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Meridian surfaces of elliptic or hyperbolic type in the four-dimensional Minkowski space*

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Abstract. Meridian surfaces of elliptic or hyperbolic type are one-parameter systems of meridians of the rotational hypersurface with timelike or spacelike axis, respectively. We classify the meridian surfaces with constant Gauss curvature or constant mean curvature, as well as the Chen meridian surfaces and the meridian surfaces with parallel normal bundle.

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1. Introduction

One of the fundamental problems of contemporary differential geometry of surfaces and hypersurfaces in standard model spaces such as the Euclidean space $\mathbb{R}^n$ and the pseudo-Euclidean space $\mathbb{R}^n_k$ is the investigation of the basic invariants characterizing the surfaces. Our aim is to study and classify various important classes of surfaces in the four-dimensional Minkowski space $\mathbb{R}^4_1$ characterized by conditions on their invariants.

Surfaces with codimension two in the Euclidean space $\mathbb{R}^4$ have been studied in [14] on the base of invariant functions and invariant figures in the tangent or normal space of the surface. Further, differential geometry of surfaces in the Euclidean space $\mathbb{R}^4$ or $\mathbb{R}^n$ was developed on the basis of second order invariants and the corresponding curvature ellipses by several authors (see e.g. [1, 19, 21]).

An invariant theory of spacelike surfaces in $\mathbb{R}^4_1$ was developed by the present authors in [8]. We introduced an invariant linear map γ of Weingarten type in the tangent plane at any point of the surface, which generates two invariant functions $k = \det \gamma$ and $\varkappa = -\dfrac{1}{2}\,\mathrm{tr}\gamma$. On the basis of the map γ we introduced principal lines and a geometrically determined moving frame field at each point of the surface. Writing derivative formulas of Frenet type for this frame field, we obtained eight

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invariant functions γ_1, γ_2, ν_1, ν_2, λ, μ, β_1, β_2 and proved a fundamental theorem of Bonnet type, stating that under some natural conditions these eight invariants determine the surface up to a rigid motion in $\mathbb{R}^4_1$.

The basic geometric classes of surfaces in $\mathbb{R}^4_1$ are characterized by conditions on these invariant functions. For example, surfaces with flat normal connection are characterized by the condition $\nu_1 = \nu_2$, minimal surfaces are described by $\nu_1 + \nu_2 = 0$, Chen surfaces are characterized by $\lambda = 0$, and surfaces with the parallel normal bundle are characterized by the condition $\beta_1 = \beta_2 = 0$.

In [7], we constructed special two-dimensional surfaces in the Euclidean 4-space $\mathbb{R}^4$, which are one-parameter systems of meridians of the rotational hypersurface and called these surfaces *meridian surfaces*. We classified meridian surfaces with constant Gauss curvature, constant mean curvature, and constant invariant k [7].

Similarly to the Euclidean case, in [9], we constructed two-dimensional spacelike surfaces in the Minkowski 4-space $\mathbb{R}^4_1$ which are one-parameter systems of meridians of the rotational hypersurface with a timelike or a spacelike axis. We called these surfaces *meridian surfaces of elliptic type* and *meridian surfaces of hyperbolic type*, respectively. Geometric construction of meridian surfaces is different from construction of standard rotational surfaces with a two-dimensional axis. Hence, the class of meridian surfaces is a new source of examples of two-dimensional surfaces in $\mathbb{R}^4_1$. In [9], we found all marginally trapped meridian surfaces of elliptic or hyperbolic type.

In [10], we continued with the study of meridian surfaces in $\mathbb{R}^4_1$ considering a rotational hypersurface with a lightlike axis and constructed two-dimensional surfaces which are one-parameter systems of meridians of the rotational hypersurface. We called these surfaces *meridian surfaces of parabolic type*. We calculated their basic invariants and found all marginally trapped meridian surfaces of parabolic type.

In the present paper, we consider meridian surfaces of elliptic or hyperbolic type in $\mathbb{R}^4_1$ and calculate the invariants γ_1, γ_2, ν_1, ν_2, λ, μ, β_1, β_2 of these surfaces. Using the invariants we describe and classify completely meridian surfaces of elliptic or hyperbolic type with constant Gauss curvature (Theorem 1), constant mean curvature (Theorem 2), and constant invariant k (Theorem 3). In Theorem 4 we classify Chen meridian surfaces and in Theorem 5 we give the classification of meridian surfaces with the parallel normal bundle.

2. Preliminaries

Let $\mathbb{R}^4_1$ be the four-dimensional Minkowski space endowed with the metric $\langle , \rangle$ of signature $(3,1)$ and let $Oe_1e_2e_3e_4$ be a fixed orthonormal coordinate system, i.e., $\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = \langle e_3, e_3 \rangle = 1$, $\langle e_4, e_4 \rangle = -1$. A surface $M^2 : z = z(u,v),\ (u,v) \in \mathcal{D}$ $(\mathcal{D} \subset \mathbb{R}^2)$ in $\mathbb{R}^4_1$ is said to be *spacelike* if $\langle , \rangle$ induces a Riemannian metric g on M^2. Thus at each point p of a spacelike surface M^2 we have the following decomposition:

$$\mathbb{R}^4_1 = T_pM^2 \oplus N_pM^2$$

with the property that the restriction of the metric $\langle , \rangle$ onto the tangent space T_pM^2 is of signature $(2,0)$, and the restriction of the metric $\langle , \rangle$ onto the normal space N_pM^2 is of signature $(1,1)$.

Denote by ∇' and ∇ the Levi Civita connections on $\mathbb{R}^4_1$ and M^2, respectively. Let x and y be vector fields tangent to M^2 and let ξ be a normal vector field. The formulas of Gauss and Weingarten give the decompositions of vector fields $\nabla'_x y$ and $\nabla'_x \xi$ into tangent and normal components:

$$\begin{array}{l}
\nabla'_x y = \nabla_x y + \sigma(x,y);\\
\nabla'_x \xi = -A_\xi x + D_x \xi,
\end{array}$$

which define the second fundamental tensor σ, the normal connection D and the shape operator A_ξ with respect to ξ. The mean curvature vector field H of M^2 is defined as $H = \dfrac{1}{2}\,\mathrm{tr}\,\sigma$. Basic facts and definitions concerning the theory of spacelike and timelike surfaces in Lorentz-Minkowski space can be found, for example, in [18].

Let $M^2 : z = z(u,v),\ (u,v) \in \mathcal{D}\ (\mathcal{D} \subset \mathbb{R}^2)$ be a local parametrization on a spacelike surface in $\mathbb{R}^4_1$. The tangent space at an arbitrary point $p = z(u,v)$ of M^2 is $T_pM^2 = \mathrm{span}\{z_u, z_v\}$, where $\langle z_u, z_u\rangle > 0$, $\langle z_v, z_v\rangle > 0$. We use standard denotations $E(u,v) = \langle z_u, z_u\rangle,\ F(u,v) = \langle z_u, z_v\rangle,\ G(u,v) = \langle z_v, z_v\rangle$ for coefficients of the first fundamental form and denote $W = \sqrt{EG - F^2}$. Let $\{n_1, n_2\}$ be a normal frame field of M^2 such that $\langle n_1, n_1\rangle = 1$, $\langle n_2, n_2\rangle = -1$, and the quadruple $\{z_u, z_v, n_1, n_2\}$ is positively oriented in $\mathbb{R}^4_1$. The coefficients of the second fundamental form II of the surface M^2 are given by the following functions

$$L = \frac{2}{W}\begin{vmatrix} c_{11}^1 & c_{12}^1 \\ c_{11}^2 & c_{12}^2 \end{vmatrix}; \quad M = \frac{1}{W}\begin{vmatrix} c_{11}^1 & c_{22}^1 \\ c_{11}^2 & c_{22}^2 \end{vmatrix}; \quad N = \frac{2}{W}\begin{vmatrix} c_{12}^1 & c_{22}^1 \\ c_{12}^2 & c_{22}^2 \end{vmatrix},$$

where

$$\begin{array}{lll}
c_{11}^1 = \langle z_{uu}, n_1\rangle; & c_{12}^1 = \langle z_{uv}, n_1\rangle; & c_{22}^1 = \langle z_{vv}, n_1\rangle;\\
c_{11}^2 = \langle z_{uu}, n_2\rangle; & c_{12}^2 = \langle z_{uv}, n_2\rangle; & c_{22}^2 = \langle z_{vv}, n_2\rangle.
\end{array}$$

Note that the second fundamental form II is well defined and invariant up to the orientation of the tangent space. It is invariant up to the orientation of the normal space, as well. The second fundamental form II is globally defined on an oriented surface.

The condition $L = M = N = 0$ characterizes points at which the space $\{\sigma(x,y) : x, y \in T_pM^2\}$ is one-dimensional. We call such points *flat points* of the surface. These points are analogous to flat points in the theory of surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$ [7]. In [8], we gave a local geometric description of spacelike surfaces consisting of flat points proving that any spacelike surface consisting of flat points whose mean curvature vector at any point is a non-zero spacelike vector or timelike vector either lies in a hyperplane of $\mathbb{R}^4_1$ or is part of a developable ruled surface in $\mathbb{R}^4_1$. Furthermore we consider surfaces free of flat points, i.e., $(L, M, N) \neq (0,0,0)$.

Using functions L, M, N and E, F, G in [8] we introduced a linear map γ of Weingarten type in the tangent space at any point of M^2. The map γ is invariant with respect to changes of parameters on M^2 as well as motions in $\mathbb{R}^4_1$. It generates two invariant functions

$$k = \frac{LN - M^2}{EG - F^2}, \qquad \varkappa = \frac{EN + GL - 2FM}{2(EG - F^2)}.$$

It turns out that the invariant $\varkappa$ is the curvature of the normal connection of the surface (see [8]). As in the theory of surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$ the invariant k divides the points of M^2 into the following types: *elliptic* ($k > 0$), *parabolic* ($k = 0$), and *hyperbolic* ($k < 0$).

The second fundamental form II determines conjugate, asymptotic, and principal tangents at a point p of M^2 in a standard way. A line $c : u = u(q),\ v = v(q);\ q \in J \subset \mathbb{R}$ on M^2 is said to be an *asymptotic line*, or a *principal line* if its tangent at any point is asymptotic, or principal. The surface M^2 is parameterized by principal lines if and only if $F = 0,\ M = 0$.

Spacelike surfaces in $\mathbb{R}^4_1$ whose normal mean curvature vector is lightlike at each point are called marginally trapped. An invariant theory of such surfaces was developed in [9]. Recently, the lightlike geometry of marginally trapped surfaces in $\mathbb{R}^4_1$ was treated in [13].

Considering spacelike surfaces in $\mathbb{R}^4_1$ whose mean curvature vector at any point is a non-zero spacelike vector or timelike vector, on the basis of the principal lines we introduced a geometrically determined orthonormal frame field $\{x, y, b, l\}$ at each point of such surface [8]. The tangent vector fields x and y are collinear with the principal directions, and the normal vector field b is collinear with the mean curvature vector field H. Writing derivative formulas of Frenet type for this frame field, we obtained eight invariant functions γ_1, γ_2, ν_1, ν_2, λ, μ, β_1, β_2, which determine the surface up to a rigid motion in $\mathbb{R}^4_1$.

The invariants γ_1, γ_2, ν_1, ν_2, λ, μ, β_1, and β_2 are determined by the geometric frame field $\{x, y, b, l\}$ as follows:

$$\begin{array}{llll}
\nu_1 = \langle \nabla'_x x, b\rangle, & \nu_2 = \langle \nabla'_y y, b\rangle, & \lambda = \langle \nabla'_x y, b\rangle, & \mu = \langle \nabla'_x y, l\rangle,\\
\gamma_1 = \langle \nabla'_x x, y\rangle, & \gamma_2 = \langle \nabla'_y y, x\rangle, & \beta_1 = \langle \nabla'_x b, l\rangle, & \beta_2 = \langle \nabla'_y b, l\rangle.
\end{array} \tag{1}$$

The invariants k, $\varkappa$, and the Gauss curvature K of M^2 are expressed by the functions $\nu_1, \nu_2, \lambda, \mu$ as follows:

$$k = -4\nu_1\,\nu_2\,\mu^2, \qquad \varkappa = (\nu_1 - \nu_2)\mu, \qquad K = \varepsilon(\nu_1\,\nu_2 - \lambda^2 + \mu^2),$$

where $\varepsilon = sign\langle H, H\rangle$. The norm $\|H\|$ of the mean curvature vector is expressed as

$$\|H\| = \frac{|\nu_1 + \nu_2|}{2} = \frac{\sqrt{\varkappa^2 - k}}{2|\mu|}.$$

If M^2 is a spacelike surface whose mean curvature vector at any point is a non-zero spacelike vector or timelike vector, then M^2 is minimal if and only if $\nu_1 + \nu_2 = 0$.

The geometric meaning of the invariant λ is connected with the notion of Chen submanifolds. Let M be an n-dimensional submanifold of an $(n + m)$-dimensional Riemannian manifold $\widetilde{M}$ and let ξ be a normal vector field of M. B.-Y. Chen [4] defined the *allied vector field* $a(\xi)$ of ξ by the formula

$$a(\xi) = \frac{\|\xi\|}{n}\sum_{k=2}^{m}\{\mathrm{tr}(A_1 A_k)\}\xi_k,$$

where $\{\xi_1 = \dfrac{\xi}{\|\xi\|}, \xi_2, \dots, \xi_m\}$ is an orthonormal base of the normal space of M, and $A_i = A_{\xi_i},\ i = 1, \dots, m$ is the shape operator with respect to ξ_i. The allied vector field $a(H)$ of the mean curvature vector field H is called the *allied mean curvature vector field* of M in $\widetilde{M}$. B.-Y. Chen defined $\mathcal{A}$-submanifolds to be those submanifolds of $\widetilde{M}$ for which $a(H)$ vanishes identically [4]. In [11, 12] $\mathcal{A}$-submanifolds are called *Chen submanifolds*. It is easy to see that minimal submanifolds, pseudo-umbilical submanifolds and hypersurfaces are Chen submanifolds. These Chen submanifolds are said to be trivial Chen submanifolds. In [8], we showed that if M^2 is a spacelike surface in $\mathbb{R}^4_1$ with a spacelike or a timelike mean curvature vector field, then the allied mean curvature vector field of M^2 is

$$a(H) = \frac{\sqrt{\varkappa^2 - k}}{2}\,\lambda\, l.$$

Hence, if M^2 is free of minimal points, then $a(H) = 0$ if and only if $\lambda = 0$. This gives the geometric meaning of the invariant λ: M^2 is a non-trivial Chen surface if and only if the invariant λ is zero.

Now we shall discuss the geometric meaning of invariants β_1 and β_2. It follows from (1) that

$$\begin{array}{ll} \nabla'_x b = -\nu_1\, x - \lambda\, y - \beta_1\, l; & \qquad \nabla'_x l = -\mu\, y - \beta_1\, b; \\ \nabla'_y b = -\lambda\, x - \nu_2\, y - \beta_2\, l; & \qquad \nabla'_y l = -\mu\, x - \beta_2\, b. \end{array}$$

Hence, $\beta_1 = \beta_2 = 0$ if and only if $D_x b = D_y b = 0$ (or equivalently, $D_x l = D_y l = 0$).

A normal vector field ξ is said to be *parallel in the normal bundle* (or simply *parallel*) [5] if $D_x \xi = 0$ holds identically for any tangent vector field x. Hence, invariants β_1 and β_2 are identically zero if and only if geometric normal vector fields b and l are parallel in the normal bundle.

Surfaces admitting a geometric normal frame field $\{b, l\}$ of parallel normal vector fields, are called *surfaces with the parallel normal bundle*. They are characterized by the condition $\beta_1 = \beta_2 = 0$. Note that if M^2 is a surface free of minimal points with a parallel mean curvature vector field (i.e., $DH = 0$), then M^2 is a surface with the parallel normal bundle, but the converse is not true in general. It is true only in the case $\|H\| = const$.

3. Invariants of meridian surfaces of elliptic or hyperbolic type

In [7], we constructed a family of surfaces lying on a standard rotational hypersurface in the four-dimensional Euclidean space $\mathbb{R}^4$. These surfaces are one-parameter systems of meridians of the rotational hypersurface, that is why we called them *meridian surfaces*. In [9], we used the idea from the Euclidean case to construct special families of two-dimensional spacelike surfaces lying on rotational hypersurfaces in $\mathbb{R}^4_1$ with a timelike or a spacelike axis. The construction was as follows.

Let $f = f(u),\ g = g(u)$ be smooth functions defined in an interval $I \subset \mathbb{R}$, such that $\dot{f}^2(u) - \dot{g}^2(u) > 0,\ u \in I$. We assume that $f(u) > 0,\ u \in I$. The standard rotational hypersurface $\mathcal{M}'$ in $\mathbb{R}^4_1$ obtained by the rotation of the meridian curve $m : u \rightarrow (f(u), g(u))$ about the Oe_4-axis is parameterized as follows:

$$\mathcal{M}' : Z(u, w^1, w^2) = f(u) \left(\cos w^1 \cos w^2 e_1 + \cos w^1 \sin w^2 e_2 + \sin w^1 e_3\right) + g(u) e_4.$$

The rotational hypersurface $\mathcal{M}'$ is a two-parameter system of meridians. Let $w^1 = w^1(v),\ w^2 = w^2(v),\ v \in J,\ J \subset \mathbb{R}$. We consider the two-dimensional surface $\mathcal{M}'_m$ lying on $\mathcal{M}'$ constructed in the following way:

$$\mathcal{M}'_m : z(u, v) = Z(u, w^1(v), w^2(v)), \quad u \in I,\, v \in J.$$

$\mathcal{M}'_m$ is a one-parameter system of meridians of $\mathcal{M}'$. We call $\mathcal{M}'_m$ a *meridian surface of elliptic type*.

If we denote $l(w^1, w^2) = \cos w^1 \cos w^2\, e_1 + \cos w^1 \sin w^2\, e_2 + \sin w^1\, e_3$, then the surface $\mathcal{M}'_m$ is parameterized by

$$\mathcal{M}'_m : z(u, v) = f(u)\, l(v) + g(u)\, e_4, \quad u \in I,\, v \in J. \tag{2}$$

Note that $l(w^1, w^2)$ is a unit position vector of the 2-dimensional sphere $S^2(1)$ lying in the Euclidean space $\mathbb{R}^3 = \operatorname{span}\{e_1, e_2, e_3\}$ and centered at the origin O.

In a similar way, we consider meridian surfaces lying on the rotational hypersurface in $\mathbb{R}^4_1$ with a spacelike axis. Let $f = f(u),\ g = g(u)$ be smooth functions defined in an interval $I \subset \mathbb{R}$ such that $\dot{f}^2(u) + \dot{g}^2(u) > 0,\ f(u) > 0,\ u \in I$. The rotational hypersurface $\mathcal{M}''$ in $\mathbb{R}^4_1$ obtained by the rotation of the meridian curve $m : u \rightarrow (f(u), g(u))$ about the Oe_1-axis is parameterized as follows:

$$\mathcal{M}'' : Z(u, w^1, w^2) = g(u) e_1 + f(u) \left(\cosh w^1 \cos w^2 e_2 + \cosh w^1 \sin w^2 e_3 + \sinh w^1 e_4\right).$$

If $w^1 = w^1(v),\ w^2 = w^2(v),\ v \in J,\ J \subset \mathbb{R}$, we construct a surface $\mathcal{M}''_m$ in $\mathbb{R}^4_1$ in the following way:

$$\mathcal{M}''_m : z(u, v) = Z(u, w^1(v), w^2(v)), \quad u \in I,\, v \in J.$$

$\mathcal{M}''_m$ is a one-parameter system of meridians of $\mathcal{M}''$. We call $\mathcal{M}''_m$ a *meridian surface of hyperbolic type*.

If we denote $l(w^1, w^2) = \cosh w^1 \cos w^2\, e_2 + \cosh w^1 \sin w^2\, e_3 + \sinh w^1\, e_4$, then the surface $\mathcal{M}''_m$ is given by

$$\mathcal{M}''_m : z(u, v) = f(u)\, l(v) + g(u)\, e_1, \quad u \in I,\, v \in J, \tag{3}$$

$l(w^1, w^2)$ being the unit position vector of the de Sitter space $S^2_1(1)$ in the Minkowski space $\mathbb{R}^3_1 = \operatorname{span}\{e_2, e_3, e_4\}$, i.e. $S^2_1(1) = \{V \in \mathbb{R}^3_1 : \langle V, V \rangle = 1\}$.

In [9], we found all marginally trapped meridian surfaces of elliptic or hyperbolic type. In the present section, we shall find geometric invariant functions $\gamma_1,\ \gamma_2,\ \nu_1,\ \nu_2,\ \lambda,\ \mu,\ \beta_1, \beta_2$ of meridian surfaces of elliptic or hyperbolic type.

Elliptic case:

First, we consider the surface $\mathcal{M}'_m$ parameterized by (2). We assume that the smooth curve $c : l = l(v) = l(w^1(v), w^2(v)),\ v \in J$ on $S^2(1)$ is parameterized by the arc-length, i.e., $\langle l'(v), l'(v)\rangle = 1$. Let $t(v) = l'(v)$ be the tangent vector field of c. Since $\langle t(v), t(v)\rangle = 1$, $\langle l(v), l(v)\rangle = 1$, and $\langle t(v), l(v)\rangle = 0$, there exists a unique (up to a sign) vector field $n(v)$, such that $\{l(v), t(v), n(v)\}$ is an orthonormal frame field in $\mathbb{R}^3$. With respect to this frame field, we have the following Frenet formulas of c on $S^2(1)$:

$$\begin{aligned} l' &= t; \\ t' &= \kappa\, n - l; \\ n' &= -\kappa\, t, \end{aligned} \tag{4}$$

where $\kappa(v) = \langle t'(v), n(v)\rangle$ is the spherical curvature of c.

Without loss of generality, we assume that $\dot{f}^2(u) - \dot{g}^2(u) = 1$. The tangent space of $\mathcal{M}'_m$ is spanned by the vector fields:

$$z_u = \dot{f}\, l + \dot{g}\, e_4; \qquad z_v = f\, t,$$

so the coefficients of the first fundamental form of $\mathcal{M}'_m$ are $E = 1$; $F = 0$; $G = f^2(u) > 0$. Hence, the first fundamental form is positive definite, i.e., $\mathcal{M}'_m$ is a spacelike surface.

Denote $X = z_u$, $Y = \dfrac{z_v}{f} = t$ and consider the following orthonormal normal frame field:

$$n_1 = n(v); \qquad n_2 = \dot{g}(u)\, l(v) + \dot{f}(u)\, e_4.$$

Thus we obtain a frame field $\{X, Y, n_1, n_2\}$ of $\mathcal{M}'_m$, such that $\langle n_1, n_1\rangle = 1$, $\langle n_2, n_2\rangle = -1$, $\langle n_1, n_2\rangle = 0$.

Taking into account (4) we get the following derivative formulas:

$$\begin{array}{ll} \nabla'_X X = \qquad\qquad\qquad \kappa_m\, n_2; & \nabla'_X n_1 = 0; \\ \nabla'_X Y = 0; & \nabla'_Y n_1 = \qquad -\dfrac{\kappa}{f}\, Y; \\ \nabla'_Y X = \qquad \dfrac{\dot{f}}{f}\, Y; & \nabla'_X n_2 = \kappa_m\, X; \\ \nabla'_Y Y = -\dfrac{\dot{f}}{f}\, X \qquad + \dfrac{\kappa}{f}\, n_1 + \dfrac{\dot{g}}{f}\, n_2; & \nabla'_Y n_2 = \qquad \dfrac{\dot{g}}{f}\, Y, \end{array} \tag{5}$$

where κ_m denotes the curvature of the meridian curve m, i.e., $\kappa_m(u) = \dot{f}(u)\ddot{g}(u) - \dot{g}(u)\ddot{f}(u)$.

The invariants k, $\varkappa$, and the Gauss curvature K are given by the following formulas [9]:

$$k = -\frac{\kappa_m^2(u)\, \kappa^2(v)}{f^2(u)}; \qquad \varkappa = 0; \qquad K = -\frac{\ddot{f}(u)}{f(u)}.$$

The equality $\varkappa = 0$ implies the following statement.

Proposition 1. *The meridian surface of elliptic type $\mathcal{M}'_m$ defined by* (2) *is a surface with flat normal connection.*

We distinguish the following three cases:

I. $\kappa(v) = 0$, i.e., the curve c is a great circle on $S^2(1)$. In this case, $n_1 = const$, and $\mathcal{M}'_m$ is a planar surface lying in the constant 3-dimensional space spanned by $\{X, Y, n_2\}$.

II. $\kappa_m(u) = 0$, i.e., the meridian curve m is part of a straight line. In such case, $k = \varkappa = K = 0$, and $\mathcal{M}'_m$ is a developable ruled surface.

III. $\kappa_m(u)\,\kappa(v) \neq 0$, i.e., c is not a great circle on $S^2(1)$ and m is not a straight line.

In the first two cases the surface $\mathcal{M}'_m$ consists of flat points. So, we consider the third (general) case, i.e., we assume that $\kappa_m \neq 0$ and $\kappa \neq 0$.

It follows from (5) that the mean curvature vector field H of $\mathcal{M}'_m$ is expressed as

$$H = \frac{\kappa}{2f}\, n_1 + \frac{\dot{g} + f\kappa_m}{2f}\, n_2.$$

Using that $\dot{g}^2(u) = \dot{f}^2(u) - 1$ and $\kappa_m(u) = \dfrac{\ddot{f}(u)}{\sqrt{\dot{f}^2(u) - 1}}$, we get

$$H = \frac{\kappa}{2f}\, n_1 + \frac{f\ddot{f} + \dot{f}^2 - 1}{2f\sqrt{\dot{f}^2 - 1}}\, n_2. \tag{6}$$

Since $\kappa \neq 0$, the surface $\mathcal{M}'_m$ is non-minimal, i.e., $H \neq 0$. The case $\mathcal{M}'_m$ is a marginally trapped surface, i.e., $H \neq 0$ and $\langle H, H\rangle = 0$ was described in [9]. So, here we consider the case $\langle H, H\rangle \neq 0$.

Note that the orthonormal frame field $\{X, Y, n_1, n_2\}$ of $\mathcal{M}'_m$ is not the geometric frame field defined in Section 2. The principal tangents of $\mathcal{M}'_m$ are

$$x = \frac{X + Y}{\sqrt{2}}; \qquad y = \frac{-X + Y}{\sqrt{2}}.$$

In the case $\langle H, H\rangle > 0$, i.e., $\kappa^2(\dot{f}^2 - 1) - (f\ddot{f} + \dot{f}^2 - 1)^2 > 0$, the geometric normal frame field $\{b, l\}$ is given by

$$\begin{aligned}
b &= \frac{1}{\sqrt{\kappa^2(\dot{f}^2 - 1) - (f\ddot{f} + \dot{f}^2 - 1)^2}} \left(\kappa\sqrt{\dot{f}^2 - 1}\, n_1 + (f\ddot{f} + \dot{f}^2 - 1)\, n_2\right);\\
l &= \frac{1}{\sqrt{\kappa^2(\dot{f}^2 - 1) - (f\ddot{f} + \dot{f}^2 - 1)^2}} \left((f\ddot{f} + \dot{f}^2 - 1)\, n_1 + \kappa\sqrt{\dot{f}^2 - 1}\, n_2\right).
\end{aligned}$$

In this case, the normal vector fields b and l satisfy $\langle b, b\rangle = 1$, $\langle b, l\rangle = 0$, $\langle l, l\rangle = -1$.

In the case $\langle H, H\rangle < 0$, i.e., $\kappa^2(\dot{f}^2 - 1) - (f\ddot{f} + \dot{f}^2 - 1)^2 < 0$, the geometric normal frame field $\{b,l\}$ is given by

$$b=-\frac{1}{\sqrt{(f\ddot{f}+\dot{f}^2-1)^2-\kappa^2(\dot{f}^2-1)}}\left(\kappa\sqrt{\dot{f}^2-1}\,n_1+(f\ddot{f}+\dot{f}^2-1)\,n_2\right);$$

$$l=\frac{1}{\sqrt{(f\ddot{f}+\dot{f}^2-1)^2-\kappa^2(\dot{f}^2-1)}}\left((f\ddot{f}+\dot{f}^2-1)\,n_1+\kappa\sqrt{\dot{f}^2-1}\,n_2\right).$$

In this case, we have $\langle b,b\rangle=-1$, $\langle b,l\rangle=0$, $\langle l,l\rangle=1$.

Applying formulas (1) to the geometric frame field $\{x,y,b,l\}$ of $\mathcal{M}'_m$ and derivative formulas (5), we obtain the following invariants of $\mathcal{M}'_m$:

$$\begin{aligned}
\gamma_1&=\gamma_2=-\frac{\dot{f}}{\sqrt{2}f};\\
\nu_1&=\nu_2=\frac{1}{2f\sqrt{\dot{f}^2-1}}\sqrt{\varepsilon(\kappa^2(\dot{f}^2-1)-(f\ddot{f}+\dot{f}^2-1)^2)};\\
\lambda&=\varepsilon\frac{\kappa^2(\dot{f}^2-1)+f^2\ddot{f}^2-(\dot{f}^2-1)^2}{2f\sqrt{\dot{f}^2-1}\sqrt{\varepsilon(\kappa^2(\dot{f}^2-1)-(f\ddot{f}+\dot{f}^2-1)^2)}};\\
\mu&=\frac{\kappa\ddot{f}}{\sqrt{\varepsilon(\kappa^2(\dot{f}^2-1)-(f\ddot{f}+\dot{f}^2-1)^2)}};\\
\beta_1&=\frac{-(\dot{f}^2-1)}{\sqrt{2}\varepsilon(\kappa^2(\dot{f}^2-1)-(f\ddot{f}+\dot{f}^2-1)^2)}\\
&\quad\times\left(\kappa\frac{d}{du}\left(\frac{f\ddot{f}+\dot{f}^2-1}{\sqrt{\dot{f}^2-1}}\right)-\frac{d}{dv}(\kappa)\,\frac{f\ddot{f}+\dot{f}^2-1}{f\sqrt{\dot{f}^2-1}}\right);\\
\beta_2&=\frac{(\dot{f}^2-1)}{\sqrt{2}\varepsilon(\kappa^2(\dot{f}^2-1)-(f\ddot{f}+\dot{f}^2-1)^2)}\\
&\quad\times\left(\kappa\frac{d}{du}\left(\frac{f\ddot{f}+\dot{f}^2-1}{\sqrt{\dot{f}^2-1}}\right)+\frac{d}{dv}(\kappa)\,\frac{f\ddot{f}+\dot{f}^2-1}{f\sqrt{\dot{f}^2-1}}\right),
\end{aligned}\tag{7}$$

where $\varepsilon=sign\langle H,H\rangle$.

Hyperbolic case:

Let $\mathcal{M}''_m$ be the surface parameterized by (3). Assume that the curve $c:l=l(v)=l(w^1(v),w^2(v))$, $v\in J$ on $S^2_1(1)$ is parameterized by the arc-length, i.e., $\langle l'(v),l'(v)\rangle=1$. Similarly to the previous case, we consider an orthonormal frame field $\{l(v),t(v),n(v)\}$ in $\mathbb{R}^3_1$, such that $t(v)=l'(v)$ and $\langle n(v),n(v)\rangle=-1$. With respect to this frame field, we have the following decompositions of vector fields

$l'(v)$, $t'(v)$, $n'(v)$:

$$\begin{aligned} l' &= t; \\ t' &= -\kappa\, n - l; \\ n' &= -\kappa\, t, \end{aligned} \tag{8}$$

which can be considered as Frenet formulas of c on $S^2_1(1)$. The function $\kappa(v) = \langle t'(v), n(v)\rangle$ is the spherical curvature of c on $S^2_1(1)$.

We assume that $\dot{f}^2(u) + \dot{g}^2(u) = 1$. Denote $X = z_u = \dot{f}\, l + \dot{g}\, e_1$, $Y = \dfrac{z_v}{f} = t$ and consider the orthonormal normal frame field defined by:

$$n_1 = \dot{g}(u)\, l(v) - \dot{f}(u)\, e_1; \qquad n_2 = n(v).$$

Thus we obtain a frame field $\{X, Y, n_1, n_2\}$ of $\mathcal{M}''_m$, such that $\langle n_1, n_1\rangle = 1$, $\langle n_2, n_2\rangle = -1$, $\langle n_1, n_2\rangle = 0$.

Taking into account (8) we get the following derivative formulas:

$$\begin{array}{ll} \nabla'_X X = \quad -\kappa_m\, n_1; & \nabla'_X n_1 = \kappa_m\, X; \\ \nabla'_X Y = 0; & \nabla'_Y n_1 = \quad \dfrac{\dot{g}}{f}\, Y; \\ \nabla'_Y X = \quad \dfrac{\dot{f}}{f}\, Y; & \nabla'_X n_2 = 0; \\ \nabla'_Y Y = -\dfrac{\dot{f}}{f}\, X - \dfrac{\dot{g}}{f}\, n_1 - \dfrac{\kappa}{f}\, n_2; & \nabla'_Y n_2 = \quad -\dfrac{\kappa}{f}\, Y, \end{array} \tag{9}$$

where κ_m is the curvature of the meridian curve m.

The invariants k, $\varkappa$, and the Gauss curvature K of the meridian surface $\mathcal{M}''_m$ are expressed by the curvatures $\kappa_m(u)$, $\kappa(v)$, and the function $f(u)$ in the same way as the invariants of the meridian surface of elliptic type, i.e.,

$$k = -\frac{\kappa_m^2(u)\, \kappa^2(v)}{f^2(u)}; \qquad \varkappa = 0; \qquad K = -\frac{\ddot{f}(u)}{f(u)}.$$

The following statement holds since $\varkappa = 0$.

Proposition 2. *The meridian surface of hyperbolic type $\mathcal{M}''_m$, defined by* (3)*, is a surface with flat normal connection.*

Again we have the following three cases:

I. $\kappa(v) = 0$. In this case, $n_2 = const$, and $\mathcal{M}''_m$ is a planar surface lying in the constant 3-dimensional space spanned by $\{X, Y, n_1\}$.

II. $\kappa_m(u) = 0$. In such case, $k = \varkappa = K = 0$, and $\mathcal{M}''_m$ is a developable ruled surface.

III. $\kappa_m(u)\, \kappa(v) \neq 0$.

In the first two cases, $\mathcal{M}''_m$ is a surface consisting of flat points. So, we consider the third (general) case, i.e., we assume that $\kappa_m \neq 0$ and $\kappa \neq 0$.

Using (9) we get that the mean curvature vector field H of $\mathcal{M}''_m$ is

$$H = -\frac{\dot{g} + f\kappa_m}{2f}\, n_1 - \frac{\kappa}{2f}\, n_2.$$

Having in mind that $\dot{g}^2(u) = 1 - \dot{f}^2(u)$ and $\kappa_m(u) = -\dfrac{\ddot{f}(u)}{\sqrt{1-\dot{f}^2(u)}}$, we obtain

$$H = \frac{f\ddot{f} + \dot{f}^2 - 1}{2f\sqrt{1-\dot{f}^2}}\, n_1 - \frac{\kappa}{2f}\, n_2. \tag{10}$$

The surface $\mathcal{M}''_m$ is non-minimal since $\kappa \neq 0$. The case $\mathcal{M}''_m$ is marginally trapped was described in [9]. So, we consider the case $\langle H, H\rangle \neq 0$, i.e., $(f\ddot{f} + \dot{f}^2 - 1)^2 - \kappa^2(1-\dot{f}^2) \neq 0$.

Similarly to the elliptic case, we find the geometric frame field $\{x, y, b, l\}$ of $\mathcal{M}''_m$. Applying formulas (1) for this frame field and using derivative formulas (9), we obtain the following invariants of $\mathcal{M}''_m$:

$$\begin{aligned}
\gamma_1 &= \gamma_2 = -\frac{\dot{f}}{\sqrt{2}f};\\
\nu_1 &= \nu_2 = \frac{1}{2f\sqrt{1-\dot{f}^2}}\sqrt{\varepsilon((f\ddot{f} + \dot{f}^2 - 1)^2 - \kappa^2(1-\dot{f}^2))};\\
\lambda &= \varepsilon\frac{-\kappa^2(1-\dot{f}^2) - f^2\ddot{f}^2 + (1-\dot{f}^2)^2}{2f\sqrt{1-\dot{f}^2}\sqrt{\varepsilon((f\ddot{f} + \dot{f}^2 - 1)^2 - \kappa^2(1-\dot{f}^2))}};\\
\mu &= \frac{\kappa\ddot{f}}{\sqrt{\varepsilon((f\ddot{f} + \dot{f}^2 - 1)^2 - \kappa^2(1-\dot{f}^2))}};\\
\beta_1 &= \frac{-(1-\dot{f}^2)}{\sqrt{2}\varepsilon((f\ddot{f} + \dot{f}^2 - 1)^2 - \kappa^2(1-\dot{f}^2))}\\
&\quad \left(\kappa\frac{d}{du}\left(\frac{f\ddot{f} + \dot{f}^2 - 1}{\sqrt{1-\dot{f}^2}}\right) - \frac{d}{dv}(\kappa)\,\frac{f\ddot{f} + \dot{f}^2 - 1}{f\sqrt{1-\dot{f}^2}}\right);\\
\beta_2 &= \frac{(1-\dot{f}^2)}{\sqrt{2}\varepsilon((f\ddot{f} + \dot{f}^2 - 1)^2 - \kappa^2(1-\dot{f}^2))}\\
&\quad \left(\kappa\frac{d}{du}\left(\frac{f\ddot{f} + \dot{f}^2 - 1}{\sqrt{1-\dot{f}^2}}\right) + \frac{d}{dv}(\kappa)\,\frac{f\ddot{f} + \dot{f}^2 - 1}{f\sqrt{1-\dot{f}^2}}\right),
\end{aligned} \tag{11}$$

where $\varepsilon = sign\langle H, H\rangle$.

In the following sections, using the invariants of meridian surfaces $\mathcal{M}'_m$ and $\mathcal{M}''_m$, we shall describe and classify some special classes of meridian surfaces of elliptic or hyperbolic type.

4. Meridian surfaces with constant Gauss curvature

The study of surfaces with constant Gauss curvature is one of the main topics in classical differential geometry. Surfaces with constant Gauss curvature in Minkowski space have drawn the interest of many geometers, see for example [6], [17], and the references therein.

In the present section, we give a classification of meridian surfaces of elliptic or hyperbolic type in $\mathbb{R}^4_1$ with constant Gauss curvature.

Let $\mathcal{M}'_m$ and $\mathcal{M}''_m$ be meridian surfaces of elliptic and hyperbolic type, respectively. The Gauss curvature in both cases depends only on the meridian curve m and is expressed by the formula

$$K = -\frac{\ddot{f}(u)}{f(u)}. \tag{12}$$

Theorem 1. *Let $\mathcal{M}'_m$ (resp. $\mathcal{M}''_m$) be a meridian surface of elliptic (resp. hyperbolic) type from the general class. Then $\mathcal{M}'_m$ (resp. $\mathcal{M}''_m$) has constant non-zero Gauss curvature K if and only if the meridian m is given by*

$$\begin{array}{ll} f(u) = \alpha \cos \sqrt{K} u + \beta \sin \sqrt{K} u, & \textit{if} \quad K > 0; \\ f(u) = \alpha \cosh \sqrt{-K} u + \beta \sinh \sqrt{-K} u, & \textit{if} \quad K < 0, \end{array}$$

where α and β are constants, $g(u)$ is defined by $\dot{g}(u) = \sqrt{\dot{f}^2(u) - 1}$ in the elliptic case and $g(u)$ is defined by $\dot{g}(u) = \sqrt{1 - \dot{f}^2(u)}$ in the hyperbolic case.

Proof. Using (12) we obtain that the Gauss curvature $K = const \neq 0$ if and only if the function $f(u)$ satisfies the following differential equation

$$\ddot{f}(u) + K f(u) = 0.$$

The general solution of the above equation is given by

$$\begin{array}{ll} f(u) = \alpha \cos \sqrt{K} u + \beta \sin \sqrt{K} u, & \text{if} \quad K > 0; \\ f(u) = \alpha \cosh \sqrt{-K} u + \beta \sinh \sqrt{-K} u, & \text{if} \quad K < 0, \end{array}$$

where α and β are constants. In the case of a meridian surface of elliptic type, the function $g(u)$ is determined by $\dot{g}(u) = \sqrt{\dot{f}^2(u) - 1}$ and in the case of a meridian surface of hyperbolic type, $\dot{g}(u) = \sqrt{1 - \dot{f}^2(u)}$.

$\square$

5. Meridian surfaces with constant mean curvature

Constant mean curvature surfaces in arbitrary spacetime are important objects for their special role in the theory of general relativity. The study of constant mean curvature surfaces (CMC surfaces) involves not only geometric methods but also PDE and complex analysis; that is why the theory of CMC surfaces is of great interest not only to mathematicians but also to physicists and engineers. Surfaces with constant mean curvature in Minkowski space have been studied intensively in the last years. See for example [15], [16], [20], [3], [2].

In this section, we classify meridian surfaces of elliptic or hyperbolic type with constant mean curvature.

Let $\mathcal{M}'_m$ and $\mathcal{M}''_m$ be meridian surfaces of elliptic and hyperbolic type, respectively. Equality (6) implies that the mean curvature of the meridian surface of elliptic type $\mathcal{M}'_m$ is given by

$$||H|| = \sqrt{\frac{\varepsilon(\kappa^2(\dot{f}^2-1) - (f\ddot{f} + \dot{f}^2 - 1)^2)}{4f^2(\dot{f}^2-1)}}. \tag{13}$$

Similarly, from (10) it follows that the mean curvature of the meridian surface of hyperbolic type $\mathcal{M}''_m$ is

$$||H|| = \sqrt{\frac{\varepsilon((f\ddot{f} + \dot{f}^2 - 1)^2 - \kappa^2(1-\dot{f}^2))}{4f^2(1-\dot{f}^2)}}. \tag{14}$$

Theorem 2. *Let $\mathcal{M}'_m$ (resp. $\mathcal{M}''_m$) be a meridian surface of elliptic (resp. hyperbolic) type from the general class.*

(i) $\mathcal{M}'_m$ has constant mean curvature $||H|| = a = const$, $a \neq 0$ if and only if the curve c on $S^2(1)$ has constant spherical curvature $\kappa = const = b$, $b \neq 0$, and the meridian m is determined by $\dot{f} = y(f)$, where

$$y(t) = \sqrt{1 + \frac{1}{t^2}\left(C \pm \frac{t}{2}\sqrt{b^2 - 4a^2t^2} \pm \frac{b^2}{4a}\arcsin\frac{2at}{b}\right)^2}, \qquad C = const,$$

$g(u)$ is defined by $\dot{g}(u) = \sqrt{\dot{f}^2(u) - 1}$.

(ii) $\mathcal{M}''_m$ has constant mean curvature $||H|| = a = const$, $a \neq 0$ if and only if the curve c on $S^2_1(1)$ has constant spherical curvature $\kappa = const = b$, $b \neq 0$, and the meridian m is determined by $\dot{f} = y(f)$, where

$$y(t) = \sqrt{1 - \frac{1}{t^2}\left(C \pm \frac{t}{2}\sqrt{b^2 - 4a^2t^2} \pm \frac{b^2}{4a}\arcsin\frac{2at}{b}\right)^2}, \qquad C = const,$$

$g(u)$ is defined by $\dot{g}(u) = \sqrt{1 - \dot{f}^2(u)}$.

Proof. (i) It follows from (13) that $||H|| = a$ if and only if

$$\kappa^2(v) = \frac{4a^2 f^2(u)(\dot{f}^2(u) - 1) + (f(u)\ddot{f}(u) + \dot{f}^2(u) - 1)^2}{\dot{f}^2(u) - 1},$$

which implies

$$\begin{array}{l} \kappa = const = b, \ b \neq 0; \\ 4a^2 f^2(u)(\dot{f}^2(u) - 1) + (f(u)\ddot{f}(u) + \dot{f}^2(u) - 1)^2 = b^2(\dot{f}^2(u) - 1). \end{array} \tag{15}$$

The first equality in (15) implies that the spherical curve c has constant spherical curvature $\kappa = b$, i.e., c is a circle on $S^2(1)$. The second equality in (15) gives the following differential equation for the meridian m:

$$(f\ddot{f} + \dot{f}^2 - 1)^2 = (\dot{f}^2 - 1)(b^2 - 4a^2 f^2). \tag{16}$$

Further, if we set $\dot{f} = y(f)$ in equation (16), we obtain that the function $y = y(t)$ is a solution of the following differential equation

$$\frac{t}{2}(y^2)' + y^2 - 1 = \pm\sqrt{y^2 - 1}\sqrt{b^2 - 4a^2t^2}.$$

The general solution of the above equation is given by the formula

$$y(t) = \sqrt{1 + \frac{1}{t^2}\left(C \pm \frac{t}{2}\sqrt{b^2 - 4a^2t^2} \pm \frac{b^2}{4a}\arcsin\frac{2at}{b}\right)^2}, \qquad C = const. \tag{17}$$

The function $f(u)$ is determined by $\dot{f} = y(f)$ and (17). The function $g(u)$ is defined by $\dot{g}(u) = \sqrt{\dot{f}^2(u) - 1}$.

(ii) Similarly to the elliptic case, from (14) it follows that $||H|| = a$ if and only if the curve c on $S^2_1(1)$ has constant curvature $\kappa = b$, and the meridian m is determined by the following differential equation:

$$(f\ddot{f} + \dot{f}^2 - 1)^2 = (1 - \dot{f}^2)(b^2 + 4a^2 f^2). \tag{18}$$

Setting $\dot{f} = y(f)$ in equation (18), we obtain

$$y(t) = \sqrt{1 - \frac{1}{t^2}\left(C \pm \frac{t}{2}\sqrt{b^2 - 4a^2t^2} \pm \frac{b^2}{4a}\arcsin\frac{2at}{b}\right)^2}, \qquad C = const,$$

In this case, the function $g(u)$ is defined by $\dot{g}(u) = \sqrt{1 - \dot{f}^2(u)}$.

□

6. Meridian surfaces with constant invariant k

Let $\mathcal{M}'_m$ and $\mathcal{M}''_m$ be meridian surfaces of elliptic and hyperbolic type, respectively. Then the invariant k is given by the formula

$$k = -\frac{\kappa_m^2(u)\,\kappa^2(v)}{f^2(u)}, \tag{19}$$

where $\kappa_m(u) = \dfrac{\ddot{f}(u)}{\sqrt{\dot{f}^2(u) - 1}}$ in the elliptic case, and $\kappa_m(u) = -\dfrac{\ddot{f}(u)}{\sqrt{1 - \dot{f}^2(u)}}$ in the hyperbolic case.

In the following theorem, we describe the meridian surfaces of elliptic or hyperbolic type with constant invariant k.

Theorem 3. *Let $\mathcal{M}'_m$ (resp. $\mathcal{M}''_m$) be a meridian surface of elliptic (resp. hyperbolic) type from the general class.*

(i) $\mathcal{M}'_m$ has a constant invariant $k = const = -a^2$, $a \neq 0$ if and only if the curve c on $S^2(1)$ has constant spherical curvature $\kappa = const = b$, $b \neq 0$, and the meridian m is determined by $\dot{f} = y(f)$, where

$$y(t) = \sqrt{1 + \left(C \pm \frac{at^2}{2b}\right)^2}, \qquad C = const,$$

$g(u)$ is defined by $\dot{g}(u) = \sqrt{\dot{f}^2(u) - 1}$.

(ii) $\mathcal{M}''_m$ has a constant invariant $k = const = -a^2$, $a \neq 0$ if and only if the curve c on $S^2_1(1)$ has constant spherical curvature $\kappa = const = b$, $b \neq 0$, and the meridian m is determined by $\dot{f} = y(f)$, where

$$y(t) = \sqrt{1 - \left(C \mp \frac{at^2}{2b}\right)^2}, \qquad C = const,$$

$g(u)$ is defined by $\dot{g}(u) = \sqrt{1 - \dot{f}^2(u)}$.

Proof. (i) It follows from (19) that $k = const = -a^2$, $a \neq 0$ if and only if

$$\kappa^2(v) = \frac{a^2 f^2(u)(\dot{f}^2(u) - 1)}{\ddot{f}^{\,2}(u)}.$$

The last equality implies

$$\begin{aligned} &\kappa = const = b, \ b \neq 0; \\ &a^2 f^2(u)(\dot{f}^2(u) - 1) = b^2 \ddot{f}^{\,2}(u). \end{aligned}$$

Hence, the curve c has constant spherical curvature $\kappa = b$ and the function $f(u)$ is a solution of the following differential equation:

$$b^2 \ddot{f}^{\,2} - a^2 f^2(\dot{f}^2 - 1) = 0 \tag{20}$$

Setting $\dot{f} = y(f)$ in equation (20), we obtain that the function $y = y(t)$ is a solution of

$$\frac{b}{2}(y^2)' = \pm at\sqrt{y^2 - 1}.$$

The general solution of the above equation is given by

$$y(t) = \sqrt{1 + \left(C \pm \frac{at^2}{2b}\right)^2}, \qquad C = const. \tag{21}$$

The function $f(u)$ is determined by $\dot{f} = y(f)$ and (21). The function $g(u)$ is defined by $\dot{g}(u) = \sqrt{\dot{f}^2(u) - 1}$.

(ii) Similarly to the elliptic case, we obtain that $\mathcal{M}''_m$ has a constant invariant $k = const = -a^2,\ a \neq 0$ if and only if c has constant curvature $\kappa = const = b,\ b \neq 0$, and the meridian m is determined by the following differential equation:

$$b^2 \ddot{f}^{\,2} - a^2 f^2 (1 - \dot{f}^2) = 0.$$

Again, setting $\dot{f} = y(f)$ we obtain

$$y(t) = \sqrt{1 - \left(C \mp \frac{at^2}{2b}\right)^2}, \qquad C = const.$$

□

7. Chen meridian surfaces

Let $\mathcal{M}'_m$ and $\mathcal{M}''_m$ be meridian surfaces of elliptic and hyperbolic type, respectively. The invariants of $\mathcal{M}'_m$ and $\mathcal{M}''_m$ are given by formulas (7) and (11), respectively. Recall that a spacelike surface in $\mathbb{R}^4_1$ is a non-trivial Chen surface if and only if $\lambda = 0$. In the following theorem, we classify all Chen meridian surfaces of elliptic or hyperbolic type.

Theorem 4. *Let $\mathcal{M}'_m$ (resp. $\mathcal{M}''_m$) be a meridian surface of elliptic (resp. hyperbolic) type from the general class.*

(i) $\mathcal{M}'_m$ is a Chen surface if and only if the curve c on $S^2(1)$ has constant spherical curvature $\kappa = const = b,\ b \neq 0$, and the meridian m is determined by $\dot{f} = y(f)$, where

$$y(t) = \frac{\pm 1}{2\,t^{\pm 1}} \sqrt{4\,t^{\pm 2} - a\left(t^{\pm 2} - \frac{b^2}{a}\right)^2}, \qquad a = const \neq 0,$$

$g(u)$ is defined by $\dot{g}(u) = \sqrt{\dot{f}^2(u) - 1}$.

(ii) $\mathcal{M}''_m$ is a Chen surface if and only if the curve c on $S^2_1(1)$ has constant spherical curvature $\kappa = const = b,\ b \neq 0$, and the meridian m is determined by $\dot{f} = y(f)$, where

$$y(t) = \frac{\pm 1}{2\,t^{\pm 1}} \sqrt{4\,t^{\pm 2} + a\left(t^{\pm 2} - \frac{b^2}{a}\right)^2}, \qquad a = const \neq 0,$$

$g(u)$ is defined by $\dot{g}(u) = \sqrt{1 - \dot{f}^2(u)}$.

Proof. (i) It follows from (7) that $\lambda = 0$ if and only if

$$\kappa^2(v) = \frac{(\dot f^2(u) - 1)^2 - f^2(u)\ddot f^{\,2}(u)}{\dot f^2(u) - 1},$$

which implies

$$\kappa = const = b, \; b \neq 0;$$
$$(\dot f^2(u) - 1)^2 - f^2(u)\ddot f^{\,2}(u) = b^2(\dot f^2(u) - 1).$$

Hence, the curve c has constant spherical curvature $\kappa = b$ and the function $f(u)$ is a solution of the following differential equation:

$$(\dot f^2 - 1)^2 - f^2\ddot f^{\,2} = b^2(\dot f^2 - 1). \tag{22}$$

The solutions of differential equation (22) can be found as follows. Setting $\dot f = y(f)$ in equation (22), we obtain that the function $y = y(t)$ is a solution of the equation:

$$\frac{t^2}{4}\left((y^2)'\right)^2 = (y^2 - 1)^2 - b^2(y^2 - 1). \tag{23}$$

We set $z(t) = y^2(t) - 1$ and obtain

$$\frac{t}{2}\,z' = \pm\sqrt{z^2 - b^2 z}.$$

The last equation is equivalent to

$$\frac{z'}{\sqrt{z^2 - b^2 z}} = \pm\frac{2}{t}. \tag{24}$$

Integrating both sides of (24), we get

$$\frac{b^2}{2} - z + \sqrt{z^2 - b^2 z} = c\,t^{\pm 2}, \qquad c = const. \tag{25}$$

It follows from (25) that

$$z(t) = -\frac{(a\,t^{\pm 2} - b^2)^2}{4a\,t^{\pm 2}}, \qquad a = 2c.$$

Hence, the general solution of differential equation (23) is given by

$$y(t) = \frac{\pm 1}{2\,t^{\pm 1}}\sqrt{4\,t^{\pm 2} - a\left(t^{\pm 2} - \frac{b^2}{a}\right)^2}, \qquad a = const \neq 0.$$

(ii) In a similar way, in the hyperbolic case we obtain that $\lambda = 0$ if and only if the curve c has constant curvature $\kappa = b$, $b \neq 0$ and the function $f(u)$ is a solution of

$$(1 - \dot f^2)^2 - f^2\ddot f^{\,2} = b^2(1 - \dot f^2).$$

By doing similar calculations as in the previous case, we obtain

$$y(t) = \frac{\pm 1}{2\,t^{\pm 1}}\sqrt{4\,t^{\pm 2} + a\left(t^{\pm 2} - \frac{b^2}{a}\right)^2}, \qquad a = const \neq 0.$$

□

8. Meridian surfaces with the parallel normal bundle

In this section, we shall describe meridian surfaces of elliptic or hyperbolic type with parallel normal bundle. Recall that a surface in $\mathbb{R}^4_1$ has parallel normal bundle if and only if $\beta_1 = \beta_2 = 0$.

Theorem 5. *Let $\mathcal{M}'_m$ (resp. $\mathcal{M}''_m$) be a meridian surface of elliptic (resp. hyperbolic) type from general class.*

(i) $\mathcal{M}'_m$ has parallel normal bundle if and only if one of the following cases holds:

(a) the meridian m is defined by

$$f(u) = \pm\sqrt{u^2 + 2cu + d};$$
$$g(u) = \pm\sqrt{c^2 - d}\,\ln|u + c + \sqrt{u^2 + 2cu + d}| + a,$$

where a, c, and d are constants, $c^2 > d$;

(b) the curve c on $S^2(1)$ has constant spherical curvature $\kappa = const = b$, $b \neq 0$, and the meridian m is determined by $\dot{f} = y(f)$, where

$$y(t) = \pm\frac{\sqrt{(a^2+1)\,t^2 + 2ac\,t + c^2}}{t}, \quad a = const \neq 0, \quad c = const,$$

$g(u)$ is defined by $\dot{g}(u) = \sqrt{\dot{f}^2(u) - 1}$.

(ii) $\mathcal{M}''_m$ has parallel normal bundle if and only if one of the following cases holds:

(a) the meridian m is defined by

$$f(u) = \pm\sqrt{u^2 + 2cu + d};$$
$$g(u) = \pm\sqrt{d - c^2}\,\ln|u + c + \sqrt{u^2 + 2cu + d}| + a,$$

where a, c, and d are constants, $d > c^2$;

(b) the curve c on $S^2_1(1)$ has constant spherical curvature $\kappa = const = b$, $b \neq 0$, and the meridian m is determined by $\dot{f} = y(f)$, where

$$y(t) = \pm\frac{\sqrt{(1-a^2)\,t^2 + 2ac\,t - c^2}}{t}, \quad a = const \neq 0, \quad c = const,$$

$g(u)$ is defined by $\dot{g}(u) = \sqrt{1 - \dot{f}^2(u)}$.

Proof. (i) By using formulas (7), we get that $\beta_1 = \beta_2 = 0$ if and only if

$$\begin{aligned}
&\kappa\,\frac{d}{du}\left(\frac{f\ddot{f} + \dot{f}^2 - 1}{\sqrt{\dot{f}^2 - 1}}\right) - \frac{d}{dv}(\kappa)\,\frac{f\ddot{f} + \dot{f}^2 - 1}{f\sqrt{\dot{f}^2 - 1}} = 0;\\
&\kappa\,\frac{d}{du}\left(\frac{f\ddot{f} + \dot{f}^2 - 1}{\sqrt{\dot{f}^2 - 1}}\right) + \frac{d}{dv}(\kappa)\,\frac{f\ddot{f} + \dot{f}^2 - 1}{f\sqrt{\dot{f}^2 - 1}} = 0.
\end{aligned} \tag{26}$$

It follows from (26) that there are two possible cases:

Case (a): $f\ddot{f} + \dot{f}^2 - 1 = 0$. The general solution of this differential equation is $f(u) = \pm\sqrt{u^2 + 2cu + d}$, $c = const$, $d = const$. Using that $\dot{g}^2 = \dot{f}^2 - 1$, we get $\dot{g}^2 = \dfrac{c^2 - d}{u^2 + 2cu + d}$, and hence $c^2 - d > 0$. By integrating both sides of the equation

$$\dot{g}(u) = \pm\frac{\sqrt{c^2 - d}}{\sqrt{u^2 + 2cu + d}},$$

we obtain $g(u) = \pm\sqrt{c^2 - d}\,\ln|u + c + \sqrt{u^2 + 2cu + d}| + a$, $a = const$. Consequently, the meridian m is defined as described in *(a)*.

Case (b): $\dfrac{f\ddot{f} + \dot{f}^2 - 1}{\sqrt{\dot{f}^2 - 1}} = a = const$, $a \neq 0$ and $\kappa = b = const$, $b \neq 0$. Hence, in this case the curve c has constant spherical curvature $\kappa = b$ and the meridian m is determined by the following differential equation:

$$f\ddot{f} + \dot{f}^2 - 1 = a\sqrt{\dot{f}^2 - 1}, \qquad a = const \neq 0. \tag{27}$$

The solutions of differential equation (27) can be found in the following way. Setting $\dot{f} = y(f)$ in equation (27), we obtain that the function $y = y(t)$ is a solution of the equation:

$$\frac{t}{2}\,(y^2)' + y^2 - 1 = a\sqrt{y^2 - 1}. \tag{28}$$

If we set $z(t) = \sqrt{y^2(t) - 1}$, we get

$$z' + \frac{1}{t}\,z = \frac{a}{t}.$$

The general solution of the above equation is given by the formula $z(t) = \dfrac{c + at}{t}$, $c = const$. Hence, the general solution of (28) is

$$y(t) = \pm\frac{\sqrt{(a^2 + 1)\,t^2 + 2ac\,t + c^2}}{t}, \quad c = const.$$

(ii) In a similar way, considering meridian surfaces of hyperbolic type we obtain that $\beta_1 = \beta_2 = 0$ if and only if one of the following cases holds.

Case (a): $f\ddot{f} + \dot{f}^2 - 1 = 0$. In this case, we get

$$f(u) = \pm\sqrt{u^2 + 2cu + d}; \qquad g(u) = \pm\sqrt{d - c^2}\,\ln|u + c + \sqrt{u^2 + 2cu + d}| + a,$$

where a, c, and d are constants, $d > c^2$.

Case (b): $\dfrac{f\ddot{f} + \dot{f}^2 - 1}{\sqrt{1 - \dot{f}^2}} = a = const$, $a \neq 0$ and $\kappa = b = const$, $b \neq 0$. By doing similar calculations as the calculations for solving (27), we obtain

$$y(t) = \pm\frac{\sqrt{(1 - a^2)\,t^2 + 2ac\,t - c^2}}{t}, \quad c = const.$$

□

Similarly to the elliptic or hyperbolic type, one can study the invariants of the meridian surfaces of parabolic type. The classes of meridian surfaces of parabolic type with constant Gauss curvature, constant mean curvature, constant invariant k, the Chen meridian surfaces of parabolic type, and the meridian surfaces of parabolic type with the parallel normal bundle can be described in an analogous way.

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