GLASNIK MATEMATIČKI
Vol. 40(60)(2005), 87 – 102

APPROXIMATION IN SMIRNOV-ORLICZ CLASSES

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Abstract. We use the approximation properties of the Faber polynomials to obtain some direct theorems of the polynomial approximation in Smirnov-Orlicz classes.

1. Introduction and Main Results

Let Γ be a rectifiable Jordan curve in the complex plane $\mathbb{C}$ and $G := Int\Gamma$, $G^- := Ext\Gamma$. Without loss of generality we may assume $0 \in G$. Let $\mathbb{T} := \{w \in \mathbb{C} : |w| = 1\}, \mathbb{D} := Int\mathbb{T}$ and $\mathbb{D}^- := Ext\mathbb{T}$. Let also $w = \varphi(z)$ be the conformal mapping of G^- onto $\mathbb{D}^-$, normalized by

$$\varphi(\infty) = \infty, \lim_{z\to\infty} \frac{\varphi(z)}{z} > 0,$$

and let ψ be its inverse.

If $1 \leq p < \infty$, we denote by $L_p(\Gamma)$ and $E_p(G)$ the set of all measurable complex valued functions f on Γ such that $|f|^p$ is Lebesgue integrable with respect to arclength, and the Smirnov class of analytic functions in G, respectively. Since Γ is rectifiable, we have that $\varphi' \in E_1(G^-)$ and $\psi' \in E_1(\mathbb{D}^-)$, and hence the functions φ' and ψ' admit non-tangential limits almost everywhere (a.e.) on Γ and on $\mathbb{T}$, and these functions belong to $L_1(\Gamma)$ and $L_1(\mathbb{T})$, respectively (see, for example [10, *p.* 419]).

Let h be a continuous function on $[0, 2\pi]$. Its modulus of continuity is defined by

$$\omega(t, h) := \sup\{|h(t_1) - h(t_2)| : t_1, t_2 \in [0, 2\pi], \ |t_1 - t_2| \leq t\}, \qquad t \geq 0.$$

2000 *Mathematics Subject Classification.* 30E10, 41A10, 41A20, 41A25, 46E30.

Key words and phrases. Dini-smooth curve, Smirnov-Orlicz classes, polynomial approximation, Faber polynomials, maximal convergence.

The function h is called Dini-continuous if

$$\int_0^{\pi} t^{-1}\omega(t,h)\,dt < \infty.$$

Definition 1.1 ([17, p. 48]). *The curve* Γ *is called Dini-smooth if it has a parametrization*

$$\Gamma : \varphi_0(\tau), \quad 0 \le \tau \le 2\pi$$

such that $\varphi_0'(\tau)$ *is Dini-continuous and* $\neq 0$.

If Γ is Dini-smooth, then [21]

$$0 < c_1 \le |\varphi'(z)| \le c_2 < \infty, \qquad z \in \Gamma \tag{1}$$

for some constants c_1 and c_2 independent of z.

A function $M(u) : \mathbb{R} \to \mathbb{R}^+$, where $\mathbb{R} := (-\infty, \infty)$ and $\mathbb{R}^+ := (0, \infty)$, is called an $N - function$ if it admits of the representation

$$M(u) = \int_0^{|u|} p(t)\,dt,$$

where the function $p(t)$ is right continuous and nondecreasing for $t \ge 0$ and positive for $t > 0$, which satisfies the conditions

$$p(0) = 0,\ p(\infty) := \lim_{t\to\infty} p(t) = \infty.$$

The function

$$N(v) := \int_0^{|v|} q(s)\,ds,$$

where

$$q(s) := \sup_{p(t)\le s} t, \qquad (s \ge 0)$$

is defined as complementary function of $M(u)$ [16, p. 11].

Let M be an $N - function$ and N be its complementary function. By $L_M(\Gamma)$ we denote the linear space of Lebesgue measurable functions $f : \Gamma \to \mathbb{C}$ satisfying the condition

$$\int_\Gamma M[\alpha|f(z)|]\,|dz| < \infty$$

for some $\alpha > 0$.

The space $L_M(\Gamma)$ becomes a Banach space with the norm

$$\|f\|_{L_M(\Gamma)} := \sup\left\{ \int_\Gamma |f(z)\,g(z)|\,|dz| : g \in L_N(\Gamma),\ \rho(g;N) \le 1 \right\},$$

where

$$\rho\left(g\ ;N\right):=\int\limits_{\Gamma}N\left[\ \left|g\left(z\right)\right|\ \right]\left|dz\right|.$$

The norm $\|\cdot\|_{L_M(\Gamma)}$ is called Orlicz norm and the Banach space $L_M(\Gamma)$ is called Orlicz space. Every function in $L_M(\Gamma)$ is integrable on Γ [18, p. 50], i.e.

$$L_M\left(\Gamma\right)\subset L_1\left(\Gamma\right). \tag{2}$$

An $N-function$ M satisfies the $\Delta_2-condition$ if

$$\limsup_{x\to\infty}\frac{M\left(2x\right)}{M\left(x\right)}<\infty.$$

The Orlicz space $L_M(\Gamma)$ is reflexive if and only if the $N-function$ M and its complementary function N both satisfy the $\Delta_2-condition$ [18, p. 113].

Let Γ_r be the image of the circle $\{w\in\mathbb{C}:|w|=r,\ 0<r<1\}$ under some conformal mapping of $\mathbb{D}$ onto G and let M be an $N-function.$

Definition 1.2. *If an analytic function f in G satisfies*

$$\int\limits_{\Gamma_r}M\left[\ \left|f\left(z\right)\right|\ \right]\left|dz\right|<\infty$$

uniformly in r, it belongs to Smirnov-Orlicz class $E_M(G)$.

If $M(x)=M(x,p):=x^p, 1<p<\infty$, then the Smirnov-Orlicz class $E_M(G)$ coincides with the usual Smirnov class $E_p(G)$.

Every function in the class $E_M(G)$ has [14] the non-tangential boundary values a.e. on Γ and the boundary function belongs to $L_M(\Gamma)$, and hence for $f\in E_M(G)$ we can define the $E_M(G)$ norm as:

$$\|f\|_{E_M(G)}:=\|f\|_{L_M(\Gamma)}.$$

For $\varsigma\in\Gamma$ we define the point $\varsigma_h\in\Gamma$ by

$$\varsigma_h:=\psi\left(\varphi\left(\varsigma\right)e^{ih}\right),\qquad h\in\left[0,2\pi\right],$$

and also the shift T_hf for $f\in L_M(\Gamma)$ as:

$$T_hf\left(\varsigma\right):=f\left(\varsigma_h\right),\qquad \varsigma\in\Gamma. \tag{3}$$

Using relation (1), it can be easily verified that, if Γ is Dini-smooth, then $L_M(\Gamma)$ is invariant under the shift T_hf.

We define the modulus of continuity for $f\in L_M(\Gamma)$ as:

$$\omega_M\left(\delta,f\right):=\sup_{|h|\leq\delta}\|f-T_hf\|_{L_M(\Gamma)},\qquad \delta\geq 0, \tag{4}$$

which satisfies the conditions

$$\omega_M\left(0,f\right)=0,$$
$$\omega_M\left(\delta,f\right)\geq 0\quad\text{for}\quad \delta>0,$$

$$\lim_{\delta\to 0}\omega_M(\delta,f)=0,$$

$$\omega_M(\delta,f+g)\leq\omega_M(\delta,f)+\omega_M(\delta,g)$$

for $f,g\in E_M(G)$.

For $f\in E_M(G)$ we put

$$E_n^M(f,G) := \inf\|f-p_n\|_{L_M(\Gamma)}$$

$$= \inf\left\{\sup\left\{\int_\Gamma|(f(\varsigma)-p_n(\varsigma))\,g(\varsigma)|\,|d\varsigma|\ ;\ \rho(g\,;N)\leq 1\right\}\right\}, \tag{5}$$

where inf is taken over the polynomials p_n of degree at most n.

In this work, we considered some problems of the polynomial approximation in Smirnov-Orlicz class $E_M(G)$. Our new results are the following.

THEOREM 1.3. *Let G be a finite simply connected domain with the Dini-smooth boundary Γ, and let $E_M(G)$ be a reflexive Smirnov-Orlicz space on G. Then for every $f\in E_M(G)$ and any natural number n there exists an algebraic polynomial $p_n(\cdot,f)$ of degree at most n such that*

$$\|f-p_n(\cdot,f)\|_{L_M(\Gamma)}\leq c\ \omega_M\left(\frac{1}{n},f\right)$$

with some constant c independent of n.

In the more general case, namely when Γ is a Carleson curve, applying the same method of summation, but using different modulus of continuity some direct theorem of approximation theory by polynomials in Smirnov-Orlicz class $E_M(G)$ is given in [8]. The modulus of continuity ω_M, used in this work, is simplier than the modulus of continuity considered in [8].

Similar problems for the spaces $L_p(\Gamma)$ and $E_p(G)\,, 1\leq p<\infty$, have been studied in [1, 2, 4, 5, 10, 11, 12, 15]. All these results were proved under different restrictive conditions on $\Gamma=\partial G$.

Some inverse problems of approximation theory in Smirnov-Orlicz classes have been investigated by Kokilashvili [14] in the case that Γ is Dini-smooth.

Now let K be a bounded continuum with the connected complement $D:=\overline{\mathbb{C}}\setminus K$ and let $f(z)$ be an analytic function on K. It is well known that [20, p. 199] the expansion

$$f(z)=\sum_{k=0}^{\infty}a_k\Phi_k(z)\,,\qquad z\in K, \tag{6}$$

with the Faber coefficients

$$a_k:=\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f(\psi(t))}{t^{k+1}}dt,\qquad k=0,1,2,\ldots,$$

converges absolutely and uniformly on K, where $\Phi_k(z)$, $k=0,1,2,\dots,$ are the Faber polynomials for K that satisfy the relation

$$\frac{\psi'(w)}{\psi(w)-z}=\sum_{k=0}^{\infty}\frac{\Phi_k(z)}{w^{k+1}}, \qquad z\in K,\ \ |w|>1. \tag{7}$$

The detailed information about the Faber polynomials and their approximation properties can be found in the monographs [8, 19, 20].

Let us introduce the value

$$R_n(z,f):=f(z)-\sum_{k=0}^{n}a_k\Phi_k(z)=\sum_{k=n+1}^{\infty}a_k\Phi_k(z), \qquad z\in K, \tag{8}$$

and put

$$\Gamma_R:=\{z\in D:|\varphi(z)|=R\ \}\ \text{ and } G_R:=Int\Gamma_R,\ R>1.$$

The following theorem characterizes the maximal convergence property of the Faber series (6) in the Smirnov-Orlicz space $E_M(G_R)$.

THEOREM 1.4. *If $f\in E_M(G_R)$, $R>1$, then*

$$|R_n(z,f)|\le\frac{c}{R^{n+1}(R-1)}E_n^M(f,G_R)\sqrt{n\ln n},\quad z\in K$$

with a constant $c>0$ independent of n and z.

From theorem 1.3 and 1.4, we have the following corollary.

COROLLARY 1.5. *Let K be a continuum with connected complement and let $E_M(G_R)$ be a reflexive Smirnov-Orlicz class on G_R, $R>1$. If $f\in E_M(G_R)$, then*

$$|R_n(z,f)|\le\frac{c}{R^{n+1}(R-1)}\omega_M\left(\frac{1}{n},f\right)\sqrt{n\ln n},\quad z\in K,$$

with $c>0$.

Theorem 1.4 in the Smirnov spaces $E_p(G)$, $p>1$, was proved in [20, p. 207].

We use $c,c_1,c_2,\dots$ to denote constants (which may, in general, differ in different relations) depending only on numbers that are not important for the question of interest.

2. Auxiliary Results

Let Γ be a rectifiable Jordan curve and $f\in L_1(\Gamma)$. The functions f^+ and f^- defined by

$$f^+(z)=\frac{1}{2\pi i}\int_\Gamma\frac{f(\varsigma)}{\varsigma-z}d\varsigma, \qquad z\in G, \tag{9}$$

and

$$f^{-}(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(\varsigma)}{\varsigma-z}d\varsigma,\qquad z\in G^{-},$$

are analytic in G and G^{-}, respectively and $f^{-}(\infty)=0$.

Let also

$$S_{\Gamma}f(z_0):=\lim_{\varepsilon\to 0}\int_{\Gamma\cap\{\varsigma:|\varsigma-z_0|\geq\varepsilon\}}\frac{f(\varsigma)}{\varsigma-z_0}d\varsigma,\qquad z_0\in\Gamma$$

be the Cauchy singular integral of $f\in L_1(\Gamma)$.

If one of the functions f^{+} or f^{-} has non-tangential limits a.e. on Γ, then $S_{\Gamma}f(z)$ exists a.e. on Γ and also the other one has non-tangential limits a.e. on Γ. Conversely, if $S_{\Gamma}f(z)$ exist a.e. on Γ, then both functions f^{+} and f^{-} have non-tangential limits a.e. on Γ. In both cases, the formulae

$$f^{+}(z)=S_{\Gamma}f(z)+\frac{1}{2}f(z) \tag{10}$$

$$f^{-}(z)=S_{\Gamma}f(z)-\frac{1}{2}f(z)$$

hold, and hence

$$f=f^{+}-f^{-}$$

a.e. on Γ [9, p. 431].

The linear operator $S_{\Gamma}:f\to S_{\Gamma}f$ is called the Cauchy singular operator.

For $z\in\Gamma$ and $\epsilon>0$, let $\Gamma(z,\epsilon)$ denote the portion of Γ which is inside the open disk of radius ϵ centered at z, i.e. $\Gamma(z,\epsilon):=\{t\in\Gamma:|t-z|<\epsilon\}$. Further, let $|\Gamma(z,\epsilon)|$ denote the length of $\Gamma(z,\epsilon)$. A rectifiable Jordan curve Γ is called a Carleson curve if

$$\sup_{\epsilon>0}\sup_{z\in\Gamma}\frac{1}{\epsilon}|\Gamma(z,\epsilon)|<\infty.$$

Theorem 2.1 ([13]). *Let Γ be a rectifiable Jordan curve and let $L_M(\Gamma)$ be a reflexive Orlicz space on Γ. Then the singular operator S_{Γ} is bounded on $L_M(\Gamma)$, i.e.*

$$\|S_{\Gamma}f\|_{L_M(\Gamma)}\leq c_3\|f\|_{L_M(\Gamma)}\ \text{ for all } f\in L_M(\Gamma) \tag{11}$$

for some constant $c_3>0$, if and only if Γ is a Carleson curve .

Theorem 2.2 ([16, p. 67]). *For every pair of real valued functions $u(z)\in L_M(\Gamma)$, $v(z)\in L_N(\Gamma)$ the inequality*

$$\int_{\Gamma}u(z)v(z)\,dz\leq\rho(u;M)+\rho(v;N) \tag{12}$$

holds.

Theorem 2.3 ([16, p. 74]). *For every pair of real valued functions* $u(z) \in L_M(\Gamma)$, $v(z) \in L_N(\Gamma)$ *the inequality*

$$\left| \int_{\Gamma} u(z)\, v(z)\, dz \right| \leq \|u\|_{L_M(\Gamma)} \|v\|_{L_N(\Gamma)} \tag{13}$$

holds.

The coefficients of the series in (6) are determined by the formulae

$$a_k := \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\varsigma)\, \varphi'(\varsigma)}{\varphi^{k+1}(\varsigma)} d\varsigma = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\psi(t))}{t^{k+1}} dt, \qquad k = 0, 1, 2, \ldots,$$

and hence the relation (8) implies that

$$R_n(z, f) = \frac{1}{2\pi i} \int_{\mathbb{T}} f(\psi(t)) \left[\sum_{k=n+1}^{\infty} \frac{\Phi_k(z)}{t^{k+1}} \right] dt.$$

If $p_n(z)$ is a polynomial of degree at most n, then

$$R_n(z, f) = \frac{1}{2\pi i} \int_{\mathbb{T}} \{ f(\psi(t)) - p_n(\psi(t)) \} \left[\sum_{k=n+1}^{\infty} \frac{\Phi_k(z)}{t^{k+1}} \right] dt. \tag{14}$$

Since

$$\Phi_k(z) = [\varphi(z)]^k + E_k(z), \qquad z \in K, \tag{15}$$

where $E_k(z)$ is analytic on the whole domain D and $E_k(\infty) = 0$, we have

$$\sum_{k=n+1}^{\infty} \frac{\Phi_k(z)}{t^{k+1}} = \sum_{k=n+1}^{\infty} \frac{[\varphi(z)]^k}{t^{k+1}} + \sum_{k=n+1}^{\infty} \frac{E_k(z)}{t^{k+1}}. \tag{16}$$

Hence from (15), taking into account (16), we get

$$|R_n(z, f)| \leq \frac{1}{2\pi} \int_{|t|=R} |f(\psi(t)) - p_n(\psi(t))| \left| \sum_{k=n+1}^{\infty} \frac{w^k}{t^{k+1}} \right| |dt|$$

$$+ \frac{1}{2\pi} \int_{|t|=R} |f(\psi(t)) - p_n(\psi(t))| \left| \sum_{k=n+1}^{\infty} E_k(\psi(w)) \frac{1}{t^{k+1}} \right| |dt|. \tag{17}$$

We shall also use the relations

$$E_k(\psi(w)) = \frac{1}{2\pi i} \int_{\Gamma} \tau^k F(\tau, w)\, d\tau, \qquad |w| \geq r > 1, \tag{18}$$

and

$$\frac{1}{2\pi i}\int_{\Gamma}|F(\tau,w)|\,|d\tau|\leq\sqrt{\frac{r^2}{r^4-1}\ln\frac{r^2}{r^2-1}},\qquad r>1,\ \ |w|\geq r>1, \tag{19}$$

given in [20, p. 63-205], where

$$F(\tau,w):=\frac{\psi'(\tau)}{\psi(\tau)-\psi(w)}-\frac{1}{\tau-w},\qquad |\tau|>1,\ \ |w|>1.$$

3. Proofs of Main Results

Proof of Theorem 1.3. Let $f\in L_M(\Gamma)$. Then by (2) we have $f\in L_1(\Gamma)$. Since Γ is Dini-smooth, we have $f\circ\psi\in L_1(\mathbb{T})$ and hence we can associate a formal series

$$\sum_{k=0}^{\infty}a_k w^k+\sum_{k=1}^{\infty}\frac{b_k}{w^k}$$

with the function $f\circ\psi\in L_1(\mathbb{T})$, i.e.

$$f(\psi(w))\backsim\sum_{k=0}^{\infty}a_k w^k+\sum_{k=1}^{\infty}\frac{b_k}{w^k}. \tag{20}$$

Let

$$K_n(\theta)=\sum_{m=-n}^{n}\lambda_m^{(n)}e^{im\theta}$$

be an even, nonnegative trigonometric polynomial satisfying the conditions

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}K_n(\theta)\,d\theta=1, \tag{21}$$

$$\int_{0}^{\pi}\theta K_n(\theta)\,d\theta\leq\frac{c_4}{n} \tag{22}$$

for every natural number n and with some constant $c_4>0$ (for example, the Jackson kernel

$$J_n(\theta):=\frac{3\left(\sin\frac{n\theta}{2}\right)^4}{n\left(2n^2+1\right)\left(\sin\frac{\theta}{2}\right)^4}$$

satisfies the above cited conditions, see [6, p. 203-204]).

Consider the integral

$$I(\theta,z):=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(\varsigma_{-\theta})}{\varsigma-z}d\varsigma,\qquad z\in G. \tag{23}$$

Using the change of variables $\varsigma=\psi\left(e^{it}\right)$, we obtain

$$I\left(\theta,z\right):=\frac{1}{2\pi i}\int\limits_{-\pi}^{\pi}f\left(\psi\left(e^{i(t-\theta)}\right)\right)\frac{\psi'\left(e^{it}\right)e^{it}}{\psi\left(e^{it}\right)-z}dt,$$

and taking into account the relations (20) and (7), we can write

$$I\left(\theta,z\right)\backsim\sum_{k=0}^{\infty}a_k\Phi_k\left(z\right)e^{-ik\theta}.$$

Since $I\left(\theta,z\right)\in L_1\left(\left[-\pi,\pi\right]\right)$ and $K_n\left(\theta\right)$ is of bounded variation, by the generalized Parseval identity [3, p. 225-228], we obtain

$$\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}K_n\left(\theta\right)I\left(\theta,z\right)d\theta=\sum_{k=0}^{n}\lambda_k^{(n)}a_k\Phi_k\left(z\right),$$

which together with (23) implies that

$$\frac{1}{4\pi^2 i}\int\limits_{-\pi}^{\pi}K_n\left(\theta\right)d\theta\int\limits_{\Gamma}\frac{f\left(\varsigma_{-\theta}\right)}{\varsigma-z}d\varsigma=\sum_{k=0}^{n}\lambda_k^{(n)}a_k\Phi_k\left(z\right),\qquad z\in G.$$

Hence we see that

$$P_n\left(z,f\right):=\frac{1}{4\pi^2 i}\int\limits_{-\pi}^{\pi}K_n\left(\theta\right)d\theta\int\limits_{\Gamma}\frac{f\left(\varsigma_{-\theta}\right)}{\varsigma-z}d\varsigma,\qquad z\in G,$$

is an algebraic polynomial of degree n.

Since the kernel $K_n\left(\theta\right)$ is an even function, we have

$$P_n\left(z,f\right)=\frac{1}{4\pi^2 i}\int\limits_{0}^{\pi}K_n\left(\theta\right)d\theta\int\limits_{\Gamma}\left[f\left(\varsigma_{\theta}\right)+f\left(\varsigma_{(-\theta)}\right)\right]\frac{d\varsigma}{\varsigma-z},$$

and by (3) and (9), we conclude that

$$\begin{aligned}P_n\left(z,f\right)&=\frac{1}{4\pi^2 i}\int\limits_{0}^{\pi}K_n\left(\theta\right)d\theta\int\limits_{\Gamma}\left[T_\theta f\left(\varsigma\right)+T_{(-\theta)}f\left(\varsigma\right)\right]\frac{d\varsigma}{\varsigma-z}\\&=\frac{1}{2\pi}\int\limits_{0}^{\pi}K_n\left(\theta\right)\left[\left(T_\theta f\right)^{+}\left(z\right)+\left(T_{(-\theta)}f\right)^{+}\left(z\right)\right]d\theta,\qquad z\in G.\end{aligned}$$

Now let $f\in E_M\left(G\right)$ and $z'\in G$. Multiplying both side of the equality

$$\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}K_n\left(\theta\right)d\theta=1$$

by $f^{+}(z')$ we get

$$f(z') = f^{+}(z') = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^{+}(z')\, K_n(\theta)\, d\theta = \frac{1}{2\pi}\int_{0}^{\pi} 2f^{+}(z')\, K_n(\theta)\, d\theta,$$

and hence

$$\begin{aligned} & f(z') - P_n(z', f) \\ & \quad = \frac{1}{2\pi}\int_{0}^{\pi} K_n(\theta)\left\{2f^{+}(z') - \left[(T_\theta f)^{+}(z') + \left(T_{(-\theta)} f\right)^{+}(z')\right]\right\} d\theta. \end{aligned}$$

Taking the limit $z' \to z \in \Gamma$ along all nontangential paths inside Γ and using (10), we obtain

$$\begin{aligned} f(z) - P_n(z, f) &= \frac{1}{2\pi}\int_{0}^{\pi} K_n(\theta)\, [2S_\Gamma f(z) + f(z) - S_\Gamma (T_\theta f)(z) - \frac{1}{2} T_\theta f(z) \\ & \quad - S_\Gamma \left(T_{(-\theta)} f\right)(z) - \frac{1}{2}\left(T_{(-\theta)} f\right)(z)] d\theta \\ &= \frac{1}{2\pi}\int_{0}^{\pi} K_n(\theta)\left[S_\Gamma (f - T_\theta f)(z) + S_\Gamma \left(f - T_{(-\theta)} f\right)(z)\right] d\theta \\ & \quad + \frac{1}{4\pi}\int_{0}^{\pi} K_n(\theta)\left[(f - T_\theta f)(z) + \left(f - T_{(-\theta)} f\right)(z)\right] d\theta \end{aligned}$$

for almost all $z \in \Gamma$.

Taking the supremum over all functions $g \in L_N(\Gamma)$ with $\rho(g; N) \leq 1$ in the last relation, we have

$$\begin{aligned} & \|f - P_n(\cdot, f)\|_{L_M(\Gamma)} = \sup \int_{\Gamma} |f(z) - P_n(z, f)|\, |g(z)|\, |dz| \\ & \leq \sup \int_{\Gamma} \left| \frac{1}{2\pi}\int_{0}^{\pi} K_n(\theta)\left[S_\Gamma (f - T_\theta f)(z) + S_\Gamma \left(f - T_{(-\theta)} f\right)(z)\right] d\theta \right| |g(z)|\, |dz| \\ & \quad + \sup \int_{\Gamma} \left| \frac{1}{4\pi}\int_{0}^{\pi} K_n(\theta)\left[(f - T_\theta f)(z) + \left(f - T_{(-\theta)} f\right)(z)\right] d\theta \right| |g(z)|\, |dz| \end{aligned}$$

$$\leq \sup \int_{\Gamma} \left\{ \frac{1}{2\pi} \int_{0}^{\pi} K_n(\theta) \left(|S_\Gamma (f - T_\theta f)(z)| + \left| S_\Gamma \left(f - T_{(-\theta)} f \right)(z) \right| \right) d\theta \right\} |g(z)| \, |dz|$$

$$+ \sup \int_{\Gamma} \left\{ \frac{1}{4\pi} \int_{0}^{\pi} K_n(\theta) \left[|(f - T_\theta f)(z)| + \left| \left(f - T_{(-\theta)} f \right)(z) \right| \right] d\theta \right\} |g(z)| \, |dz| ,$$

and by Fubini's theorem

$$\begin{aligned} &\|f - P_n(\cdot, f)\|_{L_M(\Gamma)} \\ &\leq \frac{1}{2\pi} \int_{0}^{\pi} K_n(\theta) \{ \sup \int_{\Gamma} [|S_\Gamma (f - T_\theta f)(z)| \\ &\quad + \left| S_\Gamma \left(f - T_{(-\theta)} f \right)(z) \right|] \, |g(z)| \, |dz| \} d\theta \\ &\quad + \frac{1}{4\pi} \int_{0}^{\pi} K_n(\theta) \left\{ \sup \int_{\Gamma} [|(f - T_\theta f)(z)| \right. \\ &\quad \left. + \left| \left(f - T_{(-\theta)} f \right)(z) \right|] \, |g(z)| \, |dz| \right\} d\theta \\ &\leq \frac{1}{2\pi} \int_{0}^{\pi} K_n(\theta) \left[\|S_\Gamma (f - T_\theta f)\|_{L_M(\Gamma)} + \left\| S_\Gamma \left(f - T_{(-\theta)} f \right) \right\|_{L_M(\Gamma)} \right] d\theta \\ &\quad + \frac{1}{4\pi} \int_{0}^{\pi} K_n(\theta) \left[\|f - T_\theta f\|_{L_M(\Gamma)} + \left\| f - T_{(-\theta)} f \right\|_{L_M(\Gamma)} \right] d\theta. \end{aligned}$$

Now applying (11), we get

$$\|f - P_n(\cdot, f)\|_{L_M(\Gamma)} \leq c_5 \int_{0}^{\pi} K_n(\theta) \left[\|f - T_\theta f\|_{L_M(\Gamma)} + \left\| f - T_{(-\theta)} f \right\|_{L_M(\Gamma)} \right] d\theta,$$

and recalling the definition (4) of $\omega_M(\delta, f)$, we obtain

$$\begin{aligned} \|f - P_n(\cdot, f)\|_{L_M(\Gamma)} &\leq c_6 \int_{0}^{\pi} K_n(\theta) \, \omega_M(\theta, f) \, d\theta \\ &\leq c_7 \, \omega_M\left(\frac{1}{n}, f\right) \int_{0}^{\pi} K_n(\theta) (n\theta + 1) \, d\theta. \end{aligned}$$

Consequently from (21) and (22), we have

$$\left\| f-P_n\left(\cdot,f\right)\right\|_{L_M(\Gamma)}\leq c_8\ \omega_M(\frac{1}{n},f),$$

which proves Theorem 1.3. ∎

Proof of Theorem 1.4. Let $z\in\Gamma_r, 1<r<R$ and $p_n\left(z\right)$ be the best approximating polynomial of degree at most n to the function $f\in E_M\left(G_R\right)$. Denoting

$$\begin{aligned} I_1 &:= \frac{1}{2\pi}\int\limits_{|t|=R}\left|f\left(\psi\left(t\right)\right)-p_n\left(\psi\left(t\right)\right)\right|\left|\sum_{k=n+1}^{\infty}\frac{w^k}{t^{k+1}}\right|\left|dt\right|, \\ I_2 &:= \frac{1}{2\pi}\int\limits_{|t|=R}\left|f\left(\psi\left(t\right)\right)-p_n\left(\psi\left(t\right)\right)\right|\left|\sum_{k=n+1}^{\infty}E_k\left(\psi\left(w\right)\right)\frac{1}{t^{k+1}}\right|\left|dt\right|, \end{aligned}$$

by virtue of (17), we see that

$$\left|R_n\left(z,f\right)\right|\leq I_1+I_2. \tag{24}$$

Using relations (1) and (13), we have

$$\begin{aligned} I_1 &= \frac{1}{2\pi}\int\limits_{\Gamma_R}\left|f\left(\varsigma\right)-p_n\left(\varsigma\right)\right|\left|\sum_{k=n+1}^{\infty}\frac{\left[\varphi\left(z\right)\right]^k}{\left[\varphi\left(\varsigma\right)\right]^{k+1}}\right|\left|\varphi'\left(\varsigma\right)\right|\left|d\varsigma\right| \\ &\leq \frac{c_9}{2\pi}\int\limits_{\Gamma_R}\left|f\left(\varsigma\right)-p_n\left(\varsigma\right)\right|\left|\sum_{k=n+1}^{\infty}\frac{\left[\varphi\left(z\right)\right]^k}{\left[\varphi\left(\varsigma\right)\right]^{k+1}}\right|\left|d\varsigma\right| \\ &\leq \frac{c_9}{2\pi}\left\{\sup\int\limits_{\Gamma_R}\left|f\left(\varsigma\right)-p_n\left(\varsigma\right)\right|\left|g\left(\varsigma\right)\right|\left|d\varsigma\right|\right\}\cdot \\ &\qquad\cdot\left\{\sup\int\limits_{\Gamma_R}\left|\sum_{k=n+1}^{\infty}\frac{\left[\varphi\left(z\right)\right]^k}{\left[\varphi\left(\varsigma\right)\right]^{k+1}}\right|\left|h\left(\varsigma\right)\right|\left|d\varsigma\right|\right\}, \end{aligned}$$

where the suprema are taken over all functions $g\in L_N\left(\Gamma\right)$ with $\rho\left(g;N\right)\leq 1$ and $h\in L_M\left(\Gamma\right)$ with $\rho\left(h;M\right)\leq 1$, respectively. By virtue of (5)

$$I_1\leq\frac{c_{10}E_n^M\left(f,G_R\right)}{2\pi}\sup\left\{\int\limits_{\Gamma_R}\left|\sum_{k=n+1}^{\infty}\frac{\left[\varphi\left(z\right)\right]^k}{\left[\varphi\left(\varsigma\right)\right]^{k+1}}\right|\left|h\left(\varsigma\right)\right|\left|d\varsigma\right|;\rho\left(h;M\right)\leq 1\right\}$$

$$\leq\frac{c_{10}E_n^M\left(f,G_R\right)}{2\pi}\sup\left\{\int\limits_{\Gamma_R}\frac{\left|\varphi\left(z\right)\right|^{n+1}}{\left|\varphi\left(\varsigma\right)\right|^{n+1}\left|\varphi\left(\varsigma\right)-\varphi\left(z\right)\right|}\left|h\left(\varsigma\right)\right|\left|d\varsigma\right|;\rho\left(h;M\right)\leq 1\right\}$$

$$\leq \frac{c_{11}E_n^M(f,G_R)}{2\pi}\cdot\frac{r^{n+1}}{R^{n+1}(R-r)}\cdot\sup\left\{\int_{\Gamma_R}|h(\varsigma)|\,|d\varsigma|\,;\rho(h;M)\leq 1\right\},$$

and by (12)

$$\sup\left\{\int_{\Gamma_R}|h(\varsigma)|\,|d\varsigma|\,;\rho(h;M)\leq 1\right\}\leq 1+N(1)\,mes\Gamma_R\leq c_{12}, \tag{25}$$

and therefore

$$I_1\leq\frac{c_{13}E_n^M(f,G_R)\,r^{n+1}}{2\pi R^{n+1}(R-r)}. \tag{26}$$

Now, we estimate the integral I_2. By (18) we have

$$I_2=\frac{1}{2\pi}\int_{|t|=R}|f(\psi(t))-p_n(\psi(t))|\left|\sum_{k=n+1}^{\infty}\frac{1}{2\pi}\int_{|\tau|=r}\frac{\tau^k}{t^{k+1}}F(\tau,w)\,d\tau\right||dt|$$

$$\leq\frac{1}{2\pi}\int_{|t|=R}|f(\psi(t))-p_n(\psi(t))|\left\{\frac{1}{2\pi}\int_{|\tau|=r}\left|\sum_{k=n+1}^{\infty}\frac{\tau^k}{t^{k+1}}\right||F(\tau,w)|\,|d\tau|\right\}|dt|$$

$$\leq\frac{1}{2\pi}\int_{|t|=R}|f(\psi(t))-p_n(\psi(t))|\left\{\frac{1}{2\pi}\int_{|\tau|=r}\left|\frac{\tau^{n+1}}{t^{n+1}(t-\tau)}\right||F(\tau,w)|\,|d\tau|\right\}|dt|\,.$$

Applying Fubini's theorem

$$I_2\leq\frac{r^{n+1}}{2\pi R^{n+1}}\int_{|\tau|=r}|F(\tau,w)|\left\{\frac{1}{2\pi}\int_{|t|=R}|f(\psi(t))-p_n(\psi(t))|\,\frac{|dt|}{|t-\tau|}\right\}|d\tau|$$

and changing the variables in the last integral and using (13), we have

$$
\begin{aligned}
I_2 &\leq \frac{r^{n+1}}{2\pi R^{n+1}} \int\limits_{|\tau|=r} |F(\tau,w)| \left\{ \frac{1}{2\pi} \int\limits_{\Gamma_R} |f(\varsigma) - p_n(\varsigma)| \frac{|\varphi'(\varsigma)|}{|\varphi(\varsigma) - \varphi(z)|} |d\varsigma| \right\} |d\tau| \\
&\leq \frac{r^{n+1}}{4\pi^2 R^{n+1}} \int\limits_{|\tau|=r} |F(\tau,w)| \left\{ \|f(\varsigma) - p_n(\varsigma)\|_{L_M(\Gamma_R)} \times \right. \\
&\qquad\qquad \left. \times \left\| \frac{\varphi'(\cdot)}{\varphi(\cdot) - \varphi(z)} \right\|_{L_N(\Gamma_R)} \right\} |d\tau| \\
&\leq \frac{c_{14}\ r^{n+1}}{4\pi^2 R^{n+1}(R-r)} \int\limits_{|\tau|=r} |F(\tau,w)| \left[E_n^M(f,G_R) \right. \\
&\qquad\qquad \left. \sup \left\{ \int\limits_{\Gamma_R} |H(\varsigma)| |d\varsigma| ; \rho(H;N) \leq 1 \right\} \right] |d\tau| .
\end{aligned}
$$

From this, by repeating the arguments given in (25) and using (19), we conclude that

$$
I_2 \leq \frac{c_{15}\ r^{n+1}}{2\pi R^{n+1}(R-r)} E_n^M(f,G_R) \sqrt{\frac{r^2}{r^4-1} \ln \frac{r^2}{r^2-1}}. \tag{27}
$$

Now, the inequalities (26), (27) and (24) imply that

$$
|R_n(z,f)| \leq \frac{c_{16}\ r^{n+1} E_n^M(f,G_R)}{2\pi R^{n+1}(R-r)} \sqrt{\frac{r^2}{r^4-1} \ln \frac{r^2}{r^2-1}}.
$$

Consequently, setting $z \in K$ and $r := 1 + \frac{1}{n}$ in this estimate, we obtain the inequality

$$
|R_n(z,f)| \leq \frac{c_{17}}{R^{n+1}(R-1)}\ E_n^M(f,G_R) \sqrt{n \ln n},
$$

with $c_{17} > 0$. □

Acknowledgements.
The authors are indepted to the referees for valuable suggestions.

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Received: 7.7.2004.
Revised: 27.10.2004.