c. \end{aligned}$$

On the other hand the line $l^* := \{(a, ca) : a \in \mathbb{R}\}$ contains the point $(1 + \alpha_1 t_0, c(1 + \alpha_1 t_0))$. We have

$$c(1 + \alpha_1 t_0) = c + c\alpha_1 t_0 = c + \frac{c^2 t_0}{d_0} = \psi_{\alpha_1}(1 + \alpha_1 t_0),$$

which means that the line l^* contains the point $(1 + \alpha_1 t_0, \psi_{\alpha_1}(1 + \alpha_1 t_0))$. Moreover the slope of this line is equal to c , which means that it is a supporting line for ψ_{α_1} . In view of Lemma 2.16, that means that $\phi_{x_1, y_1}^1(\arctan c) = \alpha_1$. In the same way one can prove that $\phi_{x_2, y_2}^2(\arctan c) = \alpha_2$ for $\alpha_2 = \frac{c}{d_1}$. Since $d_0 \neq d_1$, functions ϕ_{x_1, y_1}^1 and ϕ_{x_2, y_2}^2 do not coincide. \square

CORROLARY 2.18. *Let $(X, \|\cdot\|)$ be a real normed linear space. If $x, y \in X$, with $\|x\| = \|y\| = 1$ and $x \perp_{BJ} y$ are such vectors that $\phi_{x, y}(t) = \sin t$ for all $t \in \mathbb{R}$, then the subspace $\text{lin}(x, y)$ of X is an inner product space.*

CORROLARY 2.19. *Let $(X, \|\cdot\|)$ be a real normed linear space. Then the norm $\|\cdot\|$ comes from an inner product if and only if for all $x, y \in X$, $x \perp_{BJ} y$ with norm equal to 1, and for every $t \in \mathbb{R}$ one has $\phi_{x, y}(t) = \sin t$.*

CORROLARY 2.20. *Let $(X, \|\cdot\|)$ be a 2-dimensional real normed linear space. Then if for some $x, y \in X$, with $\|x\| = \|y\| = 1$ and $x \perp_{BJ} y$ we have $\phi_{x, y} = \sin$, then the same equality is true for all vectors x, y satisfying the above properties.*

THEOREM 2.21. *Let $(X, \|\cdot\|)$ be a real linear normed space. If for every $x, y \in X$, with $\|x\| = \|y\| = 1$ such that $x \perp_{BJ} y$ and for every $t \in (0, \frac{\pi}{2})$ we have*

$$(2.15) \quad \phi_{x, y}\left(\frac{\pi}{2} + t\right) = \phi_{x, y}\left(\frac{\pi}{2} - t\right)$$

then $\|\cdot\|$ comes from an inner product.

PROOF. Assume that the norm $\|\cdot\|$ does not come from an inner product. Then there exist vectors $x, y \in X$ such that $x \perp_{BJ} y$ and $x \not\perp_{JY} y$ which additionally satisfy $\|x\| = \|y\| = 1$ (see [1]). Since x and y are not James orthogonal, we have

$$\|y + x\| \neq \|y - x\|.$$

Assume that $\|y + x\| > \|y - x\|$. Similarly as before, for brevity, we shall write $\|(a, b)\|$ instead of $\|ax + by\|$. Let $f_0 : (-1, 1) \rightarrow \mathbb{R}$ be such a function that

$$\text{Gr}f_0 = \{(a, b) \in \mathbb{R}^2 : \|(a, b)\| = 1, |a| < 1, b > 0\}.$$

Extend this function to a continuous function $f : [-1, 1] \rightarrow \mathbb{R}$. Note that from $x \perp_{BJ} y$ we infer that $\|y - x\| \geq 1$. Thus $\|y + x\| > 1$. Now, define $t := \frac{1}{\|y+x\|} \in (0, 1)$; then $\|tx + ty\| = 1$ which means that $t = f(t)$. On the other hand

$$\|ty - tx\| = \left\| \frac{1}{\|x+y\|} (y-x) \right\| = \frac{\|y-x\|}{\|x+y\|} < 1,$$

i.e. $\|(-t, t)\| < 1$ and, consequently,

$$(2.16) \quad f(-t) > t = f(t).$$

Since the point (t, t) , is an element of the graph of f , we can find a supporting line for function f which contains this point. Denote this line by $l := \{(a, b) : b = ca + d_0\}$. Then the line $l^* := \{(a, b) : b = -ca + d_0\}$ is not a supporting line for f because this line contains the point $(-t, t)$ which lies below the graph of f . Now consider the line $l_1 := \{(a, b) : b = -ca + d_1\}$, where d_1 is the maximal number d such that the line $b = -ca + d$ contains some points of the graph of f , note that $d_1 > d_0$. Consequently, we have

$$(2.17) \quad f(t) = ct + d_0 \quad \text{and} \quad f(t_1) = -ct_1 + d_1,$$

for some $t_1 \in [-1, 1]$. Assume that $c < 0$ (for the positive c the proof runs similarly; the case of $c = 0$ is not possible: in such a case $l^* = l$ but we know that l is a supporting line for f , and l^* is not a supporting line for f). Summarizing, the line l contains the point $(t, t) \in \text{Gr}f$, and the line l_1 contains a point $(t_1, f(t_1))$ for some $t_1 \in [-1, 1]$. Given an $\alpha > 0$ consider the functions

$$\psi_\alpha^+(x) := \alpha f\left(\frac{1}{\alpha}(x-1)\right), \quad x \in (1-\alpha, 1+\alpha)$$

and

$$\psi_\alpha^-(x) := \alpha f\left(\frac{1}{\alpha}(x+1)\right), \quad x \in (-1-\alpha, -1+\alpha).$$

A line with the slope equal to $-c$ is a supporting line for ψ_α^+ in the point $(1 + \alpha t_1, \psi_\alpha^+(1 + \alpha t_1))$ provided that such a line contains this point (the proof is similar to the corresponding part of the proof of Theorem 2.17). Moreover,

a line with the slope equal to c containing the point $(-1 + \alpha t, \psi_{\alpha}^{-}(-1 + \alpha t))$ is a supporting line for function ψ_{α}^{-} at this point. Let us now consider the lines

$$l^{+} := \{(a, -ca) : a \in \mathbb{R}\} \quad \text{and} \quad l^{-} := \{(a, ca) : a \in \mathbb{R}\}$$

and define numbers $\alpha^{+} := -\frac{c}{d_1}$ and $\alpha^{-} := -\frac{c}{d_0}$. Note that $\alpha^{+}, \alpha^{-} \in (0, 1]$ and $\alpha^{-} \neq \alpha^{+}$, and these numbers are well defined since $d_0, d_1 \neq 0$. Indeed, d_1 is positive and c is negative thus the considered fraction is positive. On the other hand, since the line l^1 supports f , every point of this line has a norm greater than or equal to 1. Consequently, we have $\|(\frac{d_1}{c}, 0)\| \geq 1$. But $\|(\frac{d_1}{c}, 0)\| = |\frac{d_1}{c}|$ i.e. $|\frac{d_1}{c}| \geq 1$. For the number α^{-} we have a similar situation. Write

$$\begin{aligned} \psi_{\alpha^{+}}^{+}(1 + \alpha^{+}t_1) &= \alpha^{+}f\left(\frac{1}{\alpha^{+}}(1 + \alpha^{+}t_1 - 1)\right) = \alpha^{+}f(t_1) \\ &= -\frac{c}{d_1}(-ct_1 + d_1) = -c\left(1 - \frac{ct_1}{d_1}\right) = -c(1 + \alpha^{+}t_1). \end{aligned}$$

Hence the line l^+ contains the point $(1 + \alpha^{+}t_1, \psi_{\alpha^{+}}^{+}(1 + \alpha^{+}t_1))$. Since its slope is equal to $-c$, this line supports $\psi_{\alpha^{+}}^{+}$. In view of Lemma 2.16, it means that

$$\phi_{x,y}(\arctan(-c)) = \alpha^{+}.$$

Similarly

$$\phi_{x,y}(\pi - \arctan(-c)) = \alpha^{-}.$$

But we have already observed that $\alpha^{+} \neq \alpha^{-}$, which contradicts (2.15) (with $t = \frac{\pi}{2} - \arctan(-c)$). \square

It is a known fact that if in a normed space every pair x, y of Birkhoff-James orthogonal vectors is Pythagoras orthogonal then this space must be an inner product space (see [1]). Using our results we shall prove this fact under some additional assumption. It is by no means exciting. But we want to mention it because this example shows that using the properties of the function $\phi_{x,y}$ considered in this paper, one can prove facts concerning the properties of normed spaces, which do not involve the function $\phi_{x,y}$ itself.

COROLLARY 2.22. *Let $(X, \|\cdot\|)$ be a real normed linear space such that for all $x, y \in X$ with $\|x\| = \|y\|$ we have*

$$\min \|x + \lambda(x + y)\| = \min \|y + \lambda(x + y)\|.$$

If every two vectors in X which are Birkhoff-James orthogonal are also Pythagoras orthogonal, then the norm $\|\cdot\|$ comes from an inner product.

To prove this corollary it is enough to repeat the proof of Remark 2.9 to get

$$\phi_{x,y}\left(\frac{\pi}{2} + t\right) = \phi_{x,y}\left(\frac{\pi}{2} - t\right)$$

for all $x, y \in X$ with $\|x\| = \|y\|, x \perp_{BJ} y$ and for every $t \in (0, \frac{\pi}{2})$ and then apply Theorem 2.21.

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T. Szostok
Uniwersytet Śląski
Instytut Matematyki
Bankowa 14
40-007 Katowice, Poland
E-mail: szostok@ux2.math.us.edu.pl

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