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THE EXTENSION DIMENSION OF UNIVERSAL SPACES

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Abstract. Let α be an infinite cardinal, $\mathcal{T}$ denote a class of CW-complexes, $\mathcal{K}$ the class of all compact Hausdorff spaces, $\mathcal{M}_\alpha$ the class of all metrizable spaces of weight $\leq \alpha$, and $n \geq 0$. We shall prove that,

(a) if U is a universal metrizable space of covering dimension $\leq n$ and weight $\leq \alpha$, then $\text{ext-dim}_{(\mathcal{M}_\alpha, \mathcal{T})} U = [S^n]$, and

(b) if $U \in \mathcal{K}$, $K \in \mathcal{T}$, $\dim U \leq K$, and U contains a copy of every compact metrizable space X with $\dim X \leq K$, then $\text{ext-dim}_{(\mathcal{K}, \mathcal{T})} U = [K]$.

1. Introduction

We are going to detect the extension dimension of certain universal spaces, both in classes of metrizable spaces and in classes of compact Hausdorff spaces. Here, briefly, is some background information in this subject.

The rudiments of a theory of extension can be found in the work [10] of John Walsh where he proved the important Edwards-Walsh resolution theorem for integral cohomological dimension. He was comparing the notions of covering dimension and integral ($\mathbb{Z}$-) cohomological dimension, observing that each could be defined in terms of extensions of maps to certain CW-complexes. Later, A. Dranishnikov [1] formally introduced extension theory and the concept of extension dimension (see also [2]). In [4] the authors proved the existence of extension dimension in a large variety of settings.

Since we are going to determine the extension dimension of certain spaces, let us review the main ideas of extension theory and provide a definition of extension dimension.

Let X be a topological space, and K be a CW-complex. The notation $\dim X \leq K$, i.e., K is an absolute extensor of X, means that for every closed

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subspace A of X and map $f : A \to K$, there exists a map $F : X \to K$ which is an extension of f. Two other notations, $K \in \mathrm{AE}(X)$ and $X\tau K$ are used in the literature. For covering dimension, one of the results from the beginning is the Alexandroff Theorem: If X is a compact Hausdorff space, then the covering dimension of X is $\leq n$ if and only if $\dim X \leq S^n$.

Let $\mathcal{C}$ be a class of spaces, $\mathcal{T}$ a class of CW-complexes, and K, $K' \in \mathcal{T}$. If it is true that for all $X \in \mathcal{C}$, $\dim X \leq K$ implies that $\dim X \leq K'$, then we write $K \leq K'$. This defines a preorder on $\mathcal{T}$ (see [2]). One specifies $K \sim K'$ if and only if $K \leq K'$ and $K' \leq K$; then $\sim$ is an equivalence relation on $\mathcal{T}$. An equivalence class $[K] = [K]_{(\mathcal{C},\mathcal{T})}$ under this relation is called an *extension type*. We then write $\dim X \leq [K]$ to mean that $\dim X \leq K'$ for all $K' \in [K]$.

Denote by $\mathrm{ET}(\mathcal{C}, \mathcal{T})$ the class of all extension types. Then the above relation $\leq$ induces a partial order on $\mathrm{ET}(\mathcal{C}, \mathcal{T})$. When $K \leq K'$ we shall write, $[K]_{(\mathcal{C},\mathcal{T})} \leq_{(\mathcal{C},\mathcal{T})} [K']_{(\mathcal{C},\mathcal{T})}$, or sometimes just, $[K] \leq [K']$, when $\mathcal{C}$ and $\mathcal{T}$ are understood.

For a given space $X \in \mathcal{C}$ and for a given class $\mathcal{T}$ of CW-complexes, we may ask if there is an initial element[1] in the following class of extension types:

$$\{[L] \in \mathrm{ET}(\mathcal{C}, \mathcal{T}) \mid \dim X \leq [L]\}.$$

If there is an initial element $[K]$, then it is called the *extension dimension* of X relative to $(\mathcal{C}, \mathcal{T})$ and is denoted by $\text{ext-dim}_{(\mathcal{C},\mathcal{T})} X = [K]$.

With this background in mind, we may state our main results. Let α be an infinite cardinal and $n \geq 0$. Denote by $\mathcal{T}$ a class of CW-complexes, and by $\mathcal{M}_\alpha$ the class of metrizable spaces of weight $\leq \alpha$. It is known (see [6], [9]) that there exists an n-dimensional universal metrizable space U, of weight α. This means that the covering dimension of U is $\leq n$, the weight, $\mathrm{wt}\, U$ is $\leq \alpha$, and every metrizable space X with covering dimension $\leq n$ and $\mathrm{wt}\, X \leq \alpha$ can be embedded in U. We shall later prove,

Proposition 1.1. *If $S^n \in \mathcal{T}$ and U is a universal metrizable space of covering dimension $\leq n$ and weight $\leq \alpha$, then* $\text{ext-dim}_{(\mathcal{M}_\alpha,\mathcal{T})} U = [S^n]$.

It would be interesting if this proposition could be improved by replacing $\mathcal{M}_\alpha$ with the class $\mathcal{M}$ of all metrizable spaces.

Let $\mathcal{K}$ be the class of all compact Hausdorff spaces. By Corollary 1.9 of [5], if K is a compact CW-complex, then there exists $U \in \mathcal{K}$ which is universal with respect to the properties $\dim U \leq K$ and $\mathrm{wt}\, U \leq \alpha$. It is not known if this statement is true for arbitrary CW-complexes K.

We shall prove the following.

Proposition 1.2. *If $U \in \mathcal{K}$, $K \in \mathcal{T}$, $\dim U \leq K$, and U contains a copy of every compact metrizable space X with* $\dim X \leq K$, *then* $\text{ext-dim}_{(\mathcal{K},\mathcal{T})} U = [K]$.

[1] An initial element $s_0 \in S$ of a partially ordered set $(S, \leq)$ is understood in the following sense: for every $s \in S, s_0 \leq s$. Such s_0 is, of course, unique.

Proposition 1.2 has the following corollaries.

Corollary 1.3. *If* $K \in \mathcal{T}$ *and* U *is a universal Hausdorff compactum with respect to the properties,* $\dim U \leq K$ *and* $\mathrm{wt}\, U \leq \alpha$*, then* $\text{ext-dim}_{(\mathcal{K},\mathcal{T})} U = [K]$.

Corollary 1.4. *If* $S^n \in \mathcal{T}$, $U \in \mathcal{K}$ *is universal with respect to the properties, covering dimension of* U *is* $\leq n$ *and* $\mathrm{wt}\, U \leq \alpha$*, then* $\text{ext-dim}_{(\mathcal{K},\mathcal{T})} U = [S^n]$.

2. Lemmas, proofs of main results

Lemma 2.1. *Let* $\mathcal{T}$ *be a class of CW-complexes,* $\mathcal{C}$ *be a class of spaces, and* $U \in \mathcal{C}$*. Suppose that* $K \in \mathcal{T}$, $\dim U \leq K$*, and*

(a) *whenever* $X \in \mathcal{C}$, $X \subset U$, $L \in \mathcal{T}$*, and* $\dim U \leq L$*, then* $\dim X \leq L$.

Assume, moreover, that

(b) *if* $L \in \mathcal{T}$ *and* $[K] \not\leq_{(\mathcal{C},\mathcal{T})} [L]$*, then there exists* $X \in \mathcal{C}$, $X \subset U$*, such that* $\dim X \not\leq L$.

Then $\text{ext-dim}_{(\mathcal{C},\mathcal{T})} U$ *exists and equals* $[K]$.

Proof. Suppose that $\text{ext-dim}_{(\mathcal{C},\mathcal{T})} U = [K]$ is false. Since $\dim U \leq K$, there exists $L \in \mathcal{T}$ such that

(1) $\dim U \leq L$, and
(2) $[K] \not\leq_{(\mathcal{C},\mathcal{T})} [L]$.

Since $L \in \mathcal{T}$ and (2) holds, then (b) implies the existence of $X \in \mathcal{C}$, $X \subset U$, such that $\dim X \leq L$ is false. If we apply (1) alongside (a), then we get the contradictory statement, $\dim X \leq L$. ∎

The next lemma will help us to prove Proposition 1.2.

Lemma 2.2. *Let* K, L *be CW-complexes and* Y *be a compact Hausdorff space such that*

(a) $\dim Y \leq K$*, and*
(b) $\dim Y \not\leq L$.

Then there exists a metrizable compactum X *such that*

(c) $\dim X \leq K$*, and*
(d) $\dim X \not\leq L$.

Proof. Write Y as the limit of an inverse system $\mathbf{Y} = (P_a, p_a^b, \Gamma)$ of metric compacta (even compact polyhedra) P_a (consult [8], p.61). Because of (b), there exists a closed subset D of Y and a map $g : D \to L$ such that

(1) g does not extend to a map of Y to L.

For each $a \in \Gamma$, let $Q_a = p_a(D)$. Then put $\mathbf{D} = (Q_a, q_a^b, \Gamma)$ where $q_a^b = p_a^b|Q_b : Q_b \to Q_a$. Surely, $D = \lim \mathbf{D}$.

There exists ([8], p. 63) $a \in \Gamma$ and a map $g_a : Q_a \to L$ such that

(2) $g_a \circ q_a \simeq g : D \to L$.

Now P_a is a metrizable compactum, so its weight is $\leq \aleph_0$. Hence the factorization theorem [5] and (a) show that for some compactum X with $\mathrm{wt}\, X \leq \aleph_0$, there are maps $k : Y \to X$ and $l : X \to P_a$ such that,

(3) $l \circ k = p_a$, and

(4) $\dim X \leq K$.

Surely X is metrizable.

Now $E = l^{-1}(Q_a)$ is a closed subset of X, and we claim that $g_a \circ l|E : E \to L$ does not extend to a map of X to L, i.e., statement (d) holds. For suppose such an extension $F : X \to L$ did exist. Note that from (3), $k^{-1}(E) = k^{-1}(l^{-1}(Q_a)) = (l \circ k)^{-1}(Q_a) = p_a^{-1}(Q_a)$. Surely $D \subset p_a^{-1}(Q_a) = k^{-1}(E)$.

Consider the map $F \circ k : Y \to L$. For any $x \in D$, $k(x) \in E$. So, using (3), $F(k(x)) = g_a \circ l \circ k(x) = g_a \circ p_a(x) = g_a \circ q_a(x)$. This shows that $g_a \circ q_a = F \circ k|D$ extends to $F \circ k : Y \to L$. Using (2) and the homotopy extension theorem, one sees that g extends to a map of Y to L, which is impossible because of (1). □

Proof of Proposition 1.1. We want to apply Lemma 2.1 with $\mathcal{C} = \mathcal{M}_\alpha$ and $K = S^n$. The subspace theorem for metrizable spaces (see Corollary 3.7 below) shows us that (a) of 2.1 holds true.

Suppose that $L \in \mathcal{T}$ and $[S^n] \not\leq_{(\mathcal{M}_\alpha, \mathcal{T})} [L]$. Then for some $Y \in \mathcal{M}_\alpha$, $\dim Y \leq S^n$ is true and

(1) $\dim Y \not\leq L$.

Since Y is metrizable, $\dim Y \leq S^n$ implies that

(2) the covering dimension of Y is $\leq n$.

The universality of U shows that for some $X \in \mathcal{M}_\alpha$, X is topologically equivalent to Y and $X \subset U$. The former along with (1) dictate that statement (b) of Lemma 2.1 obtains. □

Proof of Proposition 1.2. We apply 2.1 with $\mathcal{C} = \mathcal{K}$. Part (a) of 2.1 holds since,

$$X \in \mathcal{K} \text{ and } X \subset U$$

implies that X is closed in U, and the subspace theorem for closed subspaces is certainly true.

Suppose that $L \in \mathcal{T}$ and $[K] \not\leq_{(\mathcal{K}, \mathcal{T})} [L]$. Then for some $Y \in \mathcal{K}$, $\dim Y \leq K$ is true and

(1) $\dim Y \not\leq L$.

Applying Lemma 2.2, there is a metrizable compactum X such that,

(2) $\dim X \leq K$, and

(3) $\dim X \not\leq L$.

The hypothesis of Proposition 1.2 along with (2) show that we may as well assume that $X \subset U$. Certainly (3) gives us (b) of 2.1. □

3. A subspace theorem

The subspace theorem for extension theory in metrizable spaces can be proved by a technique introduced in [10] (see page 107) to prove such a theorem simultaneously for covering and $\mathbb{Z}$-cohomological dimension theories. We, however, want to prove that theorem for a wider class of spaces and for extension theory in general.

Definition 3.1. *A T_1-space X is stratifiable provided there is a function (a stratification) assigning to each open subset U of X a sequence (U_n) of open subsets of X such that*

(S1) $\overline{U}_n \subset U$ *for each* n,
(S2) $\bigcup_{n=1}^{\infty} U_n = U$, *and*
(S3) $U \subset V$ *implies* $U_n \subset V_n$ *for each* n.

Certain properties of stratifiable spaces are listed in section 2 of [7]. We herewith note some of the important facts about them.

(a) Stratifiable spaces are hereditarily paracompact (hence hereditarily normal).
(b) The trace of a stratification on a subspace is again a stratification.
(c) Any finite (even countable) product of stratifiable spaces is stratifiable.
(d) Every CW-complex is stratifiable.
(e) CW-complexes are absolute neighborhood extensors for stratifiable spaces.
(f) Metrizable spaces are stratifiable.

We need some preliminaries before proving the subspace theorem for stratifiable spaces. The following may be derived from Corollary 2 of I.6.3 (p. 81) of [8].

Proposition 3.2. *Let X be a space and $A \subset X$ be a subspace. Suppose that $\mathbf{U} = (U_\lambda, p_{\lambda\lambda'}, \Lambda)$ is an inclusion system such that $\{U_\lambda \mid \lambda \in \Lambda\}$ is a basis for the neighborhoods of A in X and U_λ is paracompact for each $\lambda \in \Lambda$. Let $\mathbf{p} = (p_\lambda) : A \to \mathbf{U}$ be the morphism in pro-Top consisting of inclusions. Then $\mathbf{p}$ is a resolution of A.*

To see the impact of this, one should examine the property (R1) of resolutions (see p. 74 of [8]). This leads to the next statement.

Corollary 3.3. *If $f : A \to K$ is a map to an ANR K, then for some neighborhood U of A in X, there is a map $g : U \to K$ such that $g|A \simeq f$.*

The next two statements are well-known.

(1) Every CW-complex is homotopy equivalent to the polyhedron $|K|$ of some simplicial complex K where $|K|$ is given the metric topology, and
(2) every polyhedron $|K|$ with the metric topology is an ANR.

So we may conclude,

Corollary 3.4. *If $h : A \to K$ is a map to a CW-complex K, then for some neighborhood U of A in X, there is a map $f : U \to K$ such that $f|A \simeq h$.*

Lemma 3.5. *Let X be a stratifiable space and K be a CW-complex such that* $\dim X \le K$. *Then for each open subspace U of X,* $\dim U \le K$.

Proof. By the definition of a stratification, one sees that $U = \bigcup\{A_i \mid i \in \mathbb{N}\}$ where A_i is closed in X, hence in U, for each i. Since A_i is closed in X, then $\dim A_i \le K$.

Let A be closed in U and $f : A \to K$ be a map. Using (e) and the fact that $\dim A_1 \le K$, one may extend f to a map $f_1 : N_1 \to K$ where N_1 is a closed neighborhood of $A \cup A_1$ in U. Now, by the same reasoning, this time for A_2, extend f_1 to a map $f_2 : N_2 \to K$ where N_2 is a closed neighborhood of $N_1 \cup A_2$ in U. This process continues recursively. Finally, define $F = \bigcup\{f_i \mid i \in \mathbb{N}\} : U \to K$. It is easy to check that F is a continuous extension of f. □

Theorem 3.6 (Subspace Theorem). *Let X be a stratifiable space and K a CW-complex such that* $\dim X \le K$. *Then for each subspace Y of X,* $\dim Y \le K$.

Proof. Let A be a closed subspace of Y and $f : A \to K$ be a map. Using (a), one sees that there exists a base of neighborhoods of A in X consisting of paracompacta. So from Corollary 3.4, we find an open neighborhood U of A in X and a map $g : U \to K$ such that $g|A \simeq f : A \to K$. If we can show that $g|A$ extends to a map of Y to K, then our proof will be complete by virtue of the homotopy extension property for stratifiable spaces which may be proved using (f), (c), and (e).

The subsets A and $Y \setminus U$ are separated subsets of the stratifiable and hence (see (a)) hereditarily normal space X. Therefore ([3], 2.1.7) we may find a disjoint pair V and W of open subsets of X such that $A \subset V$ and $Y \setminus U \subset W$. The set $\overline{V} \cap U$ is closed in the open subspace $U \cup W$ of X. Applying Lemma 3.5, we obtain a map $G : U \cup W \to K$ extending g. We easily check that $Y \subset U \cup W$, so $G|Y : Y \to K$ is a map whose restriction to A is g. This completes our proof. □

This theorem and (f) yield the subspace theorem for metrizable spaces.

Corollary 3.7. *Let X be a metrizable space and K a CW-complex such that* $\dim X \le K$. *Then for each subspace Y of X,* $\dim Y \le K$.

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