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DENDROIDS AND PREORDERS

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Abstract. Let X be a dendroid and $x^* \in X$. $\Delta_0(X, x^*)$ will denote the preordered set of arc-components of $X \setminus \{x^*\}$ where the preorder is defined by $\alpha \leq \beta$ if $\alpha \subseteq \mathrm{cl}(\beta)$. In this paper we investigate conditions under which there exists a pair (X, x^*) such that $\Delta_0(X, x^*)$ is isomorphic to a given preordered set.

1. Introduction

A *continuum* is a compact and connected metric space. It is *hereditarily unicoherent* if the intersection of each two of its subcontinua is connected (or empty). A *dendroid* is an arcwise connected and hereditarily unicoherent continuum. It is well known that for every pair of points $x, y \in X$ there exists a unique arc $[x, y] \subseteq X$ joining them and that a subcontinuum of a dendroid X is also a dendroid.

By a *preordered set* or *preposet* we mean a set $\mathcal{P}$ with a transitive relation $\leq$ between pairs of elements of $\mathcal{P}$ which satisfies $p \leq p$ for each $p \in \mathcal{P}$. If additionally, the relation $\leq$ satisfies: $p \leq q$ and $q \leq p$ implies $p = q$, then we say that $\mathcal{P}$ is a *partially ordered set* or *poset*. An *induced subpreposet* (respectively, *subposet*) $\mathcal{P}'$ is a subset of a preposet (respectively, poset) $\mathcal{P}$ such that for $p, q \in \mathcal{P}'$, $p \leq q$ in $\mathcal{P}'$ if and only if $p \leq q$ in $\mathcal{P}$. A *chain* (respectively, *antichain)* is a subset of a preposet each pair of whose elements is comparable (respectively, incomparable). An *isomorphism* between preordered sets is a

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bijective function such that both it and its inverse preserve the order relation; the term *mapping* will always mean a continuous function.

The main purpose of this paper is to construct examples of dendroids. We do this by defining a preorder relation between the arc-components of $X \setminus \{x^*\}$ where X is a dendroid and $x^* \in X$. To make this more precise we need some more definitions.

The set of arc-components of a subset W of a dendroid X will be denoted by $\mathcal{A}(W)$. A *pointed dendroid* (X, x^*) consists of a dendroid X and a point $x^* \in X$. Following [4], we denote by $\Delta_0(X, x^*)$ the set $\mathcal{A}(X \setminus \{x^*\})$ with the preorder $\leq$ defined as $\alpha \leq \beta$ if $\alpha \subseteq \mathrm{cl}(\beta)$. We say that a preposet $\mathcal{P}$ is *realizable* whenever there is a pointed dendroid (X, x^*) such that $\Delta_0(X, x^*)$ is isomorphic to $\mathcal{P}$; in such a case (X, x^*) will be said to *realize* $\mathcal{P}$. The preposet $\mathcal{Q}$ is said to be *subrealizable* whenever there is a realizable preposet $\mathcal{P}$ containing $\mathcal{Q}$ as an induced subpreposet. It was shown in [4] that for any finite poset $\mathcal{P}$ there is a pointed dendroid (in fact infinitely many) (X, x^*) which realizes $\mathcal{P}$. In this paper we investigate this problem for infinite preposets, necessarily of cardinality at most $\mathfrak{c}$, since a compact metric space has at most $\mathfrak{c}$ elements.

We prove in this paper that a countable preposet containing an induced subpreposet which has no last element is not realizable. In particular ω and the preordered set $E_2 = (\{a, b\}, \leq)$ (where $a \neq b$), such that $a \leq b$ and $b \leq a$ are not realizable. Nevertheless, we prove that ω and E_2 are both subrealizable. For the latter, we construct a pointed dendroid (X, x^*) such that every element in $\mathcal{A}(X \setminus \{x^*\})$ is dense in X. We also prove that countable preposets in which all chains are bounded by a fixed natural number, are realizable.

2. Preliminaries

The following results concerning the arc-components of subsets of dendroids will be required in Section 3.

Theorem 2.1. *Let X be a dendroid, $W \subseteq X$ an open set in X and $\alpha \in \mathcal{A}(W)$. Then α is an F_σ subset of X. Moreover $\alpha = \bigcup\{F_n : n \in \omega\}$ where F_n is a dendroid for each $n \in \omega$.*

Proof. Let d be a metric on X which generates the topology and fix a point $y \in \alpha$. Let $r = d(y, X \setminus W)$. If $K_n = \{x \in X : d(x, X \setminus W) \geq r/(n+1)\}$ then K_n is a closed subset of W and $y \in K_n$ for each $n \in \omega$. Let F_n be the arc component of K_n containing y. To prove that F_n is a closed subset of X, note that $\mathrm{cl}(F_n)$ is a subcontinuum of X and hence it is a dendroid contained in K_n. Since F_n is the arc-component of y, it follows that $\mathrm{cl}(F_n) \subseteq F_n$ and hence F_n is closed. Now, if $x \in \alpha$, the arc $[x, y]$ is a closed subset of X contained in W and hence $[x, y]$ and $X \setminus W$ are disjoint compact subsets of X. Thus

$d([x,y],W) > 0$ and so $[x,y] \subseteq K_m$ for some $m \in \omega$. Hence $[x,y] \subseteq F_m$, implying that $y \in F_m$. ◻

The simple proof of the following theorem is left to the reader.

THEOREM 2.2. *Let X be a dendroid, Y a subdendroid of X and $x^* \in Y$. Then for each $\alpha \in \mathcal{A}(Y \setminus \{x^*\})$ there exists a unique $\psi(\alpha) \in \mathcal{A}(X \setminus \{x^*\})$ such that $\alpha \subseteq \psi(\alpha)$. Moreover $\psi : \mathcal{A}(Y \setminus \{x^*\}) \to \mathcal{A}(X \setminus \{x^*\})$ is an injective function and $\alpha = \psi[\alpha] \cap Y$.*

THEOREM 2.3. *Let (X, x^*) be a pointed dendroid such that each element of $\mathcal{A}(X \setminus \{x^*\})$ has empty interior. Then $\mathcal{A}(X \setminus \{x^*\})$ is uncountable.*

PROOF. Since $X = \bigcup\{\alpha : \alpha \in \mathcal{A}(X \setminus \{x^*\})\} \cup \{x^*\}$, it is a consequence of Theorem 2.1 that

$$X = \bigcup \left\{ \bigcup_{n\in\omega} F_n(\alpha) : \alpha \in \mathcal{A}(X \setminus x^*) \right\}$$

where each $F_n(\alpha)$ is closed (it is a dendroid) and has empty interior by hypothesis. It follows from the Baire Category Theorem that $\mathcal{A}(X \setminus \{x^*\})$ is uncountable. ◻

An *inverse sequence* is a sequence $(X_i, f_i)_{i\in\omega}$ of topological spaces X_i and onto mappings $f_i : X_{i+1} \to X_i$. The set

$$\varprojlim(X_i : f_i)_{i\in\omega} = \{(x_1, x_2, ...) \in \Pi_{i\in\omega} X_i : f_i(x_{i+1}) = x_i \text{ for all } i \in \omega\}$$

with the relative topology inherited from the product space $\Pi_{i\in\omega} X_i$ will be called the *inverse limit* of this sequence. For $j > k$, we denote $f_{kj} = f_k \circ \cdots \circ f_{j-1} \circ f_j : X_{j+1} \to X_k$. The following theorems concerning inverse limits will be used in our constructions.

THEOREM 2.4. [3, Theorem 2.10 (Anderson-Choquet)]. *Let $(X_i,\ f_i)_{i\in\omega}$ be an inverse sequence of metric spaces (X_i, d_i) contained in a compact space S and such that $X_i \subseteq X_{i+1}$. If*

(1) *For each $\varepsilon > 0$, there exists $k \in \omega$ such that for each $p_k \in X_k$,*

$$\operatorname{diam}\left(\bigcup_{j>k} f_{kj}^{-1}(p_k)\right) < \varepsilon.$$

(2) *For each $k \in \omega$ and each $\varepsilon > 0$, there is some $\delta > 0$ such that if $j > k$ and $u_j, w_j \in X_j$, then $d_j(u_j, w_j) < \delta \Rightarrow d_k(f_{kj}(u_j), f_{kj}(w_j)) < \varepsilon$,*

then

$$\varprojlim(X_i, f_i)_{i\in\omega} = \operatorname{cl}_S\left(\bigcup_{i\in\omega} X_i\right).$$

A mapping $f : X \to Y$ is said to be *monotone at* $p \in Y$ if the inverse image of each subcontinuum of Y containing p is connected. The following theorem can be found in [1].

THEOREM 2.5. [1; Corollary 3, p.145]. *Let* $X = \varprojlim(X_i, f_i)_{i\in\omega}$. *If each* X_i *is a dendroid and there exists a point* $p = (p_1, p_2, p_3, ...) \in X$ *such that each* f_i *is monotone at* p_i, *then* X *is a dendroid.*

THEOREM 2.6. *Let* $(X_i,\ f_i)_{i\in\omega}$ *be an inverse sequence where* X_i *is an arcwise connected continuum for each* $i \in \omega$ *and further assume that for each* $i \in \omega$ *there exists* $T_i \subseteq X_i$ *for which* $f_{i-1} \mid T_i : T_i \to X_{i-1}$ *is a homeomorphism. Let* $X_\infty = \varprojlim(X_n, f_n)$, *then the set*

$$\mathcal{T} = \{(x_1, x_2, ...) \in X_\infty : x_i \in T_i \textit{ for all but a finite number of } i \in \omega\}$$

is arcwise connected.

PROOF. Let $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...) \in \mathcal{T}$. Assume that $x_i, y_i \in T_i$ for every $i \geq k$ and let $\tau_k : I \to T_k$ be an arc from x_k to y_k.

For $i > k$, define inductively $\tau_{i+1} = (f_i \mid T_{i+1})^{-1} \circ \tau_i$ and for $i < k$, let $\tau_i = f_{i(k-1)} \circ \tau_k$. Finally, we define $\tau : I \to \mathcal{T}$ by $\pi_j(\tau(t)) = \tau_j(t)$. Since each τ_i is continuous, so is τ; moreover, $\tau_i(t) \in T_i$ for every $i \geq k$ and $\tau(0) = x$ and $\tau(1) = y$. Therefore τ is a path from x to y which is contained in $\mathcal{T}$ and which, in its turn, contains an arc from x to y. ∎

3. REALIZABLE AND NON-REALIZABLE PREPOSETS.

We begin this section by showing that certain preposets are not realizable.

THEOREM 3.1. *Let* $(\mathcal{P}, \leq)$ *be a preposet containing a preposet* $\mathcal{Q}$ *such that for* $p \in \mathcal{Q}$, *there exists* $p^* \in \mathcal{Q}$, $p^* \neq p$, $p \leq p^*$. *If* $\mathcal{P}$ *is countable then it is not realizable.*

PROOF. Suppose (X, x^*) realizes $\mathcal{P}$, that is there exists an isomorphism $\varphi : \mathcal{P} \to \Delta_0(X, x^*)$. Then $Y = \mathrm{cl}(\bigcup\{\alpha : \alpha \in \varphi[\mathcal{Q}]\})$ is a subdendroid of X in which each element of $\Delta_0(Y, x^*)$ has empty interior. Then, by Theorem 2.3, $\mathcal{A}(Y \setminus \{x^*\})$ is uncountable and by Theorem 2.2, $\mathcal{A}(X \setminus \{x^*\})$ is also uncountable. Since $\mathcal{P}$ is countable, we have a contradiction. ∎

COROLLARY 3.2. *No countable preposet containing* ω *as an induced subposet is realizable.*

Recall that E_2 denotes the preposet $(\{a, b\}, \leq)$ in which $a \leq b$ and $b \leq a$.

COROLLARY 3.3. *No countable preposet containing* E_2 *as an induced subpreposet is realizable.*

The following result is well-known; for completeness we include its easy proof.

Lemma 3.4. *A separable metric space cannot contain a strictly increasing or a strictly decreasing sequence of closed sets of length ω_1.*

Proof. Suppose $\{C_\lambda : \lambda \in \omega_1\}$ is a strictly increasing sequence of sets and either

i) all of the sets C_λ are open, or

ii) all of the sets C_λ are closed.

For each ordinal $\lambda \in \omega_1$, we choose $x_\lambda \in C_{\lambda+1} \setminus C_\lambda$ and then in case *i)*, $\{x_\lambda : \lambda \in \omega_1\}$ is not Lindelöf, while in Case *ii)* $\{x_\lambda : \lambda \in \omega_1\}$ is not separable. In either case we have a contradiction since X is both hereditarily separable and hereditarily Lindelöf. ◻

The following theorem follows immediately from Lemma 3.4.

Theorem 3.5. *A poset containing either a well-ordered or an anti-well-ordered uncountable chain is not subrealizable.*

Theorem 3.6. *(MA) Let κ be a cardinal such that $\omega < \kappa < \mathfrak{c}$. A poset of size κ is not realizable.*

Proof. Let X be a dendroid and suppose that for some $x^* \in X$ we have $(X, x^*) = \{C_\lambda : \lambda \in \kappa\}$. By Lemma 2.2, each arc-component C_λ is a countable union of dendroids, say $C_\lambda = \cup\{F_{\lambda,n} : n \in \omega\}$. Then $X = \cup\{F_{\lambda,n} : n \in \omega, \lambda \in \kappa\} \cup \{x^*\}$ and since each $F_{\lambda,n}$ is closed in X, Martin's Axiom implies that there are $n_0 \in \omega$ and $\lambda_0 \in \kappa$ such that $\mathrm{int}_X(F_{\lambda_0,n_0}) \neq \emptyset$. Now we let $X_1 = X \setminus \mathrm{int}_X(F_{\lambda_0,n_0})$; clearly X_1 is compact.

Having chosen indices $\{\lambda_\alpha : \alpha \in \beta\}$ and $\{n_\alpha : \alpha \in \beta\}$ and having constructed compact spaces $\{X_\alpha : \alpha < \beta\}$, then if β is a limit ordinal, we define $X_\beta = \cap\{X_\alpha : \alpha < \beta\}$. If on the other hand, β is a successor ordinal, $\beta = \gamma + 1$, then applying Martin's Axiom again, there is some $\lambda_\beta \in \kappa$ and $n_\beta \in \omega$ such that $\mathrm{int}_{X_\gamma}(F_{\lambda_\beta,n_\beta} \cap X_\gamma) \neq \emptyset$ and we define $X_\beta = X_\gamma \setminus \mathrm{int}_{X_\gamma}(\cup\{F_{\lambda_\alpha,n_\alpha} \cap X_\gamma : \alpha \leq \beta\})$. In this way we construct a strictly decreasing family $\{X_\lambda : \lambda \in \kappa\}$ of closed subsets of X, contradicting Lemma 3.4. ◻

We now prove some positive results concerning realizable and subrealizable preposets. In the sequel I and I_n (for $n \in \omega$) will denote the unit interval $[0,1]$ and H will denote the Hilbert cube $\prod_{n\in\omega} I_n$. The natural projection from H onto I_n will be denoted by π_n and the metric in H is defined by

$$d(x,y) = \sum_{n\in\omega} \frac{\mid \pi_n(x) - \pi_n(y) \mid}{2^n}.$$

H_n will denote the subspace $\{x \in H : \pi_j(x) = 0 \text{ for all } j > n\}$ of H which is clearly homeomorphic to $\prod_{k=1}^n I_k$. The restriction of d to H_n will be denoted by d_n. e^n will denote the element of H_n such that $\pi_n(e^n) = 1$ and $\pi_m(e^n) = 0$

if $m \neq n$. For each $k \in \omega$, we denote by $\mathbf{0}$ that element of H all of whose coordinates are 0 and by 0_k that element of H_k whose coordinates are all equal to 0. Finally, let $E_n = \{re^n : 0 \leq r \leq 1\}$ and $S = \{\frac{1}{m} : m \in \omega \setminus \{0\}\} \cup \{0\} \subseteq I$.

Now let D be a dendroid contained in H_k, such that $\mathbf{0} \in D$ and define $\mathrm{Comb}(D) = E_{k+1} \cup \bigcup_{t \in S} D \times \{t\}$. It is easy to verify that $\mathrm{Comb}(D)$ is a dendroid contained in H_{k+1}, $D \times \{0\} \subseteq \mathrm{Comb}(D) \subseteq \mathrm{cl}(\mathrm{Comb}(D) \setminus D)$ and $\mathcal{A}(\mathrm{Comb}(D) \setminus \{\mathbf{0}\}) = \mathcal{A}(D \setminus \{\mathbf{0}\}) \cup \{\mathrm{Comb}(D) \setminus D\}$. (Since D and $D \times \{0\}$ are homeomorphic, we will in future write D instead of $D \times \{0\}$.) It is clear that for each $n \in \omega$, the projection map from $H_{n+1} \to H_n$ restricted to $\mathrm{Comb}(D)$ is monotone at 0_n.

THEOREM 3.7. *ω is subrealizable.*

PROOF. We will construct a dendroid D in the Hilbert cube H such that $(D, \mathbf{0})$ subrealizes ω. Let D_2 be a dendroid which we assume to be embedded in H_2 and such that $0_2 \in D_2$. Now having constructed dendroids $D_2, D_3, \ldots, D_{n-1}$ as subspaces of $H_2, H_3, \ldots, H_{n-1}$ respectively, let $D_n = \mathrm{Comb}(D_{n-1}) \subseteq H_n$. It is clear that for $m \geq 3$, the projection map from H_m to H_{m-1} restricted to D_m, (and which we denote by ρ_{m-1}), maps onto D_{m-1} and $(D_m, \rho_m)_{m \geq 2}$ is an inverse system whose limit we will denote by D. It is easy to see that D is homeomorphic to a subspace of H under the map which sends $((x_1, x_2), (x_1, x_2, x_3), (x_1, x_2, x_3, x_4), ...)$ to $(x_1, x_2, x_3, x_4, \ldots)$. Since D is an inverse limit of dendroids and the bonding maps are monotone at the points 0_n, it follows from Theorem 2.5 that D is a dendroid. We claim that D is homeomorphic to $\mathrm{cl}_H(\bigcup_{n \geq 2} D_n)$. Since it is clear that $D_n \subseteq D_{n+1}$, we need only verify the two conditions of the Anderson-Choquet Theorem (Theorem 2.4).

(1) Let $\varepsilon > 0$ and and choose $k > 2$ such that $\frac{1}{2^{k-1}} < \varepsilon$. Then for each $p \in D_k$ and $u, w \in \bigcup_{j > k} \rho_{kj}^{-1}(p)$, $\pi_i(u) = \pi_i(w) = \pi_i(p)$ for $i \leq k$, so that
$$d(u, w) = \sum_{n \geq k} \frac{\mid \pi_n(x) - \pi_n(y) \mid}{2^n} \leq \frac{1}{2^{k-1}} < \varepsilon.$$

(2) Let $k \in \omega$ and $\varepsilon > 0$. If $j > k$ and $u, w \in D_j$, then we have $d_k(\rho_{kj}(u), \rho_{kj}(w)) \leq d_j(u, w)$. Thus we can choose $\delta = \varepsilon$.

We proceed to show that $\Delta_0(D, \mathbf{0})$ contains ω as an induced subposet. First we show that for each $m \geq 3$, $D_m \setminus D_{m-1} \in \mathcal{A}(D \setminus \{\mathbf{0}\})$. To this end, note that $D_m \setminus D_{m-1} = E_m \cup \bigcup_{t \in S \setminus \{0\}} D_{m-1} \times \{t\}$ and that this set is arcwise connected. We will prove now that given $x \in D_m \setminus D_{m-1}$ and $y \notin D_m \setminus D_{m-1}$, then for each arc joining x and y, $\mathbf{0} \in [x, y]$. It is easy to see from the definition of the dendroids D_n that $\mathbf{0} \in [x, y]$ for each $y \in \bigcup_{n \geq 2} D_n \setminus (D_m \setminus D_{m-1})$.

Now observe that:

1) If $y \in \mathrm{cl}_H(\bigcup_{n \geq 2} D_n) \setminus \bigcup_{n \geq 2} D_n$, then $\pi_n(y) \neq 0$ for infinitely many indices n.

2) If $y \in D$, $j \geq 3$ and $\pi_j(y) \notin S$, then $\pi_i(y) = 0$ for every $i \neq j$.

Now if $y \in \mathrm{cl}_H(\bigcup_{n\geq 2} D_n)\setminus\bigcup_{n\geq 2} D_n$, let $\sigma : I \to D$ be a path joining x and y. By 1), we can choose integers j, k greater than m and such that $\pi_j(y) \neq 0$ and $\pi_k(y) \neq 0$. Since $\pi_j(x) = \pi_k(x) = 0$, there is a point $u \in \pi_j[\sigma[I]] \setminus S$ and a point $v \in \pi_k[\sigma[I]] \setminus S$. By 2), there is a unique point u^* (respectively, v^*) in D whose $j-$th coordinate is u (respectively, whose $k-$th coordinate is v). Then u^* and v^* are both in $\sigma[I]$ and it is clear that the path joining these two points must contain $\mathbf{0}$. This proves that y is not an element of the arc-component of $D \setminus \{\mathbf{0}\}$ which contains x. Thus we have proved that $\{D_n \setminus D_{n-1} : n \geq 2\} \subseteq \mathcal{A}(D \setminus \{\mathbf{0}\})$. Since $D_n \setminus D_{n-1} \subseteq \mathrm{cl}(D_{n+1} \setminus D_n)$, it follows that $\{D_n \setminus D_{n-1} : n \geq 2\}$ is isomorphic to ω. ∎

Lemma 3.8. *Let $(X, \mathbf{0})$ be a pointed dendroid (which we assume to be embedded in H_k for some integer k) with the property (†) that if $\Lambda \subseteq A(X \setminus \{\mathbf{0}\})$ then $\bigcup\{\mathrm{cl}(L) : L \in \Lambda\}$ is a dendroid. If $\mathcal{B} = \{\mathcal{B}_n\}_{n\in\omega}$ is a countable family of subsets of $\mathcal{A}(X \setminus \{\mathbf{0}\})$, then there exists a dendroid $\Omega = \Omega((X,\mathbf{0}), \mathcal{B}) \subseteq H_{k+2}$ which contains (a homeomorphic copy of) X and has the following properties:*

i) *For each $\mathcal{B}_n \in \mathcal{B}$, there exists $A_n \in \mathcal{A}(\Omega \setminus \{\mathbf{0}\})$ such that $\bigcup\{B : B \in \mathcal{B}_n\} \subseteq \mathrm{cl}(A_n)$.*

ii) $\mathcal{A}(\Omega \setminus \{\mathbf{0}\}) = \mathcal{A}(X \setminus \{\mathbf{0}\}) \cup \{A_n\}_{n\in\omega}$.

iii) *Ω satisfies (†).*

Proof. For each $n \in \omega$, we choose $\xi_n \in H_{k+2}$ in such a way that $\pi_j(\xi_n) = 0$ if $j \leq k$ and $\{(\pi_{k+1}(\xi_n), \pi_{k+2}(\xi_n))\}_{n\in\omega}$ is a sequence of pairwise linearly independent vectors of $\mathbb{R}^2$ which converges to $(0, 0)$. $\mathbf{J}(\xi_n)$ will denote the set $\{r\xi_n : 0 \leq r \leq 1\} \subseteq H_{k+2}$. The set $\mathbf{B}_n = \bigcup\{\mathrm{cl}(B) : B \in \mathcal{B}_n\}$ is a dendroid by hypothesis. For each $m \in \omega$, let

$$A_{n,m} = \mathbf{B}_n \times \left\{ \left(\frac{\pi_k(\xi_n)}{m+1}, \frac{\pi_{k+1}(\xi_n)}{m+1} \right) \right\} \subseteq H_{k+2},$$

so that each $A_{n,m}$ is homeomorphic to $\mathbf{B}_n$ and $A_{m,n} \cap A_{r,s} = \emptyset$ unless $(m, n) = (r, s)$. We define

$$\Omega = (X \times \{(0, 0)\}) \cup \bigcup_{n\in\omega} \mathbf{J}(\xi_n) \cup \bigcup_{n,m\in\omega} A_{n,m}$$

(see Figure 1). It is not difficult to verify that Ω is a dendroid which satisfies (†). Conditions i) and ii) are satisfied with $A_n = \bigcup_{m\in\omega} A_{n,m} \cup \mathbf{J}(\xi_n)$. ∎

We now define $F_\omega = \bigcup_{j\geq 1} F_j \subset \mathbb{R}^2$ where

$$F_j = \left\{ r \left(\cos\frac{\pi}{2j}, \sin\frac{\pi}{2j} \right) : 0 \leq r \leq \frac{1}{2j} \right\} \subset \mathbb{R}^2.$$

Clearly F_ω is a dendroid and $(F_\omega, (0, 0))$ realizes a countably infinite set of mutually incomparable points.

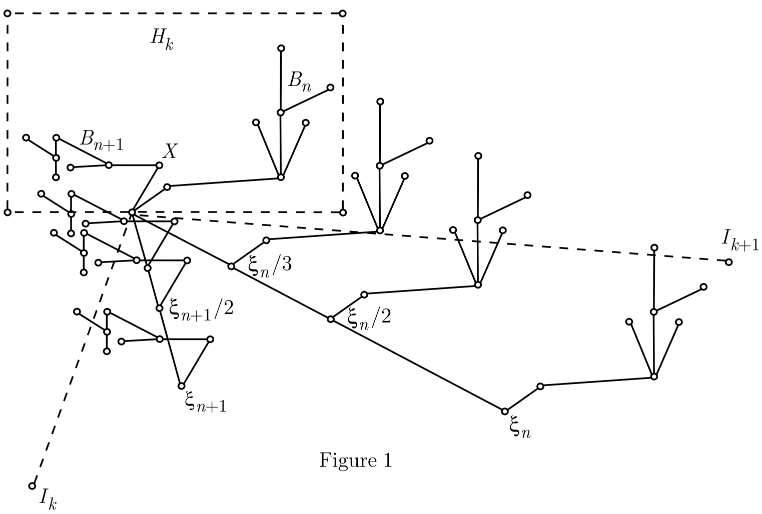

Figure 1

THEOREM 3.9. *A countable poset in which the length of every chain is bounded by some integer N is realizable.*

PROOF. Let $(P,<)$ be a countable poset in which every chain is of length at most N. The set P can be partitioned into sets $\{P_n : n \leq N\}$ where P_1 is the set of minimal elements of P and P_n is the set of minimal elements of $P \setminus \cup\{P_j : 1 \leq j \leq n-1\}$; points of P_n are mutually incomparable. Without loss of generality we assume that each P_n is countably infinite. Define $\Gamma_1 = F_\omega \subseteq H_2$. Then $(\Gamma_1, \mathbf{0})$ realizes the induced subposet $(P_1,<)$. Let $\phi : P_1 \to \Delta_0(\Gamma_1, \mathbf{0})$ be the isomorphism. Now, for each $q_m \in P_2$, let $F_m \subseteq P_1$ be the set of immediate predecesors of q_m and let $\mathcal{B}_2 = \{\phi[F_m] : m \in \omega\}$. Define $\Gamma_2 = \Omega((\Gamma_1, \mathbf{0}), \mathcal{B}_2) \subseteq H_4$. It follows from Lemma 3.8 that $(\Gamma_2, \mathbf{0})$ realizes the induced poset $(P_1 \cup P_2, \leq)$. If for some $n \leq N$, Γ_{n-1} has been constructed, we define $\Gamma_n = \Omega((\Gamma_{n-1}, \mathbf{0}), \mathcal{B}_n)$ and hence $(\Gamma_n, \mathbf{0})$ realizes the induced poset $(P_1 \cup ... \cup P_n, \leq)$. It is clear that the desired dendroid is Γ_N. ∎

As we proved in Corollary 3.3, E_2 is not realizable. However, the following theorem provides an example of a pointed dendroid (X,p) which subrealizes E_2. Moreover, we will prove that every element of $\mathcal{A}(X \setminus \{p\})$ is dense in X and so by Theorem 2.3, $\mathcal{A}(X,p)$ is not countable.

THEOREM 3.10. *The preordered set E_2 is subrealizable.*

PROOF. We will construct a dendroid X as the inverse limit, $\varprojlim(X_i, f_i)$ of trees X_i contained in $\mathbb{R}^2$ (with the metric inherited from $\mathbb{R}^2$), such that there is $p = (p_n)_{n\in\omega} \in X$ and each bonding maps f_{i-1} is monotone at $p_i \in X_i$. By Theorem 2.5, this will ensure that X is a dendroid.

In Figure 2 we illustrate the trees X_0, X_1 and X_2. X_n is the union of X_{n-1} and rectilinear segments, so that the set of end points of X_n is the set of points in the Cantor set of the form $\frac{k}{3^n}$ $(0 \le k \le 3^n)$.

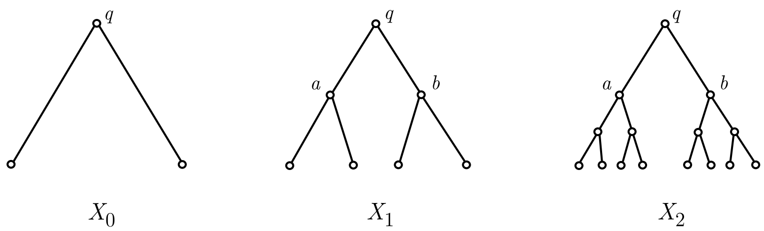

Figure 2

Let x, $y \in \mathbb{R}^2$ and denote by $[x, y]$ the rectilinear segment from x to y contained in $\mathbb{R}^2$. Let $q = (1, 1/2)$ and $T = [q, a] \cup [q, b]$ where a and b are the middle points of of the segments $[q, (0, 0)]$ and $[q, (1, 0)]$. Notice that $X_n = T \cup A_n \cup B_n$ where A_n and B_n are both homeomorphic to X_{n-1}.

The mapping $f_n : X_{n+1} \to X_n$ is defined as follows: f_n restricted to A_{n+1} and to B_{n+1} is the natural homeomorphism onto X_n and for every $x \in T$, $f_n(x) = q$. We will prove that f_n is monotone with respect to q. Let K be a subcontinuum of X_n containing q. Clearly, $f_n^{-1}(K) = K_A \cup K_B \cup N$ where $K_A = f_n^{-1}(K) \cap A_n$ and $K_B = f_n^{-1}(K) \cap B_n$. Since K_A and K_B contain a and b respectively and each of them is homeomorphic to K it follows that $f_n^{-1}(K)$ is connected. Thus X is a dendroid.

Define $\mathbf{q} \in X$ by $\pi_j(\mathbf{q}) = q$ for each $j \in \omega$. We will prove that each subset $\beta \in \mathcal{A}(X \setminus \{\mathbf{q}\})$ is dense in X. To this end, given $\varepsilon > 0$ and $x = (x_1, x_2, \dots) \in X$, we will show that there is $z = (z_1, z_2, \dots) \in \beta$ such that $d(x, z) < \varepsilon$.

Notice that any element in X has at most one coordinate contained in T. Fix $y = (y_1, y_2, \dots) \in \beta$ and choose $j_0 \in \omega$ such that $y_j \notin T$ and $\frac{1}{2^j} < \varepsilon$ for every $j \ge j_0$. To construct z, choose $z_{j_0} = x_{j_0}$; hence $z_j = x_j$ for each $j \le j_0$. If $j > j_0$, then $y_j \in A_j \cup B_j$ and we define $T_j = A_j$ if $y_j \in A_j$ or $T_j = B_j$ if $y_j \in B_j$ and then we choose z_j inductively in such a way that $z_j \in T_j$ and $f_{j-1}(z_j) = z_{j-1}$. This is possible by the definition of the maps f_j. Applying Theorem 2.6 we obtain that $z = (z_1, z_2, ...) \in \beta$ and clearly $d(x, z) < \varepsilon$. Thus we have proved that β is a dense subset of X and that E_2 is subrealizable.
$\square$

It has recently been brought to our attention that in a different context, a dendroid with similar properties to that described in Theorem 3.10 was constructed in [2].

Corollary 3.11. *A preordered set $\mathcal{P}$ of cardinality ω_1 such that for every two elements $p, q \in \mathcal{P}$ the two relations $p \leq q$ and $p \geq q$ hold, is subrealizable.*

A number of open questions remain and we mention below some of the more interesting ones.

Problem 3.12. *Is a countable poset in which every chain is finite realizable?*

Problem 3.13. *Is $\mathbb{Q}$ (the rationals) subrealizable?*

Problem 3.14. *Is every countable ordinal subrealizable?*

If the above can be answered, then:

Problem 3.15. *Characterize those countable posets which are realizable (or subrealizable).*

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