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Quantitative estimates for a certain bivariate Chlodowsky-Szasz-Kantorovich type operators*

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Abstract. In this paper, we introduce a bivariate Kantorovich variant of the combination of Chlodowsky and Szasz type operators and study local approximation properties of these operators. We estimate the approximation order in terms of Peetre's K-functional and partial moduli of continuity. We also give some numerical error estimates and illustrations.
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1. Introduction

Bernstein-Chlodowsky polynomials are a generalization of a classical Bernstein polynomial on the interval $[0, \alpha_n]$ with $\alpha_n \to \infty$ as $n \to \infty$. These polynomials are defined by

$$B_n(f;x) = \sum_{k=0}^{n} C_n^k \left(\frac{x}{\alpha_n}\right)^k \left(1-\frac{x}{\alpha_n}\right)^{n-k} f\left(\frac{k\alpha_n}{n}\right),$$

where $x \in [0, \alpha_n]$ and $\lim_{n\to\infty} \frac{\alpha_n}{n} = 0$. The approximation properties of these operators were investigated for univariate and bivariate continuous functions in [1, 9, 10, 13, 18]. Moreover, a q-generalization of Bernstein-Chlodowsky polynomials [7], Voronoskaja type theorems related to these polynomials [6, 15] and a Bezier variant of these polynomials [19] were discussed.

The modified Szasz operators defined as

$$S_m(f;y) = e^{-\beta_m y} \sum_{j=0}^{\infty} \frac{(\beta_m y)^j}{j!} f\left(\frac{j}{\gamma_m}\right); m \in \mathbb{N}, y \in [0,\infty)$$

were introduced by Walczak [20] and studied the approximation properties in one and two-dimensional weighted spaces. İspir and Atakut [12] estimated the rate of

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convergence by weighted modulus of continuity on the positive real axis in univariate and bivariate cases.

Recently, Gazanfer and Buyukyazici [11] introduced a bivariate operator associated with a combination of Chlodowsky and modified Szasz type operators as follows

$$L_{n,m}\left(f;x,y\right)=\sum_{k=0}^{n}\sum_{j=0}^{\infty}P\left(\frac{x}{\alpha_n}\right)Q_j\left(\beta_m y\right)f\left(\frac{k}{n}\alpha_n,\frac{j}{\gamma_m}\right) \tag{1}$$

for all $n,m\in\mathbb{N}$, $f\in C\left(A_{\alpha_n}\right)$ with $A_{\alpha_n}=\left\{(x,y):0\leq x\leq\alpha_n,0\leq y<\infty\right\}$ and $C\left(A_{\alpha_n}\right)=\left\{f:A_{\alpha_n}\to\mathbb{R}\text{ is continuous}\right\}$. Here (α_n) is an unbounded sequence of positive numbers such that $\lim_{n\to\infty}(\alpha_n/n)=0$ and also $(\gamma_m),(\beta_m)$ denote the unbounded sequences of positive numbers such that

$$\begin{aligned}\lim_{m\to\infty}\gamma_m^{-1}&=0,\\ \beta_m/\gamma_m&=1+O\left(1/\gamma_m\right),\end{aligned}$$

and

$$\begin{aligned}P_{n,k}\left(x\right)&=C_n^k x\left(1-x\right)^{n-k},\\ Q_j\left(y\right)&=e^{-y}\left(y^j/j!\right).\end{aligned}$$

Notice that the operator $L_{n,m}:C\left(A_{\alpha_n}\right)\to C\left(A_{\alpha_n}\right)$ is the tensorial product of ${}_xB_n$ and ${}_yS_m$, i.e., $L_{n,m}={}_x B_n\circ_y S_m$m where

$${}_xB_n\left(f;x,y\right)=\sum_{k=0}^{n}C_n^k\left(\frac{x}{\alpha_n}\right)^k\left(1-\frac{x}{\alpha_n}\right)^{n-k}f\left(\frac{k\alpha_n}{n},y\right)$$

and

$${}_yS_m\left(f;x,y\right)=e^{-\beta_m y}\sum_{j=0}^{\infty}\frac{\left(\beta_m y\right)^j}{j!}f\left(x,\frac{j}{\gamma_m}\right).$$

In [11], the authors studied some approximation properties of $L_{n,m}$ operators given by (1) in a space of continuous functions on a compact subset of $A_{\alpha_n}$and given the degree of this approximation by means of total and partial modulus of continuity. Furthermore, they investigated the weighted approximation properties of the operators $L_{n,m}$ for continuous functions with polynomial growth on $[0,\infty)\times[0,\infty)$.

Summation type operators are not suitable for approximating a function $f\in L_p[0,1]$ in the L_p norm. This problem is eliminated with integral type operators, one of which is the Kantorovich operator. The Kantorovich operators allow us to investigate the approximation properties in the uniform norm and also in the L_p norm. Some generalizations of Szasz-Kantorovich and Chlodowsky integral type operators are studied in [2, 4, 8, 14, 16, 17].

To approximate integrable functions, we define a Kantorovich variant of the operators given by (1) as follows

$$L_{n,m}^{*}\left(f;x,y\right)=\frac{n}{\alpha_n}\gamma_m\sum_{k=0}^{n}\sum_{j=0}^{\infty}P_{n,k}\left(\frac{x}{\alpha_n}\right)Q_j\left(\beta_m y\right)\int\limits_{j/\gamma_m}^{(j+1)/\gamma_m}\int\limits_{k\alpha_n/n}^{(k+1)\alpha_n/n}f\left(t,s\right)dtds,$$

where the sequences $(\alpha_n), (\beta_m), (\gamma_m)$ are defined as above and have the conditions

$$\lim_{n\to\infty} \frac{\alpha_n}{n} = 0, \tag{2}$$

and

$$\lim_{m\to\infty} \gamma_m^{-1} = 0,\ \frac{\beta_m}{\gamma_m} = 1 + O\left(\frac{1}{\gamma_m}\right). \tag{3}$$

In this paper, we investigate local approximation properties of operators $L_{n,m}^*$ in terms of the partial modulus of continuity and Petree's K-functional corresponding to the second modulus of continuity. We also give some numerical examples associated with the order of approximation and some illustrations for the rate of convergence.

2. Basic results

In this section, we give some results using the test functions $e_{i,j}(t,s) = t^i s^j$ ($i,j = 0,1,2$). To express our results we give the following auxiliary lemmas.

Lemma 1 (See [11]). *For $f \in C(A_{\alpha_n})$, we have*

$$\begin{aligned}
L_{n,m}(e_{0,0};x,y) &= 1,\\
L_{n,m}(e_{1,0};x,y) &= x,\\
L_{n,m}(e_{0,1};x,y) &= \frac{\beta_m}{\gamma_m}y,\\
L_{n,m}(e_{2,0};x,y) &= \left(1-\frac{1}{n}\right)x^2 + \frac{\alpha_n}{n}x,\\
L_{n,m}(e_{0,2};x,y) &= \frac{\beta_m^2}{\gamma_m^2}y^2 + \frac{\beta_m}{\gamma_m^2}y.
\end{aligned}$$

Lemma 2. *For $f \in C(A_{\alpha_n})$ and operators $L_{n,m}^*$ satisfy the following equalities*

$$\begin{aligned}
L_{n,m}^*(e_{0,0};x,y) &= 1,\\
L_{n,m}^*(e_{1,0};x,y) &= x + \frac{\alpha_n}{2n},\\
L_{n,m}^*(e_{0,1};x,y) &= \frac{\beta_m}{\gamma_m}y + \frac{1}{2\gamma_m},\\
L_{n,m}^*(e_{2,0};x,y) &= \left(1-\frac{1}{n}\right)x^2 + 2\frac{\alpha_n}{n}x + \frac{\alpha_n^2}{3n^2},\\
L_{n,m}^*(e_{0,2};x,y) &= \frac{\beta_m^2}{\gamma_m^2}y^2 + 2\frac{\beta_m}{\gamma_m^2}y + \frac{1}{3\gamma_m^2}.
\end{aligned}$$

Remark 1. *By applying Lemma 2 we have*

$$
\begin{aligned}
L_{n,m}^{*}(t-x;x,y) &= \frac{\alpha_n}{2n} \\
L_{n,m}^{*}\left((t-x)^2;x,y\right) &= \left(1-\frac{1}{n}\right)x^2+2\frac{\alpha_n}{n}x+\frac{\alpha_n^2}{3n^2}-2x^2-\frac{\alpha_n}{n}x+x^2 \\
&= -\frac{1}{n}x^2+\frac{\alpha_n}{n}x+\frac{\alpha_n^2}{3n^2}=\frac{x(\alpha_n-x)}{n}+\frac{\alpha_n^2}{3n^2}. \qquad (4) \\
L_{n,m}^{*}(s-y;x,y) &= \left(\frac{\beta_m}{\gamma_m}-1\right)y+\frac{1}{2\gamma_m} \\
L_{n,m}^{*}\left((s-y)^2;x,y\right) &= \frac{\beta_m^2}{\gamma_m^2}y^2+2\frac{\beta_m}{\gamma_m^2}y+\frac{1}{3\gamma_m^2}-2y\frac{\beta_m}{\gamma_m}y-2y\frac{1}{2\gamma_m}+y^2 \\
&= \left(\frac{\beta_m}{\gamma_m}-1\right)^2y^2+\frac{1}{\gamma_m}\left(2\frac{\beta_m}{\gamma_m}-1\right)y+\frac{1}{3\gamma_m^2}. \qquad (5)
\end{aligned}
$$

Hence, for all $(x,y)\in A_{\alpha_n}$ and sufficiently large n,m, we get

$$L_{n,m}^{*}\left((t-x)^2;x,y\right)=O\left(\frac{\alpha_n}{n}\right)\left(x^2+x+1\right) \qquad (6)$$

and

$$L_{n,m}^{*}\left((s-y)^2;x,y\right)=O\left(\frac{1}{\gamma_m}\right)\left(y^2+y+1\right). \qquad (7)$$

Lemma 3. *For $f\in C(A_{\alpha_n})$, we have*

$$\left\|L_{n,m}^{*}(f;x,y)\right\|\leq\|f\|,$$

where $\|.\|$ is the uniform norm on $C(A_{\alpha_n})$.

Proof. Considering the definition of $L_{n,m}^{*}(f;x,y)$ and by Lemma 2,

$$
\begin{aligned}
\left\|L_{n,m}^{*}(f;x,y)\right\| &\leq \frac{n}{\alpha_n}\gamma_m\sum_{k=0}^{n}\sum_{j=0}^{\infty}P_{n,k}\left(\frac{x}{\alpha_n}\right)Q_j(\beta_m y)\int_{j/\gamma_m}^{(j+1)/\gamma_m}\int_{k\alpha_n/n}^{(k+1)\alpha_n/n}|f(t,s)|\,dtds \\
&\leq \|f\|\left\|L_{n,m}^{*}(e_{0,0};x,y)\right\|=\|f\|.
\end{aligned}
$$

□

For $A_{ab}=[0,a]\times[0,b]$, let $C(A_{ab})$ denote the space of all real valued continuous functions on A_{ab}, equipped with the norm given by $||f||_{C(A_{ab})}=\sup\limits_{(x,y)\in A_{ab}}|f(x,y)|$.

Theorem 1. *Let $f\in C(A_{\alpha_n})$; then the operators $L_{n,m}^{*}$ given by (1) converge uniformly to f on the compact set $A_{ab}=[0,a]\times[0,b]$, as $n,m\to\infty$.*

Proof. From Lemma 2 and taking into account conditions (2) and (3), we get

$$\lim_{n,m\to\infty}\left\|L_{n,m}^{*}(e_{i,j})-e_{i,j}\right\|_{C(A_{ab})}=0,\quad i,j=0,1,2.$$

The well known Volkov Theorem implies $\lim_{n,m\to\infty}\left\|L_{n,m}^{*}(f)-f\right\|_{C(A_{ab})}=0.$ □

Example 1. *For* $n, m = 5,\ 50$ *the convergence of* $L^*_{n,m}(f; x, y)$ *to* $f(x, y) = xy^2 - x^2y$ *will be illustrated in Figure 1 and Figure 2.*

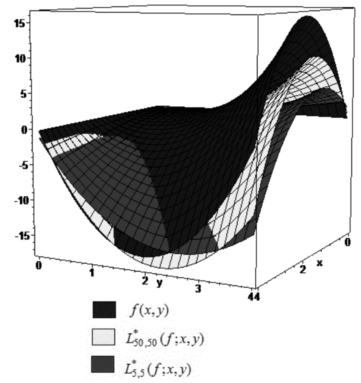

Figure 1: *The convergence of* $L^*_{n,m}(f; x, y)$ *to* $f(x, y)$, *for* $\alpha_n = \sqrt{n}$, $\beta_n = n$, *and* $\gamma_n = n + \sqrt{n}$

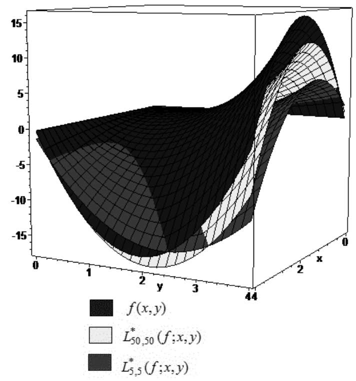

Figure 2: *The convergence of* $L^*_{n,m}(f; x, y)$ *to* $f(x, y)$, *for* $\alpha_n = \sqrt{n}$, $\beta_n = n$, *and* $\gamma_n = n + \ln(n+1)$

Example 2. *For* $n, m = 50$, *the comparison of convergences of* $L^*_{n,m}(f; x, y)$ *and the modified Szasz-Kantorovich operators defined by*

$$S^*_{n,m}(f; x, y) = \gamma_n \gamma_m \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} Q_k(\beta_n x) Q_j(\beta_m y) \int\limits_{k/\gamma_n}^{(k+1)/\gamma_n} \int\limits_{j/\gamma_m}^{(j+1)/\gamma_m} f(t, s) dt ds$$

to $f(x, y) = xy(x - 2)^2 e^{-y}$ *will be illustrated in Figure 3.*

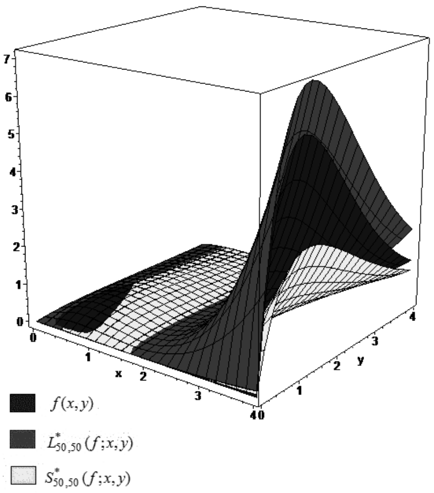

Figure 3: *The convergence of* $L^*_{n,m}(f;x,y)$ *and* $S^*_{n,m}(f;x,y)$ *to* $f(x,y)$, *for* $\alpha_n=\sqrt{n}$, $\beta_n=n$, *and* $\gamma_n=n+\sqrt{n}$

3. Degree of approximation by $L^*_{n,m}$

For $f\in C(A_{ab})$, the complete modulus of continuity for a bivariate case is defined as follows:

$$\omega(f;\delta)=\sup\left\{|f(t,s)-f(x,y)|:\sqrt{(t-x)^2+(s-y)^2}\le\delta\right\},$$

for every $(t,s),(x,y)\in\ A_{ab}$.

Further, partial moduli of continuity with respect to x and y are defined as

$$\omega^1(f;\delta)=\sup\left\{|f(x_1,y)-f(x_2,y)|:y\in[0,b]\ \ and\ \ |x_1-x_2|\le\delta\right\},$$
$$\omega^2(f;\delta)=\sup\left\{|f(x,y_1)-f(x,y_2)|:x\in[0,a]\ \ and\ \ |y_1-y_2|\le\delta\right\}.$$

It is clear that they satisfy the properties of the usual modulus of continuity [3]. Now, we give an estimate of the rate of convergence of operators $L^*_{n,m}$.

Theorem 2. *Let* $f\in C(A_{ab})$*; then for all* $(x,y)\in A_{ab}$, *we have*

$$|L^*_{n,m}(f;x,y)-f(x,y)|\le 2\omega\left(f;\delta_{n,m}\right),$$

where $\delta_{n,m}=\left(O\left(\alpha_n/n\right)(x+1)^2+O\left(1/\gamma_m\right)(y+1)^2\right)^{1/2}$.

Proof. Taking into account the complete modulus of continuity of $f(x,y)$, we can write

$$\begin{aligned}|L^*_{n,m}(f;x,y)-f(x,y)| &\leq L^*_{n,m}(|f(t,s)-f(x,y)|;x,y)\\ &\leq L^*_{n,m}\left(\omega\left(f;\sqrt{(t-x)^2+(s-y)^2}\right);x,y\right)\\ &\leq \omega\left(f;\delta_{n,m}\right)\left[1+\frac{1}{\delta_{n,m}}L^*_{n,m}\left(\sqrt{(t-x)^2+(s-y)^2};x,y\right)\right].\end{aligned}$$

Applying Cauchy-Scwartz inequality, by (6) and (7), we obtain

$$\begin{aligned}&|L^*_{n,m}(f;x,y)-f(x,y)|\\ &\quad\leq \omega\left(f;\delta_{n,m}\right)\left[1+\frac{1}{\delta_{n,m}}\left\{L^*_{n,m}\left((t-x)^2+(s-y)^2;x,y\right)\right\}^{1/2}\right]\\ &\quad\leq \omega\left(f;\delta_{n,m}\right)\left[1+\frac{1}{\delta_{n,m}}\left\{L^*_{n,m}\left((t-x)^2;x,y\right)+L^*_{n,m}\left((s-y)^2;x,y\right)\right\}^{1/2}\right]\\ &\quad\leq \omega\left(f;\delta_{n,m}\right)\left[1+\frac{1}{\delta_{n,m}}\left(O\left(\alpha_n/n\right)(x+1)^2+O\left(1/\gamma_m\right)(y+1)^2\right)^{1/2}\right].\end{aligned}$$

Taking $\delta_{n,m}=\left(O\left(\alpha_n/n\right)(x+1)^2+O\left(1/\gamma_m\right)(y+1)^2\right)^{1/2}$, for all $(x,y)\in A_{ab}$, we reach the result. $\square$

Theorem 3. *Let $f\in C(A_{ab})$; then the following inequalities satisfy*

$$|\ L^*_{n,m}(f;x,y)-f(x,y)|\leq 2\left(\omega^1(f;\delta_n(x))+\omega^2(f;\delta_m(y))\right),$$

where

$$\delta_n(x)=\left(\frac{x\left(\alpha_n-x\right)}{n}+\frac{\alpha_n^2}{3n^2}\right)^{1/2}$$

and

$$\delta_m(y)=\left(\frac{\left(\beta_m-\gamma_m\right)^2}{\gamma_m^2}y^2+\frac{2\beta_m-\gamma_m}{\gamma_m^2}y+\frac{1}{3\gamma_m^2}\right)^{1/2},$$

for all $(x,y)\in A_{ab}$.

Proof. Using the definition of partial moduli of continuity of $f(x,y)$ and applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&|L^*_{n,m}(f;x,y)-f(x,y)| \\
&\quad \leq L^*_{n,m}(|f(t,s)-f(x,y)|;x,y) \\
&\quad \leq L^*_{n,m}\left(|f(t,s)-f(x,s)|;x,y\right)+L^*_{n,m}(|f(x,s)-f(x,y)|;x,y) \\
&\quad \leq L^*_{n,m}\left(|\omega^1(f;|t-x|)|;x,y\right)+L^*_{n,m}\left(|\omega_2(f;|s-y|)|;x,y\right) \\
&\quad \leq \omega^1(f;\delta_n)\left[1+\frac{1}{\delta_n}L^*_{n,m}\left(|t-x|\,;x,y\right)\right] \\
&\qquad +\omega^1(f;\delta_m)\left[1+\frac{1}{\delta_m}L^*_{n,m}\left(|s-y|\,;x,y\right)\right] \\
&\quad \leq \omega^2(f;\delta_n)\left[1+\frac{1}{\delta_n}\left(L^*_{n,m}\left((t-x)^2\,;x,y\right)\right)^{1/2}\right] \\
&\qquad +\omega^2(f;\delta_m)\left[1+\frac{1}{\delta_m}\left(L^*_{n,m}\left((s-y)^2\,;x,y\right)\right)^{1/2}\right].
\end{aligned}$$

Taking into account (4) and (5), choosing

$$\delta_n=\left(\frac{x\left(\alpha_n-x\right)}{n}+\frac{\alpha_n^2}{3n^2}\right)^{1/2}$$

and

$$\delta_m=\left(\frac{\left(\beta_m-\gamma_m\right)^2}{\gamma_m^2}y^2+\frac{2\beta_m-\gamma_m}{\gamma_m^2}y+\frac{1}{3\gamma_m^2}\right)^{1/2},$$

for all $(x,y)\in A_{ab}$ we obtain the desired result. □

Some numerical examples

1. The error of the approximation of $f(x,y)=xye^{-y}$ by using the partial moduli of continuity of function f are listed in Table 1.

$(n=m)$	$\alpha_n=\sqrt{n}$, $\beta_n=n$, $\gamma_n=n+\sqrt{n}$	$\alpha_n=\sqrt{n}$, $\beta_n=n$, $\gamma_n=n+\ln(n+1)$
100	2.6203	2.0571
200	2.1422	1.5441
1000	1.2156	0.7726
1100	1.1721	0.7422
1200	1.1335	0.7155
2000	0.9284	0.5788
2100	0.9106	0.5673
2200	0.8940	0.5566
3000	0.7900	0.4908

Table 1:

2. The error of the approximation of $f(x,y)=\arctan(x+y)$ by using the partial moduli of continuity of function f are listed in Table 2.

$(n=m)$	$\alpha_n=\sqrt{n}$, $\beta_n=n$, $\gamma_n=n+\sqrt{n}$	$\alpha_n=\sqrt{n}$, $\beta_n=n$, $\gamma_n=n+\ln(n+1)$
100	1.8969	1.6346
200	1.5505	1.3182
1000	0.9558	0.8207
1100	0.9288	0.7988
1200	0.9048	0.7794
2000	0.7762	0.6756
2100	0.7650	0.6665
2200	0.7545	0.6579
3000	0.6879	0.6040

Table 2:

Now we estimate the degree of approximation by operators $L_{n,m}^*$ for the functions satisfying the Lipschitz condition. If $f \in Lip_M(\eta_1,\eta_2)$, then

$$|f(t,s)-f(x,y)| \le M\,|t-x|^{\eta_1}\,|s-y|^{\eta_2}$$

for $\eta_1,\eta_2 \in (0,1]$.

Theorem 4. *Let $f \in Lip_M(\eta_1,\eta_2)$; then we have*

$$||L_{n,m}^*(f;x,y)-f(x,y)|| \le M\delta_n^{\eta_1/2}\delta_m^{\eta_2/2},$$

where δ_n and δ_m are defined as in Theorem 3.

Proof. Since $f \in Lip_M(\eta_1,\eta_2)$, we can write

$$\begin{aligned}\left|L_{n,m}^*(f;x,y)-f(x,y)\right| &\le L_{n,m}^*\left(|f(t,s)-f(x,y)|\,;x,y\right)\\ &\le L_{n,m}^*\left(M\,|t-x|^{\eta_1}\,|s-y|^{\eta_2}\,;x,y\right)\\ &\le M\ {}_xB_n\left(|t-x|^{\eta_1}\,;x,y\right)\ {}_yS_m\left(|s-y|^{\eta_2}\,;x,y\right).\end{aligned}$$

Now, using the Hölder's inequality with

$$w_1=\frac{2}{\eta_1},u_1=\frac{2}{2-\eta_1},\ and\ w_2=\frac{2}{\eta_2},u_2=\frac{2}{2-\eta_2},$$

we have

$$\begin{aligned}\left|L_{n,m}^*(f;x,y)-f(x,y)\right| &\le M\ {}_xB_n\left(e_{0,0};x,y\right)^{(2-\eta_1)/2}\\ &\qquad\times\ {}_yS_m\left(\left(s-y\right)^2;x,y\right)^{\eta_2/2}\ {}_yS_m\left(e_{0,0};x,y\right)^{(2-\eta_2)/2}\\ &\le M\delta_n^{\eta_1/2}\delta_m^{\eta_2/2},\end{aligned}$$

which implies the desired result. □

Now, let $C^{(j)}(A_{ab})=\{f\in C(A_{ab}):f^{(i,j)}\in C(A_{ab}),1\le i,j\le 2\}$, where $f^{(i,j)}$ is the (i,j) th-order partial derivative with respect to x,y of f, endowed with the norm

$$||f||_{C^{(2)}(A_{ab})}=||f||_{C(A_{ab})}+||f||^1_{C(A_{ab})}+||f||^2_{C(A_{ab})},$$

where

$$||f||^{1}_{C(A_{ab})} = \sup_{(x,y)\in A_{ab}} \left\{ |f(x,y)|, |f^{(1,0)}(x,y)|, |f^{(0,1)}(x,y)| \right\}$$

and

$$||f||^{2}_{C(A_{ab})} = \sup_{(x,y)\in A_{ab}} \Big\{ |f(x,y)|, |f^{(1,0)}|x,y), |f^{(0,1)}|(x,y), |f^{(2,0)}(x,y)|, |f^{(1,1)}(x,y)|, \\ |f^{(0,2)}(x,y)| \Big\}.$$

Theorem 5. *Let $f \in C^{(1)}(A_{ab})$. Then we have*

$$|L^{*}_{n,m}(f;x,y) - f(x,y)| \leq \|f_x\| \, \delta_n + \|f_y\| \, \delta_m,$$

where δ_n, δ_m are defined as in Theorem 3.

Proof. From the hypothesis we can write

$$f(t,s) - f(x,y) = \int_x^t f_w(w,s)dw + \int_y^s f_u(x,u)du.$$

Applying $L^{*}_{n,m}(.;x,y)$ on both sides, we get

$$\begin{aligned} &|L^{*}_{n,m}(f;x,y) - f(x,y)| \\ &\quad \leq L^{*}_{n,m}\left(\int_x^t |f_w(w,s)| \, dw; x,y \right) + L^{*}_{n,m}\left(\int_y^s |f_u(x,u)| \, du; x,y \right). \end{aligned}$$

Since

$$\int_x^t |f_w(w,s)| \, dw \leq \|f_x\| \, |t-x| \ \text{ and } \int_y^s |f_u(x,u)| \, du \leq \|f_y\| \, |s-y|,$$

we have

$$|L^{*}_{n,m}(f;x,y) - f(x,y)| \leq \|f_x\| \, L^{*}_{n,m}(|t-x|;x,y) + \|f_y\| \, L^{*}_{n,m}(|s-y|;x,y).$$

Applying the Cauchy-Schwarz inequality and by (4) and (5) we obtain

$$\begin{aligned} |L^{*}_{n,m}(f;x,y) - f(x,y)| &\leq \|f_x\| \left(L^{*}_{n}\left((t-x)^2;x,y\right)\right)^{1/2} \left(L^{*}_{n}(e_{0,0};x,y)\right)^{1/2} \\ &\quad + \|f_y\| \left(L^{*}_{m}\left((s-y)^2;x,y\right)\right)^{1/2} \left(L^{*}_{m}(e_{0,0};x,y)\right)^{1/2} \\ |L^{*}_{n,m}(f;x,y) - f(x,y)| &\leq \|f_x\| \, \delta_n + \|f_y\| \, \delta_m. \end{aligned}$$

□

Next, we shall give an estimate for the order of approximation of the sequence $\left\{L^{*}_{n,m}(f)\right\}$ to the function $f \in C(A_{ab})$ in terms of the Peetre's K-functional defined as

$$\mathcal{K}(f;\delta) = \inf_{g\in C^{(2)}(A_{ab})} \{||f-g||_{C(A_{ab})} + \delta ||g||_{C^{(2)}(A_{ab})}, \delta > 0\}.$$

It is also known that the following inequality

$$\mathcal{K}(f;\delta) \leq M_1\{\overline{\omega_2}(f;\sqrt{\delta}) + \min(1,\delta)||f||_{C(A_{ab})}\} \tag{8}$$

holds for all $\delta > 0$ ([5], p. 192). The constant M_1 is independent of δ and f and $\overline{\omega_2}(f;\sqrt{\delta})$ is the second order complete modulus of continuity.

Theorem 6. *If $f \in C(A_{ab})$, then we have*

$$\begin{aligned}&\left|L^*_{n,m}(f;x,y) - f(x,y)\right| \\ &\quad \leq 4\mathcal{K}(f;F_{n,m}(x,y)) + \omega\left(\sqrt{\left(\frac{2nx+\alpha_n}{2n} - x\right)^2 + \left(\frac{2\beta_m y+1}{2\gamma_m} - y\right)^2}\right),\end{aligned}$$

where $F_{n,m}(x,y) = O\left(\frac{\alpha_n}{n}\right)\left(x^2+x+1\right) + O\left(\frac{1}{\gamma_m}\right)\left(y^2+y+1\right)$.

Furthermore,

$$\begin{aligned}&\left|L^*_{n,m}(f;x,y) - f(x,y)\right| \\ &\quad \leq M\left\{\overline{\omega_2}\left(f;\sqrt{F_{n,m}(x,y)}\right) + \min\{1,F_{n,m}(x,y)\}\|f\|_{C(A_{ab})}\right\} \\ &\quad +\omega\left(\sqrt{\left(\frac{2nx+\alpha_n}{2n} - x\right)^2 + \left(\frac{2\beta_m y+1}{2\gamma_m} - y\right)^2}\right).\end{aligned}$$

Proof. We define the following auxiliary operator

$$\widehat{L}^*_{n,m}(f;x,y) = L^*_{n,m}(f;x,y) - f\left(\frac{2nx+\alpha_n}{2n}, \frac{2\beta_m y+1}{2\gamma_m}\right) + f(x,y).$$

Then, by Lemma 2, we get

$$\widehat{L}^*_{n,m}(e_{1,0};x,y) = x,\ \widehat{L}^*_{n,m}(e_{0,1};x,y) = y.$$

Hence,

$$\widehat{L}^*_{n,m}((t-x);x,y) = 0, \widehat{L}^*_{n,m}((s-y);x,y) = 0.$$

Let $g \in C^{(2)}(A_{ab})$ and $t,s \in A_{ab}$. Using Taylor's theorem, we can write

$$\begin{aligned}g(t,s) - g(x,y) &= g(t,y) - g(x,y) + g(t,s) - g(t,y) \\ &= \frac{\partial g(x,y)}{\partial x}(t-x) + \int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du + \frac{\partial g(x,y)}{\partial y}(s-y) \\ &\quad + \int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv.\end{aligned}$$

Applying the operator $\widehat{L}^*_{n,m}(f;x,y)$, we obtain

$$\begin{aligned}
\widehat{L}^*_{n,m}(g;x,y)-g\left(x,y\right) &= \widehat{L}^*_{n,m}\left(\int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du;x,y\right) \\
&\quad +\widehat{L}^*_{n,m}\left(\int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv;x,y\right) \\
&= L^*_{n,m}\left(\int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du;x,y\right) \\
&\quad -\int_x^{\frac{2nx+\alpha_n}{2n}}\left(\frac{2nx+\alpha_n}{2n}-u\right)\frac{\partial^2 g(u,y)}{\partial u^2}du \\
&\quad +L^*_{n,m}\left(\int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv;x,y\right) \\
&\quad -\int_y^{\frac{2\beta_m y+1}{2\gamma_m}}\left(\frac{2\beta_m y+1}{2\gamma_m}-v\right)\frac{\partial^2 g(x,v)}{\partial v^2}dv.
\end{aligned}$$

Hence, using (6) and (7) and taking into account conditions (2) and (3), we have

$$\begin{aligned}
\left|\widehat{L}^*_{n,m}(g;x,y)-g\left(x,y\right)\right| &\le L^*_{n,m}\left(\int_x^t |t-u| \;\left|\frac{\partial^2 g(u,y)}{\partial u^2}\right|du;x,y\right) \\
&\quad +\int_x^{\frac{2nx+\alpha_n}{2n}}\left|\frac{2nx+\alpha_n}{2n}-u\frac{2nx+\alpha_n}{2n}-u\right|\left|\frac{\partial^2 g(u,y)}{\partial u^2}\right|du \\
&\quad +L^*_{n,m}\left(\int_y^s |s-v| \;\left|\frac{\partial^2 g(x,v)}{\partial v^2}\right|dv;x,y\right) \\
&\quad +\int_y^{\frac{2\beta_m y+1}{2\gamma_m}}\left|\frac{2\beta_m y+1}{2\gamma_m}-v\right|\left|\frac{\partial^2 g(x,v)}{\partial v^2}\right|dv \\
&\le \left\{L^*_{n,m}\left(\left(t-x\right)^2;x,y\right)+\left(\frac{2nx+\alpha_n}{2n}-x\right)^2\right\}\|g\|^2_{C^{(2)}(A_{ab})} \\
&\quad +\left\{L^*_{n,m}\left(\left(s-y\right)^2;x,y\right)+\left(\frac{2\beta_m y+1}{2\gamma_m}-y\right)^2\right\}\|g\|^2_{C^{(2)}(A_{ab})} \\
&\le \left(O\left(\frac{\alpha_n}{n}\right)+\left(\frac{\alpha_n}{n}\right)^2+O\left(\frac{1}{\gamma_m}\right)+\left(\frac{2\left(\beta_m-\gamma_m\right)y+1}{2\gamma_m}\right)^2\right) \\
&\quad \times\|g\|^2_{C^{(2)}(A_{ab})} \\
&\le \left(O\left(\frac{\alpha_n}{n}\right)+O\left(\frac{1}{\gamma_m}\right)\right)\|g\|^2_{C^{(2)}(A_{ab})}.
\end{aligned}$$

Also, from Lemma 3,

$$\begin{aligned}
\left|\widehat{L}^*_{n,m}(f;x,y)\right| &\le \left|L^*_{n,m}(f;x,y)\right|+\left|f\left(\frac{2nx+\alpha_n}{2n},\frac{2\beta_m y+1}{2\gamma_m}\right)\right|+\left|(f;x,y)\right| \\
&\le 3\left\|f\right\|_{C(A_{ab})}.
\end{aligned}$$

Therefore, for $f \in C\left(A_{ab}\right)$,

$$\begin{aligned}
&\left|L_{n,m}^{*}(f;x,y)-f\left(x,y\right)\right| \\
&\quad\leq \left|\widehat{L}_{n,m}^{*}(f;x,y)-f\left(x,y\right)+f\left(\frac{2nx+\alpha_n}{2n},\frac{2\beta_m y+1}{2\gamma_m}\right)-f\left(x,y\right)\right| \\
&\quad\leq \left|\widehat{L}_{n,m}^{*}(\left(f-g\right);x,y)\right|+\left|\widehat{L}_{n,m}^{*}(g;x,y)-g\left(x,y\right)\right| \\
&\qquad+\left|g\left(x,y\right)-f\left(x,y\right)\right|+\left|\widehat{L}_{n,m}^{*}(g;x,y)-g\left(x,y\right)\right| \\
&\qquad+\left|f\left(\frac{2nx+\alpha_n}{2n},\frac{2\beta_m y+1}{2\gamma_m}\right)-f\left(x,y\right)\right| \\
&\quad\leq 4\left\|f-g\right\|_{C(A_{ab})}+\left(O\left(\frac{\alpha_n}{n}\right)\left(x^2+x+1\right)+O\left(\frac{1}{\gamma_m}\right)\left(y^2+y+1\right)\right)\left\|g\right\|_{C^{(2)}(A_{ab})}^{2} \\
&\qquad+\left|f\left(\frac{2nx+\alpha_n}{2n},\frac{2\beta_m y+1}{2\gamma_m}\right)-f\left(x,y\right)\right| \\
&\quad\leq 4\left\|f-g\right\|_{C(A_{ab})}+\left(O\left(\frac{\alpha_n}{n}\right)\left(x^2+x+1\right)+O\left(\frac{1}{\gamma_m}\right)\left(y^2+y+1\right)\right)\left\|g\right\|_{C^{(2)}(A_{ab})}^{2} \\
&\qquad+\omega\left(\sqrt{\left(\frac{2nx+\alpha_n}{2n}-x\right)+\left(\frac{2\beta_m y+1}{2\gamma_m}-y\right)}\right).
\end{aligned}$$

Taking the infimum on the right-hand side over all $g \in C^{(2)}\left(A_{ab}\right)$ and using inequality (8), we obtain

$$\begin{aligned}
&\left|L_{n,m}^{*}(f;x,y)-f\left(x,y\right)\right| \\
&\quad\leq 4\mathcal{K}\left(f;F_{n,m}\left(x,y\right)\right)+\omega\left(\sqrt{\left(\frac{2nx+\alpha_n}{2n}-x\right)^2+\left(\frac{2\beta_m y+1}{2\gamma_m}-y\right)^2}\right) \\
&\quad\leq M\left\{\overline{\omega_2}\left(f;\sqrt{F_{n,m}\left(x,y\right)}\right)+\min\left\{1,F_{n,m}\left(x,y\right)\right\}\left\|f\right\|_{C(A_{ab})}\right\} \\
&\qquad+\omega\left(\sqrt{\left(\frac{2nx+\alpha_n}{2n}-x\right)^2+\left(\frac{2\beta_m y+1}{2\gamma_m}-y\right)^2}\right),
\end{aligned}$$

where $F_{n,m}\left(x,y\right)=O\left(\frac{\alpha_n}{n}\right)\left(x^2+x+1\right)+O\left(\frac{1}{\gamma_m}\right)\left(y^2+y+1\right)$. □

References

[1] A. E. Abdullayeva, A. N. Mammadova, *On order of approximation function by generalized Szasz operators and Bernstein-Chlodowsky polynomials*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **38**(2013), 3–8.

[2] F. Altomare, M. C. Montano, V. Leonessa, *On a generalization of Szász - Mirakjan-Kantorovich operators*, Results Math. **63**(2013), 837–863.

[3] G. A. Anastassiou, S. G. Gal, *Approximation theory: moduli of continuity and global smoothness preservation*, Birkhäuser, Boston, 2000.

[4] B. Baxhaku, F. Berisha, *Statistical Approximation of Kantorovich Type q-Bernstein-Schurer-Chlodowsky Operators*, International Journal of Education, Science, Technology, Innovation, Health and Environment **1**(2015), 95–102.

[5] P. L. Butzer, H. Berens, *Semi-groups of operators and approximation*, Springer, New York, 1967.

[6] P. L. Butzer, H. Karsli, *Voronovskaya-type theorems for derivatives of the Bernstein-Chlodovsky polynomials and the Szász-Mirakyan operator*, Comment. Math. **49**(2009), 33–58.

[7] İ. Büyükyazici, *On the approximation properties of two-dimensional q-Bernstein-Chlodowsky polynomials*, Math. Commun. **14**(2009), 255–269.

[8] İ. Büyükyazici, H. Sharma, *Approximation properties of two-dimensional q-Bernstein-Chlodowsky-Durrmeyer operators*, Numer. Funct. Anal. Optim. **33**(2012), 1351–1371.

[9] A. D. Gadjiev, R. O. Efendiev, E. İbikli, *Generalized Bernstein-Chlodowsky polynomials*, Rocky Mountain J. Math. **28**(1998), 1267–1277.

[10] E. A. Gadjieva, E. İbikli, *Weighted approximation by Bernstein-Chlodowsky polynomials*, Indian J. Pure Appl. Math. **30**(1999), 83–87.

[11] A. K. Gazanfer, İ. Büyükyazici, *Approximation by certain linear positive operators of two variables*, Abstr. Appl. Anal. 2014, Art. ID 782080, 6 pp.

[12] N. İspir, C. Atakut, *Approximation by modified Szász-Mirakjan operators on weighted spaces*, Proc. Indian Acad. Sci. Math. Sci. **112**(2002), 571–578.

[13] A. İzgi, İ. Büyükyazici, *On a generalization of Bernstein-Chlodovsky polynomials for two variables*, Int. Math. Forum **1**2006, 1001–1015.

[14] A. Izgi, *Approximation by composition of Szász-Mirakyan and Durrmeyer-Chlodowsky operators*, Eurasian Math. J. **3**(2012), 63–71.

[15] H. Karsli, *A Voronovskaya-type theory for the second derivative of the Bernstein-Chlodovsky polynomials*, Proc. Est. Acad. Sci. **61**(2012), 9–19.

[16] N. Mahmudov, V. Gupta, *On certain q-analogue of Szász-Kantorovich operators*, J. Appl. Math. Comput. **37**(2011), 407–419.

[17] M. Örkcü, *Approximation properties of bivariate extension of q-Szász-Mirakjan-Kantorovich operators*, J. Inequal. Appl. 2013, 10 pp.

[18] A. E. Piriyeva, *On order of approximation of functions by generalized Bernstein-Chlodowsky polynomials*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **21**(2004), 157–164.

[19] P. Pych-Taberska, H. Karsli, *On the rates of convergence of Bernstein-Chlodovsky polynomials and their Bézier-type variants*, Appl. Anal. **90**(2011), 403–416.

[20] Z. Walczak, *On certain modified Szász-Mirakyan operators for functions of two variables*, Demonstratio Math. **33**(2000), 91–100.