Math. Commun. **21**(2016), 45–64

Stability analysis of Hilfer fractional differential systems

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Received March 1, 2015; accepted October 29, 2015

Abstract. In this paper, we present some remarks on the stability of fractional order systems with the Hilfer derivative. Using the Laplace transform, some sufficient conditions on the stability and asymptotic stability of autonomous and non-autonomous fractional differential systems are given. The results are obtained via the properties of Mittag-Leffler functions and the non-standard Gronwall inequality.
AMS subject classifications: 26A33, 65L20
Key words: fractional order system, stability analysis, Hilfer derivative, Mittag-Leffler function

1. Introduction

In the last decades, the theory of the fractional calculus and fractional differential systems have received a lot of attention due to their potential applications in science and engineering [20, 16, 8, 10, 4, 5, 19]. There are several definitions of fractional integrals and derivatives in the literature, but the most popular definitions are in the sense of the Riemann-Liouville and Caputo.

The stability results of fractional differential systems have been the main goal in contributions. In 1996, Matignon studied for the first time the stability of autonomous linear fractional differential systems with the Caputo derivative from the control point of view [18]. Later, other research on the stability of fractional-order systems were presented. For example, Qian et al. [22] studied the stability theorems for fractional differential systems with the Riemann-Liouville derivative. In [11], authors derived the same results to [18] for the different case of order of the fractional derivative. Moreover, Deng et al. [7] studied the stability of n-dimensional linear fractional differential equations with time delays. The stability of distributed order fractional differential systems with respect to the nonnegative density function have also been studied [25, 24, 2, 3].

Recently, Hilfer has introduced a generalized form of the Riemann-Liouville fractional derivative [14]. In short, Hilfer fractional derivative ${}_{0^+}D_t^{\alpha,\beta}(t)$ is an interpolation between the Riemann-Liouville and Caputo fractional derivatives with applications in fractional evolutions equations [15], and physical problems [29]. Also,

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other researchers demonstrated new results for this type of a fractional derivative [12, 13, 27, 28, 17, 1].

In this paper, we intend to study the stability of linear autonomous and non-autonomous Hilfer fractional differential systems (commensurate and in commensurate) by using the properties of Mittag-Leffler functions. We used the non-standard Gronwall inequality, the Laplace transform and the final-value theorem to state our results in stability analysis of this type of fractional differential systems. The rest of the paper is organized as follows. In Section 2, we recall some necessary definitions and lemmas. Stability analysis for autonomous linear fractional differential systems with the Hilfer derivative is presented in Section 3. In Section 4, we investigate stability analysis for non-autonomous linear Hilfer fractional differential systems. Finally, the main conclusions are drawn in Section 5.

2. Preliminaries

In this section, we briefly mention some important notations, definitions and lemmas, which we use later.

Definition 1. *The Riemann-Liouville fractional integral of order α for an absolutely integrable function $f(t)$ is given by [20]*

$$\left({}_{0^+}I_t^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \quad t>0, \alpha>0, \tag{1}$$

where Γ is the gamma function and ${}_{0^+}I_t^0 f(t) = f(t)$.

Definition 2. *The Riemann-Liouville fractional derivative of order $0<\alpha<1$ for an absolutely integrable function $f(t)$ is defined by [20]*

$$({}_{0^+}D_t^{\alpha} f)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha}} d\tau = \left(\frac{d}{dt}\right) \left({}_{0^+}I_t^{1-\alpha} f\right)(t), \quad t>0. \tag{2}$$

Definition 3. *The Caputo fractional derivative of order $0<\alpha<1$ for the function $f(t)$ whose first derivative is absolutely integrable is defined by [20]*

$$\left({}_{0^+}^{C}D_t^{\alpha} f\right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau = \left({}_{0^+}I_t^{1-\alpha} \frac{df}{dt}\right)(t), \quad t>0. \tag{3}$$

Definition 4. *The Hilfer fractional derivative of order α and type β for an absolutely integrable function $f(t)$ is defined by [14]*

$$\left({}_{0^+}D_t^{\alpha,\beta} f\right)(t) = \left({}_{0^+}I_t^{\beta(1-\alpha)} \frac{d}{dt} {}_{0^+}I_t^{(1-\beta)(1-\alpha)} f\right)(t), \quad 0<\alpha<1, 0\leq\beta\leq 1. \tag{4}$$

Lemma 1. *The following fractional derivative formula holds [26]*

$${}_{0^+}D_t^{\alpha,\beta}(t^{\gamma}) = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}(t^{\gamma-\alpha}), \quad t>0, \gamma>-1, \tag{5}$$

where $0<\alpha<1$ and $0\leq\beta\leq 1$.

From Lemma 1, we easily have the following lemma.

Lemma 2. *By integrating relation* (5), *we obtain [1]*

$$\int_0^t {}_{0^+}D_t^{\alpha,\beta}(t^\gamma)\,dt = \frac{\Gamma(1+\gamma)}{\Gamma(1-\alpha+\gamma+1)}(t^{1-\alpha+\gamma}), \quad t>0, \gamma>-1, \tag{6}$$

where $0<\alpha<1$ *and* $0\le\beta\le 1$.

Remark 1. *The Caputo derivative represents a type of regularization in the time domain (origin) for the Riemann-Liouville derivative.*

Remark 2. *The Hilfer fractional derivative interpolates between the Riemann-Liouville fractional derivative and the Caputo fractional derivative.*

(i) When $\beta=0$, $0<\alpha<1$, *the Hilfer fractional derivative corresponds to the classical Riemann-Liouville fractional derivative*

$${}_{0^+}D_t^{\alpha,0}f(t) = \frac{d}{dt}{}_{0^+}I_t^{1-\alpha}f(t) = {}_{0^+}D_t^{\alpha}f(t).$$

(ii) When $\beta=1$, $0<\alpha<1$, *the Hilfer fractional derivative corresponds to the classical Caputo fractional derivative*

$${}_{0^+}D_t^{\alpha,1}f(t) = {}_{0^+}I_t^{1-\alpha}\frac{d}{dt}f(t) = {}_{0^+}^{C}D_t^{\alpha}f(t).$$

Remark 3. *Let* $0<\alpha<1, 0\le\beta\le 1, f(t)\in L^1[0,b], 0<t<b\le+\infty$,

$$f(t) * \frac{t^{(1-\alpha)(1-\beta)-1}}{\Gamma\left((1-\alpha)(1-\beta)\right)} \in AC^1[0,b].$$

The Laplace transform of the Hilfer derivative is given by [14]

$$\mathcal{L}\left\{{}_{0^+}D_t^{\alpha,\beta}f(t)\right\} = s^\alpha\mathcal{L}\left\{f(t)\right\} - s^{\beta(\alpha-1)}\left({}_{0^+}I_t^{(1-\beta)(1-\alpha)}f\right)(0^+), \tag{7}$$

where

$$\left({}_{0^+}I_t^{(1-\beta)(1-\alpha)}f\right)(0^+) = \lim_{t\to 0^+}\left({}_{0^+}I_t^{(1-\beta)(1-\alpha)}f\right)(t). \tag{8}$$

It is clear that the initial conditions that must be considered as the fractional integral of order $(1-\beta)(1-\alpha)$, $\left({}_{0^+}I_t^{(1-\beta)(1-\alpha)}f\right)(0^+)$.

Definition 5. *The Mittag-Leffler function with parameter* α *is given by [16]*

$$E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \quad \Re(\alpha)>0, z\in\mathbb{C}. \tag{9}$$

It is obvious that $E_\alpha(z)=e^z$ *for* $\alpha=1$.

Definition 6. *A generalization of the Mittag-Leffler function with two parameters α and β is given by [16]*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \Re(\alpha) > 0, \Re(\beta) > 0, z \in \mathbb{C}. \tag{10}$$

By means of the series representation, a generalization of (9) and (10) was introduced by Prabhakar [21] as

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{k!\Gamma(\alpha k + \beta)}, \quad \Re(\alpha) > 0, \Re(\beta) > 0, \gamma > 0, z \in \mathbb{C}, \tag{11}$$

where

$$(\gamma)_k = \gamma(\gamma+1)\dots(\gamma+k-1) = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}, \quad (\gamma)_0 = 1, \gamma \neq 0.$$

One of the widely used relations in this paper are the Laplace transforms of the matrix Mittag-Leffler function. The following lemma allows us to find such transforms.

Lemma 3. *Let $\alpha > 0$, $\beta > 0$ be two arbitrary real numbers, A an arbitrary square matrix of dimension n, and $I - As^{-\alpha}$ an invertible matrix. Then the following relation holds [6]*

$$\mathcal{L}\{t^{\beta-1}E_{\alpha,\beta}(At^{\alpha})\} = s^{-\beta}(I - As^{-\alpha})^{-1}, \quad \Re(s) > \|A\|^{1/\alpha}, \tag{12}$$

where I is the identity matrix of dimension n and $\|.\|$ denotes the l_2-norm.

The Mittag-Leffler function has the following asymptotic expression.

Lemma 4. *Let $0 < \alpha < 2$, $\beta > 0$ be an arbitrary real number, $\beta - \alpha k$ is not a negative integer for $k = 1, 2, \dots$. Then the following asymptotic expansions hold [16]*

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp(z^{1/\alpha}) - \sum_{k=1}^{p} \frac{1}{\Gamma(\beta - \alpha k)} \frac{1}{z^k} + O(\frac{1}{|z|^{p+1}}), \tag{13}$$

as $|z| \longrightarrow \infty$, $|\arg(z)| \leq \frac{\alpha\pi}{2}$ and

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{p} \frac{1}{\Gamma(\beta - \alpha k)} \frac{1}{z^k} + O(\frac{1}{|z|^{p+1}}), \tag{14}$$

as $|z| \longrightarrow \infty$, $|\arg(z)| > \frac{\alpha\pi}{2}$.

Remark 4. *In Lemma 4, the function $O(\frac{1}{|z|^{p+1}})$ has the following form*

$$\sum_{k=0}^{\infty} a_k \frac{1}{|z|^{k+p+1}}, \quad a_0 \neq 0, a_k \in \mathbb{R}.$$

Thus, the series can be differentiated (integrated) term by term.

Remark 5. *In Lemma 4, for the case $\alpha = \beta$ we have [22]*

$$E_{\alpha,\alpha}(z) = \frac{1}{\alpha} z^{(1-\alpha)/\alpha} \exp(z^{1/\alpha}) - \sum_{k=2}^{p} \frac{1}{\Gamma(\alpha - \alpha k)} \frac{1}{z^k} + O(\frac{1}{|z|^{p+1}}), \qquad (15)$$

as $|z| \longrightarrow \infty$, $|\arg(z)| \leq \frac{\alpha\pi}{2}$ and

$$E_{\alpha,\alpha}(z) = -\sum_{k=2}^{p} \frac{1}{\Gamma(\alpha - \alpha k)} \frac{1}{z^k} + O(\frac{1}{|z|^{p+1}}), \qquad (16)$$

when $|z| \longrightarrow \infty$, $|\arg(z)| > \frac{\alpha\pi}{2}$.

Lemma 5 (Non-standard Gronwall Inequality, [23])**.** *Suppose that $g(t)$ and $u(t)$ are continuous functions on the interval $[t_0, t]$ such that $g(t) \geq 0$. Also, $\lambda \geq 0$ and $r \geq 0$ are two constants and*

$$u(t) \leq \lambda + \int_{t_0}^{t} [g(\tau)u(\tau) + r] d\tau.$$

Then

$$u(t) \leq (\lambda + r(t_1 - t_0)) \exp(\int_{t_0}^{t} g(\tau) d\tau), \quad t_0 \leq t \leq t_1. \qquad (17)$$

3. Stability analysis of linear autonomous Hilfer fractional differential systems

In this section, we discuss the stability of the following linear autonomous Hilfer fractional differential system

$$\begin{aligned} {}_{0^+}D_t^{\boldsymbol{\alpha},\beta} x(t) &= A\, x(t), \quad t > 0, 0 \leq \beta \leq 1, \\ {}_{0^+}I_t^{\mathbf{1}-\boldsymbol{\gamma}} x(0^+) &= x_0, \end{aligned} \qquad (18)$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ is a matrix, $x_0 = (x_{10}, x_{20}, \ldots, x_{n0})^T$, $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, $\mathbf{1} = (1, 1, \ldots, 1)$ and $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_n]$ such that $0 < \alpha_i < 1$, $\gamma_i = \alpha_i + \beta - \alpha_i\beta$, for $i = 1, 2, \ldots, n$.

Remark 6. *If $\alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_n$, system* (18) *is called a commensurate order system; otherwise, system* (18) *is an in commensurate order system.*

Definition 7. *Linear Hilfer fractional differential system* (18) *is said to be*

(i) stable if for any initial value x_0, there exists an $\epsilon > 0$ such that $\|x(t)\| \leq \epsilon$ for all $t > 0$,

(ii) asymptotically stable if at first it is stable and $\|x(t)\| \to 0$ as $t \to \infty$.

In what follows, first we prove stability theorems for commensurate order systems, then we give conditions for their asymptotic stability.

Theorem 1. *The solution of linear commensurate order system* (18) *is given by*

$$x(t) = x_0 t^{\gamma-1} E_{\alpha,\gamma}(At^{\alpha}), \quad \gamma = \alpha + \beta - \alpha\beta. \tag{19}$$

Proof. Using formula (7) and taking the Laplace transform of (18), we have

$$s^{\alpha} X(s) - s^{\beta(\alpha-1)} x_0 = A\, X(s).$$

Thus

$$\begin{aligned} X(s) &= x_0\, s^{\beta(\alpha-1)}\, (Is^{\alpha} - A)^{-1} = x_0\, s^{\beta(\alpha-1)-\alpha}\, (I - As^{-\alpha})^{-1} \\ &= x_0\, s^{-\gamma}\, (I - As^{-\alpha})^{-1}. \end{aligned} \tag{20}$$

At this point, by applying the inverse of the Laplace transform on both sides of the above relation and using relation (12), we obtain the claimed result. □

Next, we discuss the asymptotic stability of system (18) when A has non-zero eigenvalues.

Theorem 2. *If all eigenvalues of A satisfy*

$$|\arg(\lambda(A))| > \frac{\alpha\pi}{2}, \tag{21}$$

then fractional differential system (18) *is asymptotically stable.*

Proof. According to the above theorem, we can state that the solution of (18) is described by (19). For A, there exists an invertible matrix P such that $P^{-1}AP = J$, where J is the Jordan canonical form of the matrix A with eigenvalues on the diagonal. For this decomposition, we consider two cases as follows.

Case 1. For $J = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$, we have

$$E_{\alpha,\gamma}(At^{\alpha}) = P\,[E_{\alpha,\gamma}(Jt^{\alpha})]\, P^{-1} = P\,[diag(E_{\alpha,\gamma}(\lambda_1 t^{\alpha}), \ldots, E_{\alpha,\gamma}(\lambda_n t^{\alpha}))]\, P^{-1},$$

where $E_{\alpha,\gamma}(\lambda_i t^{\alpha})$ is given as (according to (14))

$$E_{\alpha,\gamma}(\lambda_i t^{\alpha}) = -\sum_{k=1}^{p} \frac{1}{\Gamma(\gamma - \alpha k)} \frac{1}{(\lambda_i t^{\alpha})^k} + O(\frac{1}{|\lambda_i t^{\alpha}|^{p+1}}) \to 0, \text{ as } t \to +\infty, 1 \le i \le n.$$

Hence

$$\lim_{t\to+\infty} \|E_{\alpha,\gamma}(Jt^{\alpha})\| = \lim_{t\to+\infty} \|diag(E_{\alpha,\gamma}(\lambda_1 t^{\alpha}), \ldots, E_{\alpha,\gamma}(\lambda_n t^{\alpha}))\| = 0,$$

and

$$\begin{aligned} \lim_{t\to+\infty} \|x(t)\| &= \lim_{t\to+\infty} \left\| x_0 t^{\gamma-1} E_{\alpha,\gamma}(At^{\alpha}) \right\| \\ &= \lim_{t\to+\infty} \left\| P[x_0 t^{\gamma-1} E_{\alpha,\gamma}(Jt^{\alpha})]P^{-1} \right\| = 0. \end{aligned}$$

Case 2. We consider $J = diag(J_1, J_2, \ldots, J_s)$, where $J_i,\ 1 \le i \le s$ has the following form

$$J_i = \begin{pmatrix} \lambda_i & 1 & & & \\ & \lambda_i & \ddots & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{pmatrix}_{n_i \times n_i}, \quad \lambda_i \in \mathbb{C},$$

and $\sum_{i=1}^{s} n_i = n$. In this case, we have

$$\begin{aligned} E_{\alpha,\gamma}(At^\alpha) &= P[E_{\alpha,\gamma}(Jt^\alpha)]P^{-1} = P\,[\sum_{k=0}^{\infty} \frac{diag(J_1^k, J_2^k, \ldots, J_s^k)t^{\alpha k}}{\Gamma(\alpha k + \gamma)}]\,P^{-1} \\ &= P\,[diag(E_{\alpha,\gamma}(J_1 t^\alpha), E_{\alpha,\gamma}(J_2 t^\alpha), \ldots, E_{\alpha,\gamma}(J_s t^\alpha))]\,P^{-1}, \end{aligned}$$

where the matrix $E_{\alpha,\gamma}(J_i t^\alpha),\ 1 \le i \le s$, can be written as

$$\begin{aligned} &E_{\alpha,\gamma}(J_i t^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{(J_i t^\alpha)^k}{\Gamma(\alpha k + \gamma)} = \sum_{k=0}^{\infty} \frac{(t^\alpha)^k}{\Gamma(\alpha k + \gamma)} \begin{pmatrix} \lambda_i^k & C_k^1 \lambda_i^{k-1} & \ldots & C_k^{n_i-1} \lambda_i^{k-n_i+1} \\ 0 & \lambda_i^k & \ddots & \vdots \\ \vdots & \ddots & \ddots & C_k^1 \lambda_i^{k-1} \\ 0 & \ldots & 0 & \lambda_i^k \end{pmatrix} \\ &= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{(t^\alpha)^k}{\Gamma(\alpha k+\gamma)} \lambda_i^k & \sum_{k=0}^{\infty} \frac{(t^\alpha)^k}{\Gamma(\alpha k+\gamma)} C_k^1 \lambda_i^{k-1} & \ldots & \sum_{k=0}^{\infty} \frac{(t^\alpha)^k}{\Gamma(\alpha k+\gamma)} C_k^{n_i-1} \lambda_i^{k-n_i+1} \\ 0 & \sum_{k=0}^{\infty} \frac{(t^\alpha)^k}{\Gamma(\alpha k+\gamma)} \lambda_i^k & \ddots & \vdots \\ \vdots & \ddots & \ddots & \sum_{k=0}^{\infty} \frac{(t^\alpha)^k}{\Gamma(\alpha k+\gamma)} C_k^1 \lambda_i^{k-1} \\ 0 & \ldots & 0 & \sum_{k=0}^{\infty} \frac{(t^\alpha)^k}{\Gamma(\alpha k+1)} \lambda_i^k \end{pmatrix} \\ &= \begin{pmatrix} E_\alpha(\lambda_i t^\alpha) & \frac{1}{1!}\frac{\partial}{\partial \lambda_i} E_{\alpha,\gamma}(\lambda_i t^\alpha) & \ldots & \frac{1}{(n_i-1)!}\left(\frac{\partial}{\partial \lambda_i}\right)^{n_i-1} E_{\alpha,\gamma}(\lambda_i t^\alpha) \\ 0 & E_{\alpha,\gamma}(\lambda_i t^\alpha) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{1}{1!}\frac{\partial}{\partial \lambda_i} E_{\alpha,\gamma}(\lambda_i t^\alpha) \\ 0 & \ldots & 0 & E_{\alpha,\gamma}(\lambda_i t^\alpha) \end{pmatrix}, \end{aligned}$$

and coefficients C_k^j, $1 \le j \le n_i, 1 \le i \le s$, the binomial coefficients such that

$$C_k^j = \begin{pmatrix} k \\ j \end{pmatrix} = \begin{cases} \frac{k!}{j!\,(k-j)!}, & \text{for } 0 \le j \le k, \\ 0, & \text{otherwise} \end{cases}.$$

The non-zero elements of $E_{\alpha,\gamma}(J_i t^\alpha)$ can be described uniformly as

$$\frac{1}{(j-1)!}\left\{\left(\frac{\partial}{\partial\lambda}\right)^{j-1} E_{\alpha,\gamma}(\lambda t^\alpha)\right\}\Bigg|_{\lambda=\lambda_i} \qquad j = 1, 2, \ldots, n_i, 1 \le i \le s.$$

If $|\arg(\lambda_i)| > \alpha\pi/2$ and $t \to +\infty$, then from Lemma 5 we have

$$E_{\alpha,\gamma}(\lambda_i t^\alpha) = -\sum_{k=1}^{p} \frac{1}{\Gamma(\gamma-\alpha k)} \frac{1}{(\lambda_i t^\alpha)^k} + O(\frac{1}{|\lambda_i t^\alpha|^{p+1}}),$$

which implies $|E_{\alpha,\gamma}(\lambda_i t^\alpha)| \to 0$ as $t \to +\infty$, and

$$\begin{aligned}
&\frac{1}{(j-1)!}\left(\frac{\partial}{\partial\lambda_i}\right)^{j-1} E_{\alpha,\gamma}(\lambda_i t^\alpha) \\
&\quad = \frac{1}{(j-1)!}\left(\frac{\partial}{\partial\lambda_i}\right)^{j-1} \left\{-\sum_{k=1}^{p} \frac{1}{\Gamma(\gamma-\alpha k)} \frac{1}{(\lambda_i t^\alpha)^k} + O(\frac{1}{|\lambda_i t^\alpha|^{p+1}})\right\} \\
&\quad = -\sum_{k=1}^{p} \frac{(-1)^{j-1}(k+j-2)\ldots(k+1)k}{(j-1)!\Gamma(\gamma-\alpha k)} \frac{1}{\lambda_i^{k+j-1} t^{\alpha k}} + O(\frac{1}{|\lambda_i|^{p+j}\,|t^\alpha|^{p+1}}) \\
&\quad = -\sum_{k=1}^{p} \frac{(-1)^{j-1}(k+j-2)!}{(j-1)!(k-1)!\Gamma(\gamma-\alpha k)} \frac{1}{\lambda_i^{k+j-1} t^{\alpha k}} + O(\frac{1}{|\lambda_i|^{p+j}\,|t^\alpha|^{p+1}}),
\end{aligned}$$

which leads to

$$\left|\frac{1}{(j-1)!}\left(\frac{\partial}{\partial\lambda}\right)^{j-1} E_{\alpha,\gamma}(\lambda_i t^\alpha)\right| \to 0, \quad 1 \le j \le n_i \text{ as } t \to +\infty.$$

It

$$\lim_{t\to+\infty} \|x(t)\| = \lim_{t\to+\infty} \left\|x_0 t^{\gamma-1} E_{\alpha,\gamma}(At^\alpha)\right\| = 0,$$

for any non-zero initial value x_0. The proof is complete. $\square$

Remark 7. *If A has an eigenvalue λ_0 such that $|\arg(\lambda_0)| < \alpha\pi/2$, then system* (18) *is unstable.*

Proof. First, we suppose A is similar to a diagonal matrix. According to (13), we have

$$\begin{aligned}
E_{\alpha,\gamma}(\lambda_0 t^\alpha) &= \frac{1}{\alpha}\lambda_0^{(1-\gamma)/\alpha} t^{(1-\gamma)} \exp(\lambda_0^{1/\alpha} t) - \sum_{k=1}^{p} \frac{1}{\Gamma(\gamma-\alpha k)} \frac{1}{\lambda_0^k t^{\alpha k}} + O(\frac{1}{|\lambda_0 t^\alpha|^{p+1}}) \\
&\to +\infty \text{ as } t \to +\infty,
\end{aligned}$$

and $\lim_{t\to+\infty} \|x(t)\| = \lim_{t\to+\infty} \left\|x_0 t^{\gamma-1} E_{\alpha,\gamma}(At^{\alpha})\right\| = +\infty$.

Next, we consider the case when A is similar to a Jordan form (see the proof of Theorem 2). From the asymptotic expansion (13), we have

$$\begin{aligned}
&\frac{1}{(j-1)!}\left\{\left(\frac{\partial}{\partial\lambda_0}\right)^{j-1} E_{\alpha,\gamma}(\lambda_0 t^{\alpha})\right\}\\
&\quad=\frac{1}{(j-1)!}\left\{\left(\frac{\partial}{\partial\lambda_0}\right)^{j-1}\left(\frac{1}{\alpha}\lambda_0^{(1-\gamma)/\alpha} t^{(1-\gamma)}\exp(\lambda_0^{1/\alpha}t)\right.\right.\\
&\qquad\left.\left.-\sum_{k=1}^{p}\frac{1}{\Gamma(\gamma-\alpha k)}\frac{1}{\lambda_0^k t^{\alpha k}}+O(\frac{1}{|\lambda_0 t^{\alpha}|^{p+1}})\right)\right\}\\
&\quad=\frac{1}{(j-1)!}\left\{\frac{(1-\gamma)(1-\gamma-\alpha)\dots(1-\gamma-(j-2)\alpha)}{\alpha^j}\lambda_0^{(1-\gamma-(j-1)\alpha)/\alpha}t^{1-\gamma}\right.\\
&\qquad+\dots+\frac{((j-1)j-(j-1)\gamma-(j-1)(j-2)\alpha)}{2\alpha^j}\lambda_0^{(j-1-\gamma-(j-1)\alpha)/\alpha}t^{j-\gamma-1}\\
&\qquad\left.+\frac{1}{\alpha^j}\lambda_0^{(j-\gamma-(j-1)\alpha)/\alpha}t^{j-\gamma}\right\}\exp(\lambda_0^{1/\alpha}t)\\
&\qquad-\frac{1}{(j-1)!}\sum_{k=1}^{p}\frac{(-k)(-k-1)\dots(-k-j+2)}{\Gamma(\gamma-\alpha k)}\frac{1}{\lambda_0^{k+j-1}t^{\alpha k}}+O(\frac{1}{|\lambda_0|^{p+j}\,t^{(p+1)\alpha}}).
\end{aligned}$$

Thus for t large enough

$$\begin{aligned}
&\left|\frac{t^{\gamma-1}}{(j-1)!}\left\{\left(\frac{\partial}{\partial\lambda_0}\right)^{j-1} E_{\alpha,\gamma}(\lambda_0 t^{\alpha})\right\}\right|\\
&\quad\geq\frac{1}{(j-1)!}\left\{\ \left|\frac{1}{\alpha^j}\lambda_0^{(j-\gamma-(j-1)\alpha)/\alpha}t^{j-1}\right|\right.\\
&\qquad-\left|\frac{(1-\gamma)(1-\gamma-\alpha)\dots(1-\gamma-(j-2)\alpha)}{\alpha^j}\lambda_0^{(1-\gamma-(j-1)\alpha)/\alpha}\right|\\
&\qquad\left.-\dots-\left|\frac{((j-1)j-(j-1)\gamma-(j-1)(j-2)\alpha)}{2\alpha^j}\lambda_0^{(j-1-\gamma-(j-1)\alpha)/\alpha}t^{j-2}\right|\right\}\\
&\qquad\times\exp(|\lambda_0|^{1/\alpha}\cos\left(\frac{\arg(\lambda_0)}{\alpha}\right)t)\\
&\qquad-\frac{1}{(j-1)!}\sum_{k=1}^{p}\frac{|(-k)(-k-1)\dots(-k-j+1)|}{|\Gamma(\gamma-\alpha k)|}\frac{1}{\left|\lambda_0^{k+j}t^{\alpha k-j+1}\right|}\\
&\qquad+O(\frac{1}{|\lambda_0|^{p+j}\,t^{(p+1)\alpha-\gamma+1}})
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(j-1)!}\Bigg\{\frac{t^{j-1}}{\alpha^j}\,|\lambda_0|^{(j-\gamma-(j-1)\alpha)/\alpha} \\
&\quad -\frac{(1-\gamma)(1-\gamma-\alpha)\ldots(1-\gamma-(j-2)\alpha)}{\alpha^j}\,|\lambda_0|^{(1-\gamma-(j-1)\alpha)/\alpha} \\
&\quad -\cdots-\frac{((j-1)j-(j-1)\gamma-(j-1)(j-2)\alpha)t^{j-2}}{2\alpha^j}\,|\lambda_0|^{(j-1-\gamma-(j-1)\alpha)/\alpha}\Bigg\} \\
&\quad \times \exp(|\lambda_0|^{1/\alpha}\cos\left(\frac{\arg(\lambda_0)}{\alpha}\right)t) \\
&\quad -\frac{1}{(j-1)!}\sum_{k=1}^{p}\frac{|(-k)(-k-1)\ldots(-k-j+1)|}{|\Gamma(\gamma-\alpha k)|}\frac{1}{\left|\lambda_0^{k+j}t^{\alpha k-j+1}\right|} \\
&\quad +O(\frac{1}{|\lambda_0|^{p+j}\,t^{(p+1)\alpha-\gamma+1}})\to+\infty \quad \text{as} \quad t\to+\infty.
\end{aligned}
$$

Since $\left|\frac{\arg(\lambda_0)}{\alpha}\right| < \pi/2$, we have $\cos(\frac{\arg(\lambda_0)}{\alpha}) > 0$. Therefore, the first term of the last equality is the highest order term. Hence system (18) is unstable. $\square$

Remark 8. *If A has zero eigenvalue, then system* (18) *is unstable.*

Proof. By induction we always have

$$
\begin{aligned}
\left(\frac{\partial}{\partial\lambda}\right)^{j-1}E_{\alpha,\gamma}(\lambda t^\alpha) &= \sum_{k=0}^{\infty}\frac{(k+j-1)!\lambda^k\,t^{\alpha(k+j-1)}}{k!\Gamma(\alpha k+\alpha(j-1)+\gamma)} \\
&= \frac{(j-1)!\,t^{\alpha(j-1)}}{\Gamma(\alpha(j-1)+\gamma)}+\sum_{k=1}^{\infty}\frac{(k+j-1)!\lambda^k\,t^{\alpha(k+j-1)}}{k!\Gamma(\alpha k+\alpha(j-1)+\gamma)}.
\end{aligned} \tag{22}
$$

Substituting $\lambda = 0$ into (22), we obtain

$$
\left(\frac{\partial}{\partial\lambda}\right)^{j-1}E_{\alpha,\gamma}(\lambda t^\alpha) = \frac{(j-1)!\,t^{\alpha(j-1)}}{\Gamma(\alpha(j-1)+\gamma)}.
$$

Now, multiplying $\frac{t^{\gamma-1}}{(j-1)!}$ on both sides of the above equality yields

$$
\frac{t^{\gamma-1}}{(j-1)!}\left\{\left(\frac{\partial}{\partial\lambda}\right)^{j-1}E_{\alpha,\gamma}(\lambda t^\alpha)\right\} = \frac{t^{(j-1)\alpha+\gamma-1}}{\Gamma((j-1)\alpha+\gamma)}.
$$

Now it is obvious that

$$
\lim_{t\to+\infty}\frac{t^{(j-1)\alpha+\gamma-1}}{\Gamma((j-1)\alpha+\gamma)} = +\infty \quad \text{for} \quad j\geq 1.
$$

Thus, $\lim\limits_{t\to+\infty}\|x(t)\| = +\infty$. $\square$

Theorem 3. *If all eigenvalues of A satisfy*

$$|\arg(\lambda(A))| \geq \frac{\alpha\pi}{2}, \tag{23}$$

and the critical eigenvalues satisfying $|\arg(\lambda(A))| = \frac{\alpha\pi}{2}$ have the same algebraic and geometric multiplicities, then system (18) *is stable but not asymptotically stable.*

Proof. Without loss of generality, suppose there exists a critical eigenvalue with algebraic and geometric multiplicity both equal to one (say λ_i satisfying $|\arg(\lambda_i)| = \alpha\pi/2$). Then, from (19) the solution of system (18) is given by

$$\begin{aligned} x(t) &= x_0 t^{\gamma-1} E_{\alpha,\gamma}(At^\alpha) \\ &= x_0 t^{\gamma-1} P\left[diag(E_{\alpha,\gamma}(J_1 t^\alpha), E_{\alpha,\gamma}(J_2 t^\alpha), \ldots, E_{\alpha,\gamma}(J_s t^\alpha))\right] P^{-1}, \end{aligned}$$

where J_k's are Jordan block matrices with order k, $|\arg(\lambda_k)| > \frac{\alpha\pi}{2}$, and

$$\sum_{k=1}^{i-1} n_k + \sum_{k=i+1}^{r} n_k + 1 = n, \quad k = 1, \ldots, i-1, i+1, \ldots, r.$$

Next, from (13) we have

$$E_{\alpha,\gamma}(\lambda_i t^\alpha) = \frac{1}{\alpha}\lambda_i^{(1-\gamma)/\alpha} t^{(1-\gamma)} \exp(\lambda_i^{1/\alpha} t) - \sum_{k=1}^{p} \frac{1}{\Gamma(\gamma-\alpha k)} \frac{1}{\lambda_i^k t^{\alpha k}} + O(\frac{1}{|\lambda_i t^\alpha|^{p+1}}).$$

Now, if we set $\lambda_i = r\left(\cos\frac{\alpha\pi}{2} + i\,\sin\frac{\alpha\pi}{2}\right)$, where r is the modulus of λ_i, then we get

$$\begin{aligned} & t^{\gamma-1} E_{\alpha,\gamma}(\lambda_i t^\alpha) \\ &= \frac{1}{\alpha}\left(r\left(\cos\frac{\alpha\pi}{2} + i\,\sin\frac{\alpha\pi}{2}\right)\right)^{(1-\gamma)/\alpha} \exp\left\{\left(r\left(\cos\frac{\alpha\pi}{2} + i\,\sin\frac{\alpha\pi}{2}\right)\right)^{1/\alpha} t\right\} \\ &\quad - \sum_{k=1}^{p} \frac{(r(\cos\frac{\alpha\pi}{2} + i\,\sin\frac{\alpha\pi}{2}))^{-k} t^{\gamma-\alpha k-1}}{\Gamma(\gamma-\alpha k)} \\ &\quad + O\left(\left|r\left(\cos\frac{\alpha\pi}{2} + i\,\sin\frac{\alpha\pi}{2}\right)\right|^{-p-1} |t|^{\gamma-\alpha p-\alpha-1}\right) \\ &= \frac{1}{\alpha} r^{(1-\gamma)/\alpha}\left(\cos\frac{(1-\gamma)\pi}{2} + i\,\sin\frac{(1-\gamma)\pi}{2}\right) \exp\left\{r^{1/\alpha} t\left(\cos\frac{\pi}{2} + i\,\sin\frac{\pi}{2}\right)\right\} \\ &\quad - \sum_{k=1}^{p} \frac{r^{-k} t^{\gamma-\alpha k-1}\left(\cos\frac{-\alpha k\pi}{2} + i\,\sin\frac{-\alpha k\pi}{2}\right)}{\Gamma(\gamma-\alpha k)} + O\left(t^{\gamma-\alpha p-\alpha-1}\right) \\ &= \frac{1}{\alpha} r^{(1-\gamma)/\alpha}\left(\sin\frac{\gamma\pi}{2} + i\,\cos\frac{\gamma\pi}{2}\right) \exp\left\{i r^{1/\alpha} t\right\} \\ &\quad - \sum_{k=1}^{p} \frac{r^{-k} t^{\gamma-\alpha k-1}\left(\cos\frac{\alpha k\pi}{2} - i\,\sin\frac{\alpha k\pi}{2}\right)}{\Gamma(\gamma-\alpha k)} + O\left(t^{\gamma-\alpha p-\alpha-1}\right). \end{aligned}$$

The absolute value of the first term of the right-hand side of the above equality equals $\frac{1}{\alpha} r^{(1-\gamma)/\alpha}$, whereas the rest of the terms tends to zero as $t \to +\infty$. From the proof of Theorem 2, $E_{\alpha,\gamma}(J_k t^\alpha)$ tends to zero as $t \to +\infty$ for $k = 1, \ldots, i-1, 1, i+1, \ldots, r$. Therefore, we deduce that system (18) is stable but not asymptotically stable. $\square$

Now, we consider an in commensurate linear Hilfer fractional differential system

$$\begin{cases} {}_{0^+}D_t^{\alpha_1,\beta}\, x_1(t) \;=\; a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t), \\ {}_{0^+}D_t^{\alpha_2,\beta}\, x_2(t) \;=\; a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t), \\ \qquad\qquad \vdots \\ {}_{0^+}D_t^{\alpha_n,\beta}\, x_n(t) \;=\; a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t), \end{cases} \tag{24}$$

with initial conditions

$${}_{0^+}I_t^{1-\gamma_i} x_i(0^+) = x_{i0}, \gamma_i = \alpha_i + \beta - \alpha_i\beta, 0 < \alpha_i < 1, \text{ for } i = 1, \ldots, n,$$

where $0 \leq \beta \leq 1$.

We study the stability of system (24) by applying the Laplace transforms on both sides of this system. We have

$$s^{\alpha_i} X_i(s) - s^{\beta(\alpha-1)} x_{i0} = \sum_{j=1}^{n} a_{ij} X_i(s), \tag{25}$$

for $i = 1, \ldots, n$, where $X_i(s)$ is the Laplace transform of $x_i(t)$. We can rewrite (25) as follows

$$\Delta(s). \begin{pmatrix} X_1(s) \\ X_2(s) \\ \vdots \\ X_n(s) \end{pmatrix} = s^{\beta(\alpha-1)} x_0, \tag{26}$$

where

$$\Delta(s) = \begin{pmatrix} \Delta_{11}(s) & \Delta_{12}(s) & \ldots & \Delta_{1n}(s) \\ \Delta_{21}(s) & \Delta_{22}(s) & \ldots & \Delta_{2n}(s) \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n1}(s) & \Delta_{n2}(s) & \cdots & \Delta_{nn}(s) \end{pmatrix},$$

and

$$\Delta_{ij}(s) = \begin{cases} s^{\alpha_i} - a_{ii}, & \text{if } i = j, \\ -a_{ij}, & \text{otherwise} \end{cases}.$$

For simplicity, we name $\Delta(s)$ a characteristic matrix of (24) with respect to $\boldsymbol{\alpha}$. Moreover, $\det(\Delta(s)) = 0$ is the characteristic equation of system (24) with respect to $\boldsymbol{\alpha}$. Now, we express the main theorem for the stability of system (24), but first we recall the following theorem.

Theorem 4 (Final Value Theorem [9])**.** *Let $F(s)$ be the Laplace transform of the function $f(t)$ and all poles of $sF(s)$ are in the open left half plane; then*

$$\lim_{t\to+\infty} f(t) = \lim_{s\to 0} sF(s). \tag{27}$$

Theorem 5. *If all roots of* $\det(\Delta(s)) = 0$ *have negative real parts, then system* (24) *is asymptotically stable.*

Proof. Multiplying s on both sides of (26) gives

$$\Delta(s).\begin{pmatrix} sX_1(s) \\ sX_2(s) \\ \vdots \\ sX_n(s) \end{pmatrix} = s^{\beta(\alpha-1)+1}x_0. \tag{28}$$

If all roots of $\det(\Delta(s)) = 0$ lie in the open left half complex plane (i.e., $\Re(s) < 0$), then we consider (28) in $\Re(s) \geq 0$. In this restricted area, relation (28) has a unique solution $sX(s) = (sX_1(s), sX_2(s), \dots, sX_n(s))$. Since for $i = 1, \dots, n$, $\lim\limits_{s\to 0} s^{\beta(\alpha-1)+1} = 0$, we have

$$\lim_{s\to 0, \Re(s)\geq 0} sX_i(s) = 0, \quad i = 1, 2, \dots, n,$$

and from the final value theorem we get

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty}(x_1(t), x_2(t), \dots, x_n(t)) = \lim_{s\to 0}(sX_1(s), sX_2(s), \dots, sX_n(s)) = \mathbf{0}.$$

The above result shows that system (24) is asymptotically stable. $\square$

Definition 8. *The eigenvalues of A with respect to $\boldsymbol{\alpha}$ are the roots of the characteristic equation of system* (24)*.*

The inertia of a matrix is the triplet of the numbers of eigenvalues of A with positive, negative and zero real parts. Now, we generalize the inertia concept for analyzing the stability of the fractional linear system.

Definition 9. *The inertia of system* (24) *is the triple*

$$I_{n(\boldsymbol{\alpha})}(A) = (\pi_{n(\boldsymbol{\alpha})}(A), \nu_{n(\boldsymbol{\alpha})}(A), \delta_{n(\boldsymbol{\alpha})}(A)),$$

where $\pi_{n(\boldsymbol{\alpha})}(A), \nu_{n(\boldsymbol{\alpha})}(A)$ and $\delta_{n(\boldsymbol{\alpha})}(A)$ are the numbers of roots of $\det(\Delta(s)) = 0$ *with positive, negative and zero real parts, respectively,.*

Definition 10. *The matrix A is called a stable matrix with respect to $\boldsymbol{\alpha}$, if all eigenvalues of A with respect to $\boldsymbol{\alpha}$ have negative real parts.*

Theorem 6. *Autonomous linear Hilfer fractional differential system* (24) *is asymptotically stable if any of the following equivalent conditions holds.*

(*i*) *The matrix A is stable with respect to $\boldsymbol{\alpha}$.*

(*ii*) $\pi_{n(\boldsymbol{\alpha})}(A) = \delta_{n(\boldsymbol{\alpha})}(A) = 0$.

(*iii*) *All roots of the characteristic equation of system* (24) *satisfy* $|\arg(s)| > \pi/2$.

Proof. We first show that (i) $\Rightarrow$ (ii). Since the matrix A is stable with respect to $\boldsymbol{\alpha}$, according to Definition 10, all eigenvalues of A with respect to α have negative real parts, hence $\pi_{n(\boldsymbol{\alpha})}(A) = \delta_{n(\boldsymbol{\alpha})}(A) = 0$.

If (ii) holds, then eigenvalues of A with respect to $\boldsymbol{\alpha}$ have negative real parts. Therefore, we have (iii).

Assume now that (iii) holds. According to Definition 8, the matrix A is stable with respect to $\boldsymbol{\alpha}$. Hence (i) follows. $\square$

Particularly, if $\boldsymbol{\alpha} = \mathbf{1}$ (i.e. $\alpha_i = 1$, for $i = 1, \ldots, n$), where $0 \leq \beta \leq 1$, we have a linear system $x'(t) = Ax(t)$. In this case, the characteristic matrix and the characteristic equation of (24) are reduced to $sI - A$ and $det(sI - A) = 0$, respectively. Also, the inertia of matrix A is a triplet $I(A) = (\pi(A), \nu(A), \delta(A))$, where $\pi(A), \nu(A)$ and $\delta(A)$ are the numbers of eigenvalues of A with positive, negative and zero real parts, respectively. This result is a special case of Definition 9, which corresponds to typical definitions for typical differential equations.

Based on the previous theorem, we can obtain the following corollaries.

Corollary 1. *Suppose that* $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha \in (0, 1)$. *If all roots of the equation* $\det(\lambda I - A) = 0$ *satisfy* $|\arg(\lambda)| > \alpha\pi/2$, *then system* (18) *is asymptotically stable.*

Proof. The characteristic equation of system (18) becomes $\det(s^{\alpha} I - A) = 0$. Let λ be s^{α}; then $s = \lambda^{1/\alpha}$. Now since all roots of equation $\det(\lambda I - A) = 0$ satisfy $|\arg(\lambda)| > \alpha\pi/2$, it follows that $|\arg(s)| = \left|\arg(\lambda^{1/\alpha})\right| > \pi/2$. Therefore, all characteristic roots of system (18) have negative real parts. This completes the proof. $\square$

Corollary 2. *Suppose that all* α_i *are rational numbers between* 0 *and* 1, *for* $i = 1, 2, \ldots, n$. *Also,* M *is the lowest common multiple of the denominators* u_i *of* α_i, *where* $\alpha_i = v_i/u_i$, $(v_i, u_i) = 1$, $u_i, v_i \in Z^{+}$, $i = 1, 2, \ldots, n$ *and* $\omega = 1/M$. *The characteristic equation of the following relation*

$$\det\begin{pmatrix} \lambda^{M\alpha_1} - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda^{M\alpha_2} - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda^{M\alpha_n} - a_{nn} \end{pmatrix} = 0 \tag{29}$$

can be transformed into an integer order polynomial equation if all α_i *are rational numbers. Therefore, system* (24) *is asymptotically stable if all roots* λ *of characteristic equation* (29) *satisfy*

$$|\arg(\lambda)| > \omega\pi/2.$$

Proof. Obviously, the characteristic equation is

$$\det\begin{pmatrix} s^{\alpha_1} - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & s^{\alpha_2} - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & s^{\alpha_n} - a_{nn} \end{pmatrix} = 0. \tag{30}$$

Denote λ by s^{ω}; then $s = \lambda^{1/\omega}$, hence (30) is changed to (29).

$$|\arg(s)| = \left|\arg(\lambda^{\frac{1}{\omega}})\right| > \frac{\pi}{2}$$

due to the argument assumption of equation (29). The conclusion holds. $\square$

4. The stability of linear non-autonomous Hilfer fractional differential systems

We consider the linear non-autonomous Hilfer fractional system

$$\begin{aligned} {}_{0^+}D_t^{\alpha,\beta}x(t) &= Ax(t) + B(t)x(t), \quad t > 0, 0 < \alpha < 1, 0 \leq \beta \leq 1, \\ {}_{0^+}I_t^{1-\gamma}x(0^+) &= x_0, \gamma = \alpha + \beta - \alpha\beta, \end{aligned} \tag{31}$$

where $B(t) : [0, \infty] \to \mathbb{R}^{n\times n}$ is a continuous matrix and $x_0 = (x_{10}, x_{20}, \dots, x_{n0})^T$.

Theorem 7. *Suppose $\|B(t)\|$ is bounded ($\|B(t)\| \leq K$ for some $K > 0$) and all eigenvalues of A satisfy*

$$|\arg(\lambda(A))| > \alpha\pi/2. \tag{32}$$

Then system (31) is asymptotically stable.

Proof. Using the Laplace transform and the inverse Laplace transform, the solution of equations (31) can be written as

$$x(t) = x_0 t^{\gamma-1}E_{\alpha,\gamma}(At^\alpha) + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t-\tau)^\alpha)B(\tau)x(\tau)d\tau,$$

which leads to

$$\|x(t)\| \leq \left\|x_0 t^{\gamma-1}E_{\alpha,\gamma}(At^\alpha)\right\| + \int_0^t (t-\tau)^{\alpha-1} \|E_{\alpha,\alpha}(A(t-\tau)^\alpha)\| \,.\, \|B(\tau)\| \,.\, \|x(\tau)\| \, d\tau.$$

Using Lemma 5, we have

$$\begin{aligned} \|x(t)\| &\leq \left\|x_0 t^{\gamma-1}E_{\alpha,\gamma}(At^\alpha)\right\| \exp\left\{\int_0^t \left\|(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t-\tau)^\alpha)\right\| \,.\, \|B(\tau)\| \, d\tau\right\} \\ &= \left\|x_0 t^{\gamma-1}E_{\alpha,\gamma}(At^\alpha)\right\| \exp\left\{\int_0^t \left\|\tau^{\alpha-1}E_{\alpha,\alpha}(A\tau^\alpha)\right\| \,.\, \|B(t-\tau)\| \, d\tau\right\} \\ &\leq \left\|x_0 t^{\gamma-1}E_{\alpha,\gamma}(At^\alpha)\right\| \exp\left\{K\int_0^t \left\|\tau^{\alpha-1}E_{\alpha,\alpha}(A\tau^\alpha)\right\| d\tau\right\}. \end{aligned}$$

Now, first suppose that A is similar to a diagonal matrix and

$$\begin{aligned} &\int_0^t \left\|\tau^{\alpha-1}E_{\alpha,\alpha}(A\tau^\alpha)\right\| d\tau \\ &\quad = \int_0^t \left\|P[\, diag(\tau^{\alpha-1}E_{\alpha,\alpha}(\lambda_1\tau^\alpha), \dots, \tau^{\alpha-1}E_{\alpha,\alpha}(\lambda_n\tau^\alpha))]\, P^{-1}\right\| d\tau. \end{aligned}$$

We shall now show that there exists a positive constant H such that

$$\int_0^t \left|\tau^{\alpha-1}E_{\alpha,\alpha}(\lambda_i\tau^\alpha)\right| d\tau \leq H, \quad 1 \leq i \leq n.$$

Indeed, using (14) we find for $t > t_0 > 0$,

$$\begin{aligned}
&\int_0^t \left|\tau^{\alpha-1}E_{\alpha,\alpha}(\lambda_i\tau^\alpha)\right| d\tau \\
&\quad = \int_0^{t_0} \left|\tau^{\alpha-1}E_{\alpha,\alpha}(\lambda_i\tau^\alpha)\right| d\tau + \int_{t_0}^t \left|\tau^{\alpha-1}E_{\alpha,\alpha}(\lambda_i\tau^\alpha)\right| d\tau \\
&\quad = \int_0^{t_0} \left|\tau^{\alpha-1}E_{\alpha,\alpha}(\lambda_i\tau^\alpha)\right| d\tau + \int_{t_0}^t \left|\tau^{\alpha-1}\left(-\sum_{k=2}^{p}\frac{(\lambda_i\tau^\alpha)^{-k}}{\Gamma(\alpha-\alpha k)} + O(\frac{1}{|\lambda_i\tau^\alpha|^{p+1}})\right)\right| d\tau \\
&\quad = \int_0^{t_0} \left|\tau^{\alpha-1}E_{\alpha,\alpha}(\lambda_i\tau^\alpha)\right| d\tau + \int_{t_0}^t \left|-\sum_{k=2}^{p}\frac{\lambda_i^{-k}\tau^{-\alpha k+\alpha-1}}{\Gamma(\alpha-\alpha k)} + O(\frac{1}{|\lambda_i|^{p+1}\,\tau^{\alpha p+1}})\right| d\tau \\
&\quad \le \int_0^{t_0} \tau^{\alpha-1}E_{\alpha,\alpha}(|\lambda_i|\,\tau^\alpha) d\tau + \int_{t_0}^t \left\{\sum_{k=2}^{p}\frac{|\lambda_i|^{-k}\,\tau^{-\alpha k+\alpha-1}}{|\Gamma(\alpha-\alpha k)|} + O(\frac{1}{|\lambda_i|^{p+1}\,\tau^{\alpha p+1}})\right\} d\tau \\
&\quad = \sum_{k=0}^{\infty}\frac{|\lambda_i|^k}{\Gamma(\alpha k+\alpha)}\int_0^{t_0}\tau^{\alpha k+\alpha-1}d\tau + \sum_{k=2}^{p}\frac{|\lambda_i|^{-k}}{|\Gamma(\alpha-\alpha k)|}\int_{t_0}^t -\frac{1}{\tau^{\alpha k-\alpha+1}}d\tau \\
&\qquad + O(\frac{1}{|\lambda_i|^{p+1}\,t^{\alpha p}}) \\
&\quad = \sum_{k=0}^{\infty}\frac{|\lambda_i|^k\,t_0^{\alpha k+\alpha}}{\Gamma(\alpha k+\alpha+1)} + \sum_{k=2}^{p}\frac{|\lambda_i|^{-k}\,t^{-\alpha k+\alpha}}{(\alpha-\alpha k)\,|\Gamma(\alpha-\alpha k)|} - \sum_{k=2}^{p}\frac{|\lambda_i|^{-k}\,t_0^{-\alpha k+\alpha}}{(\alpha-\alpha k)\,|\Gamma(\alpha-\alpha k)|} \\
&\qquad + O(\frac{1}{|\lambda_i|^{p+1}\,t^{\alpha p}}) \to t_0^\alpha E_{\alpha,\alpha+1}(|\lambda_i|\,t_0^\alpha) + \sum_{k=2}^{p}\frac{|\lambda_i|^{-k}\,t_0^{-\alpha k+\alpha}}{|\Gamma(\alpha-\alpha k+1)|} \le H \text{ as } t\to+\infty.
\end{aligned}$$

It immediately follows that $\int_0^t \left\|\tau^{\alpha-1}E_{\alpha,\alpha}(A\tau^\alpha)\right\| d\tau \le C_1$ for any $t \ge 0$.

Next, we consider the case when A is similar to a Jordan form. For $t > t_0 > 0$, we find

$$\begin{aligned}
&\int_0^t \left|\tau^{\alpha-1}\frac{1}{(j-1)!}\left(\frac{\partial}{\partial\lambda_i}\right)^{j-1}E_{\alpha,\alpha}(\lambda_i\tau^\alpha)\right| d\tau \\
&\quad = \int_0^{t_0} \left|\frac{\tau^{\alpha-1}}{(j-1)!}\left(\frac{\partial}{\partial\lambda_i}\right)^{j-1}E_{\alpha,\alpha}(\lambda_i\tau^\alpha)\right| d\tau \\
&\qquad + \int_{t_0}^t \left|\frac{\tau^{\alpha-1}}{(j-1)!}\left(\frac{\partial}{\partial\lambda_i}\right)^{j-1}E_{\alpha,\alpha}(\lambda_i\tau^\alpha)\right| d\tau \\
&\quad \le \int_0^t \sum_{k=0}^{\infty}\frac{k(k-1)\cdots(k-j+2)\,|\lambda_i|^{k-j+1}\,\tau^{\alpha k+\alpha-1}}{(j-1)!\Gamma(\alpha k+\alpha)} d\tau \\
&\qquad + \int_{t_0}^t \left|\tau^{\alpha-1}\frac{1}{(j-1)!}\left(\frac{\partial}{\partial\lambda_i}\right)^{j-1}\left\{-\sum_{k=2}^{p}\frac{1}{\Gamma(\alpha-\alpha k)}\frac{1}{(\lambda_i t^\alpha)^k} + O(\frac{1}{|\lambda_i t^\alpha|^{p+1}})\right\}\right| d\tau
\end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{k(k-1)\cdots(k-j+2)\,|\lambda_i|^{k-j+1}}{(j-1)!\Gamma(\alpha k+\alpha)} \int_0^t \tau^{\alpha k+\alpha-1} d\tau$$
$$+\int_{t_0}^t \left| \tau^{\alpha-1} \left\{ \sum_{k=2}^{p} \frac{(-1)^{j-1}(k+j-2)!}{(j-1)!(k-1)!\Gamma(\alpha-\alpha k)} \frac{1}{\lambda_i^{k+j-1}\tau^{\alpha k}} + O(\frac{1}{|\lambda_i|^{p+j}\,\tau^{\alpha(p+1)}}) \right\} \right| d\tau$$
$$\leq \sum_{k=0}^{\infty} \frac{k(k-1)\cdots(k-j+2)\,|\lambda_i|^{k-j+1}\, t_0^{\alpha k+\alpha}}{(j-1)!\Gamma(\alpha k+\alpha+1)}$$
$$+\int_{t_0}^t \left\{ \sum_{k=2}^{p} \frac{(k+j-2)!}{(j-1)!(k-1)!\,|\Gamma(\alpha-\alpha k)|} \frac{1}{|\lambda_i|^{k+j-1}\,\tau^{\alpha k-\alpha+1}} + O(\frac{1}{|\lambda_i|^{p+j}\,\tau^{\alpha p+1}}) \right\} d\tau$$
$$= t_0^{\alpha} \frac{1}{(j-1)!} \left(\frac{\partial}{\partial\,|\lambda_i|} \right)^{j-1} E_{\alpha,\alpha+1}(|\lambda_i|\, t_0^{\alpha})$$
$$+\sum_{k=2}^{p} \frac{(k+j-2)!}{(j-1)!(k-1)!\,|\Gamma(\alpha-\alpha k)|} \frac{1}{|\lambda_i|^{k+j-1}} \left(\frac{t^{-\alpha k+\alpha}}{\alpha-\alpha k} - \frac{t_0^{-\alpha k+\alpha}}{\alpha-\alpha k} \right)$$
$$+O(\frac{1}{|\lambda_i|^{p+j}\,\tau^{\alpha p}})$$
$$\to \frac{t_0^{\alpha}}{(j-1)!} \left(\frac{\partial}{\partial\,|\lambda_i|} \right)^{j-1} E_{\alpha,\alpha+1}(|\lambda_i|\, t_0^{\alpha}) + \sum_{k=2}^{p} \frac{(k+j-2)!\,|\lambda_i|^{-k-j+1}\, t_0^{-\alpha k+\alpha}}{(j-1)!(k-1)!\,|\Gamma(\alpha-\alpha k+1)|} \leq C_2$$

as $t \to +\infty$, where $1 \leq j \leq n_i$. Thus,

$$\exp\left\{ K \int_0^t \left\| \tau^{\alpha-1} E_{\alpha,\alpha}(A\tau^{\alpha}) \right\| d\tau \right\}$$

is bounded.

Further, we note that $\lim_{t\to+\infty} \left\| x_0 t^{\gamma-1} E_{\alpha,\gamma}(At^{\alpha}) \right\| = 0$. Hence, we have

$$\lim_{t\to+\infty} \|x(t)\| = 0,$$

which completes the proof. □

Theorem 8. *If all eigenvalues of A satisfy*

$$|\arg(\lambda(A))| \geq \alpha\pi/2, \tag{33}$$

and the critical eigenvalues have the same algebraic and geometric multiplicities and $\int_0^{\infty} \|B(t)\|\, dt$ is bounded, then system (31) *is stable.*

Proof. From the proof of the previous theorem, we have

$$\|x(t)\| \leq \left\| x_0 t^{\gamma-1} E_{\alpha,\gamma}(At^{\alpha}) \right\| + \int_0^t (t-\tau)^{\alpha-1} \left\| E_{\alpha,\alpha}(A(t-\tau)^{\alpha}) \right\| . \|B(\tau)\| . \|x(\tau)\|\, d\tau.$$

Now according to the proof of Theorem 3, the matrix is bounded. Therefore, there exists a positive number M such that $\left\| t^{\gamma-1} E_{\alpha,\gamma}(At^{\alpha}) \right\| \leq M$. Moreover, for $\beta = 0$, there exists a positive number N such that $\left\|t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})\right\| \leq N$. Then we have

$$\|x(t)\| \leq M\|x_0\| + \int_0^t N\|B(\tau)\|.\|x(\tau)\|\,d\tau.$$

Applying Lemma 5, we have

$$\|x\| \leq (\|x_0\|M)\exp\left(N\int_0^t \|B(\tau)\|\,d\tau\right).$$

Thus $\|x(t)\|$ is bounded according to the condition

$$\int_0^\infty \|B(t)\|\,dt < \infty$$

and system (31) is stable. □

5. Conclusion

In this paper, we studied the stability and the asymptotic stability of linear autonomous and non-autonomous fractional differential systems with the Hilfer fractional derivative. Although in this paper we just surveyed the systems with $0 < \alpha < 1$, higher order systems can be discussed based on the analysis of this paper. This will be the main goal of investigation in future work.

Acknowledgement

We are very grateful to anonymous referees, Dr. Alireza Ansari and Dr. Maziar Salahi for their careful reading and valuable comments which led to the improvement of this paper.

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