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Induced representations of Hilbert modules over locally C*-algebras and the imprimitivity theorem

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Abstract. We study induced representations of Hilbert modules over locally C*-algebras and their non-degeneracy. We show that if V and W are Morita equivalent Hilbert modules over locally C*-algebras A and B, respectively, then there exists a bijective correspondence between equivalence classes of non-degenerate representations of V and W.

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1. Introduction

Morita equivalence and induced representations of C*-algebras were first introduced by Rieffel [16, 17]. Two C*-algebras A and B are Morita equivalent if there exists a full Hilbert A-module E such that B is isomorphic to the C*-algebra $K_A(E)$ of all compact operators on E. Some properties of C*-algebras that are preserved under Morita equivalence were investigated in [2, 4, 15, 21]. Indeed, Rieffel defined induced representations of C*-algebras, that are now known as Rieffel induced representations, by using tensor products of Hilbert modules and established an equivalence between the categories of non-degenerate representations of Morita equivalent C*-algebras. Joita [10, 11] defined the notions of Morita equivalence and induced representations in the category of locally C*-algebras. Joita and Moslehian [12] have recently introduced a notion of Morita equivalence in the category of Hilbert C*-modules considered to obtain induced representations of Hilbert modules over locally C*-algebras. This enables us to prove the imprimitivity theorem for induced representations of Hilbert modules over locally C*-algebras.

Let us quickly recall the definition of locally C*-algebras and Hilbert modules over them. A locally C*-algebra is a complete Hausdorff complex topological $*$-algebra A whose topology is determined by its continuous C*-seminorms in the sense that the net $\{a_i\}_{i\in I}$ converges to 0 if and only if the net $\{p(a_i)\}_{i\in I}$ converges to 0 for every continuous C*-seminorm p on A. Such algebras appear in the study of

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certain aspects of C*-algebras such as tangent algebras of C*-algebras, a domain of closed $*$-derivations on C*-algebras, multipliers of Pedersen's ideal, noncommutative analogues of classical Lie groups, and K-theory. These algebras were first introduced by Inoue [6] as a generalization of C*-algebras and studied more in [5, 14] with different names. A (right) *pre-Hilbert module* over a locally C*-algebra A is a right A-module E compatible with the complex algebra structure and equipped with an A-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to A$, $(x, y) \mapsto \langle x, y \rangle$, which is A-linear in the second variable y and has the properties:

$$\langle x, y \rangle = \langle y, x \rangle^*, \text{ and } \langle x, x \rangle \geq 0 \text{ with equality if and only if } x = 0.$$

A pre-Hilbert A-module E is a Hilbert A-module if E is complete with respect to the topology determined by the family of seminorms $\{\overline{p}_E\}_{p \in S(A)}$, where $\overline{p}_E(\xi) = \sqrt{p(\langle \xi, \xi \rangle)}$, $\xi \in E$. Hilbert modules over locally C*-algebras have been studied systematically in the book [8] and the papers [7, 14, 20].

Joita and Moslehian [12], and Skeide [18] defined Morita equivalence for Hilbert C*-modules with two different methods. In the recent sense of Joita and Moslehian, two Hilbert modules V and W over C*-algebras A and B, respectively, are called Morita equivalent if $K_A(V)$ and $K_B(W)$ are strong Morita equivalent as C*-algebras. We consider this definition, which is weaker than Skeide's definition and also fitted to our paper.

In this paper, we first present some definitions and basic facts about locally C*-algebras and Hilbert modules over them. In [19], Skeide proved that if E is a Hilbert module over a C*-algebra A, then every representation of A induces a representation of E. We use this fact to reformulate the induced representations of Hilbert C*-modules and some of their properties which have been studied in [1]. These enable us to obtain the notion of induced representations of Hilbert modules over locally C*-algebras. We finally define the concept of Morita equivalence for Hilbert modules over locally C*-algebras. We prove that two full Hilbert modules over locally C*-algebras are Morita equivalent if and only if their underlying locally C*-algebras are strong Morita equivalent and then we give a module version of the imprimitivity theorem. Indeed, we show that for Morita equivalent Hilbert modules V and W over locally C*-algebras A and B, respectively, there is a bijective correspondence between equivalence classes of non-degenerate representations of V and W.

2. Preliminaries

Let A be a locally C*-algebra, $S(A)$ the set of all continuous C*-seminorms on A and $p \in S(A)$. We set $N_p = \{a \in A : \ p(a) = 0\}$, then $A_p = A/N_p$ is a C*-algebra in the norm induced by p. For $p, q \in S(A)$ with $p \geq q$, the surjective morphisms $\pi_{pq} : A_p \to A_q$ defined by $\pi_{pq}(a + N_p) = a + N_q$ induce the inverse system $\{A_p; \pi_{pq}\}_{p,q \in S(A),\, p \geq q}$ of C*-algebras and $A = \varprojlim_p A_p$, i.e., the locally C*-algebra A can be identified with $\varprojlim_p A_p$. The canonical map from A onto A_p is denoted by π_p and a_p is reserved to denote $a + N_p$. A morphism of locally C*-algebras is a continuous morphism of $*$-algebras. An isomorphism of locally C*-algebras is a morphism of locally C*-algebras which possesses an inverse morphism of locally C*-algebras.

A representation of a locally C*-algebra A is a continuous $*$-morphism $\varphi : A \to B(H)$, where $B(H)$ is the C*-algebra of all bounded linear maps on a Hilbert space H. If (φ, H) is a representation of A, then there is $p \in S(A)$ such that $\|\varphi(a)\| \leq p(a)$, for all $a \in A$. The representation (φ_p, H) of A_p, where $\varphi_p \circ \pi_p = \varphi$ is called a representation of A_p associated to (φ, H). We refer to [5, 11] for basic facts and definitions about the representation of locally C*-algebras.

Suppose E is a Hilbert A-module and $\langle E, E\rangle$ is the closure of linear span of $\{\langle x, y\rangle : \ x, y \in E\}$. The Hilbert A-module E is called *full* if $\langle E, E\rangle = A$. One can always consider any Hilbert A-module as a full Hilbert module over locally C*-algebra $\langle E, E\rangle$. For each $p \in S(A)$, $N_p^E = \{\xi \in E : \ \bar{p}_E(\xi) = 0\}$ is a closed submodule of E and $E_p = E/N_p^E$ is a Hilbert A_p-module with the action $(\xi + N_p^E)\pi_p(a) = \xi a + N_p^E$ and the inner product $\langle \xi + N_p^E, \eta + N_p^E\rangle = \pi_p(\langle \xi, \eta\rangle)$. The canonical map from E onto E_p is denoted by σ_p^E and ξ_p is reserved to denote $\sigma_p^E(\xi)$. For $p, q \in S(A)$ with $p \geq q$, the surjective morphisms $\sigma_{pq}^E : E_p \to E_q$ defined by $\sigma_{pq}^E(\sigma_p^E(\xi)) = \sigma_q^E(\xi)$ induce the inverse system $\{E_p;\ A_p;\ \sigma_{pq}^E,\ \pi_{pq}\}_{p,q\in S(A),\, p\geq q}$ of Hilbert C*-modules in the following sense:

- $\sigma_{pq}^E(\xi_p a_p) = \sigma_{pq}^E(\xi_p)\pi_{pq}(a_p),\ \xi_p \in E_p,\ a_p \in A_p,\ p, q \in S(A),\ p \geq q,$
- $\langle \sigma_{pq}^E(\xi_p), \sigma_{pq}^E(\eta_p)\rangle = \pi_{pq}(\langle \xi_p, \eta_p\rangle),\ \xi_p, \eta_p \in E_p,\ p, q \in S(A),\ p \geq q,$
- $\sigma_{qr}^E \circ \sigma_{pq}^E = \sigma_{pr}^E$ if $p, q, r \in S(A)$ and $p \geq q \geq r,$
- $\sigma_{pp}^E(\xi_p) = \xi_p,\ \xi \in E,\ p \in S(A).$

In this case, $\varprojlim_p E_p$ is a Hilbert A-module which can be identified with E. Let E and F be Hilbert A-modules and $T : E \to F$ an A-module map. The module map T is called bounded if for each $p \in S(A)$ there is $k_p > 0$ such that $\bar{p}_F(Tx) \leq k_p\ \bar{p}_E(x)$ for all $x \in E$. The module map T is called adjointable if there exists an A-module map $T^* : F \to E$ with the property $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in E, y \in F$. It is well-known that every adjointable map is bounded. The set $L_A(E, F)$ of all bounded adjointable A-module maps from E into F becomes a locally convex space with the topology defined by the family of seminorms $\{\tilde{p}\}_{p\in S(A)}$, where $\tilde{p}(T) = \|(\pi_p)_*(T)\|_{L_{A_p}(E_p,F_p)}$ and $(\pi_p)_* : L_A(E, F) \to L_{A_p}(E_p, F_p)$ is defined by $(\pi_p)_*(T)(\xi + N_p^E) = T\xi + N_p^F$ for all $T \in L_A(E, F),\ \xi \in E$. For $p, q \in S(A)$ with $p \geq q$, the morphisms $(\pi_{pq})_* : L_{A_p}(E_p, F_p) \to L_{A_q}(E_q, F_q)$ defined by $(\pi_{pq})_*(T_p)(\sigma_q^E(\xi)) = \sigma_{pq}^F(T_p(\sigma_p^E(\xi)))$ induce the inverse system

$$\{L_{A_p}(E_p, F_p); (\pi_{pq})_*\}_{p,q\in S(A),\, p\geq q}$$

of Banach spaces such that $\varprojlim_p L_{A_p}(E_p, F_p)$ can be identified to $L_A(E, F)$. In particular, topologizing, $L_A(E, E)$ becomes a locally C*-algebra which is abbreviated by $L_A(E)$. The set of all compact operators $K_A(E)$ on E is defined as the closed linear subspace of $L_A(E)$ spanned by $\{\theta_{x,y} : \ \theta_{x,y}(\xi) = x\langle y, \xi\rangle$ for all $x, y, \xi \in E\}$. This is a locally C*-subalgebra and a two-sided ideal of $L_A(E)$; moreover, $K_A(E)$ can be identified to $\varprojlim_p\ K_{A_p}(E_p)$.

Let V and W be Hilbert modules over locally C*-algebras A and B, respectively, and $\Psi : A \to L_B(W)$ a continuous $*$-morphism. We can regard W as a left A-module by $(a, y) \to \Psi(a)y,\ a \in A,\ y \in W$. The right B-module $V \otimes_A W$ is a pre-Hilbert module with the inner product given by $\langle x \otimes y, z \otimes t\rangle = \langle y, \Psi(\langle x, z\rangle)t\rangle$. We denote by $V \otimes_\Psi W$ the completion of $V \otimes_A W$, cf. [9] for more detailed information.

3. Induced representations of Hilbert modules

In this section, we first study induced representations of Hilbert C*-modules and then we reformulate them in the context of Hilbert modules over locally C*-algebras.

Let H and K be Hilbert spaces. Then the space $B(H, K)$ of all bounded operators from H into K can be considered as a Hilbert $B(H)$-module with the module action $(T, S) \to TS$, $T \in B(H, K)$ and $S \in B(H)$ and the inner product defined by $\langle T, S\rangle = T^*S$, $T, S \in B(H, K)$. Murphy [13] showed that any Hilbert C*-module can be represented as a submodule of the concrete Hilbert module $B(H, K)$ for some Hilbert spaces H and K. This allows us to extend the notion of a representation from the context of C*-algebras to the context of Hilbert C*-modules. Let V and W be two Hilbert modules over C*-algebras A and B, respectively, and $\varphi : A \to B$ be a morphism of C*-algebras. A map $\Phi : V \to W$ is said to a φ-morphism if $\langle \Phi(x), \Phi(y)\rangle = \varphi(\langle x, y\rangle)$ for all $x, y \in V$. A φ-morphism $\Phi : V \to B(H, K)$, where $\varphi : A \to B(H)$ is a representation of A is called a representation of V. When Φ is a representation of V, we assume that an associated representation of A is denoted by the same lowercase letter φ, so we will not explicitly mention φ. Let $\Phi : V \to B(H, K)$ be a representation of a Hilbert A-module V. We say Φ is a non-degenerate representation if $\overline{\Phi(V)(H)} = K$ and $\overline{\Phi(V)^*(K)} = H$. Two representations $\Phi_i : V \to B(H_i, K_i)$ of V, $i = 1, 2$ are said to be unitarily equivalent if there are unitary operators $U_1 : H_1 \to H_2$ and $U_2 : K_1 \to K_2$, such that $U_2\Phi_1(v) = \Phi_2(v)U_1$ for all $v \in V$. Representations of Hilbert modules have been investigated in [1, 3, 19].

Lemma 1. *Let V be a full Hilbert A-module and $\Phi_1 : V \to B(H_1, K_1)$ and $\Phi_2 : V \to B(H_2, K_2)$ two non-degenerate representations of V. If Φ_1 and Φ_2 are unitarily equivalent, then φ_1 and φ_2 are unitarily equivalent.*

Proof. Let $U_1 : H_1 \to H_2$ and $U_2 : K_1 \to K_2$ be unitary operators and $U_2\Phi_1(x) = \Phi_2(x)U_1$ for all $x \in V$. Then we have

$$U_1\varphi_1(\langle x, y\rangle)h = U_1\Phi_1(x)^*\Phi_1(y)h = \Phi_2(x)^*\Phi_2(y)U_1h = \varphi_2(\langle x, y\rangle)U_1h,$$

for every $x, y \in V$ and $h \in H_1$. Since V is full, we conclude that $U_1\varphi_1(a)h = \varphi_2(a)U_1h$ for every $a \in A$ and $h \in H_1$, and consequently, φ_1 and φ_2 are unitarily equivalent. □

Skeide [19] recovered the result of Murphy by embedding every Hilbert A-module E into a matrix C*-algebra as a lower submodule. He proved that every representation of B induces a representation of E. We describe his induced representation as follows.

Construction 1. *Let B be a C*-algebra and E a Hilbert B-module and $\varphi : B \to B(H)$ a $*$-representation of B. Define a sesquilinear form $\langle .,. \rangle$ on the vector space $E \otimes_{alg} H$ by $\langle x \otimes h, y \otimes k \rangle = \langle h, \varphi(\langle x, y\rangle)k\rangle_H$, where $\langle .,. \rangle_H$ denotes the inner product on the Hilbert space H. By [19, Proposition 3.8], the sesquilinear form is positive and so $E \otimes_{alg} H$ is a semi-Hilbert space. Then $(E \otimes_{alg} H)/N_\varphi$ is a pre-Hilbert space with the inner product defined by*

$$\langle x \otimes h + N_\varphi \ , \ y \otimes k + N_\varphi \rangle = \langle x \otimes h, y \otimes k\rangle,$$

where N_φ is the vector subspace of $E \otimes_{alg} H$ generated by $\{x \otimes h \in E \otimes_{alg} H : \langle x \otimes h, x \otimes h \rangle = 0\}$. The completion of $(E \otimes_{alg} H)/N_\varphi$ with respect to the above inner product is denoted by ${}_EH$. We identify the elements $x \otimes h$ with the equivalence classes $x \otimes h + N_\varphi \in {}_EH$. Suppose $x \in E$ and $L_x h = x \otimes h$ then $\|L_x h\|^2 = \langle h, \varphi(\langle x, x\rangle)h\rangle \le \|h\|^2\|x\|^2$, i.e. $L_x \in B(H, {}_E H)$. We define $\eta_\varphi : E \to B(H, {}_E H)$ by $\eta_\varphi(x) = L_x$. Then for $x, x' \in E$, $h, h' \in H$ and $b \in B$ we have $\langle \eta_\varphi(x), \eta_\varphi(x')\rangle = \varphi(\langle x, x'\rangle)$ and $\eta_\varphi(xb) = \eta_\varphi(x)\varphi(b)$, and so η_φ is a representation of E.

Lemma 2. *Let $\varphi_1 : B \to B(H_1)$ and $\varphi_2 : B \to B(H_2)$ be two non-degenerate representations of B. If φ_1 and φ_2 are unitarily equivalent, then η_{φ_1} and η_{φ_2} are unitarily equivalent.*

Proof. Suppose $U : H_1 \to H_2$ is a unitary operator such that $U\varphi_1(b) = \varphi_2(b)U$ for all $b \in B$. Then $id_E \otimes U : E \otimes_{alg} H_1 \to E \otimes_{alg} H_2$ given by $x \otimes h_1 \mapsto x \otimes h_2$ can be extended to a unitary operator V from ${}_EH_1$ onto ${}_EH_2$ and $V\eta_{\varphi_1}(x) = \eta_{\varphi_2}(x)U$ for all $x \in E$. Hence, η_{φ_1} and η_{φ_2} are unitarily equivalent. □

The above argument enables us to extend the Rieffel induced representations from the case of C*-algebras to the context of Hilbert C*-modules. For this, let V and W be two full Hilbert modules over C*-algebras A and B, respectively. Let E be a Hilbert B-module and A acts as adjointable operators on the Hilbert C*-module E, and $\Phi : W \to B(H, K)$ is a non-degenerate representation of W. Using [15, Proposition 2.66], the formula ${}^A_E\varphi(x \otimes h) = (a.x) \otimes h$ extends to obtain a (Rieffel induced) representation of A as bounded operators on Hilbert space ${}_EH$. In view of Construction 1, the representation ${}^A_E\varphi : A \to B({}_EH)$ of the C*-algebra A obtains the representation $\eta_{{}^A_E\varphi} : V \to B({}_EH, \ {}_V({}_EH))$ of the Hilbert A-module V. The constructed representation $\eta_{{}^A_E\varphi}$ is called the *Rieffel induced representation* from W to V via E and denoted by ${}^V_E\Phi$. The following result can be found in [1, Proposition 3.3] that we derive from Lemmas 1 and 2. Our argument seems to be shorter.

Lemma 3. *Let W be a full Hilbert B-module and $\Phi_1 : W \to B(H_1, K_1)$ and $\Phi_2 : W \to B(H_2, K_2)$ two non-degenerate representations of W. If Φ_1 and Φ_2 are unitarily equivalent, then ${}^V_E\Phi_1$ and ${}^V_E\Phi_2$ are unitarily equivalent.*

Corollary 1. *If $\Phi : W \to B(H, K)$ and $\oplus_{i\in I}\Phi_i : W \to B(\oplus_{i\in I}H_i, \oplus_{i\in I}K_i)$ are unitarily equivalent, then ${}^V_E\Phi$ and $\oplus_{i\in I}\ {}^V_E\Phi_i$ are unitary equivalent.*

Now, we reformulate representations of the Hilbert module from the case of C*-algebras to the case of locally C*-algebras. Let V and W be two Hilbert modules over locally C*-algebras A and B, respectively, and $\varphi : A \to B$ a morphism of locally C*-algebras. A map $\Phi : V \to W$ is said to be a φ-morphism if $\langle \Phi(x), \Phi(y)\rangle = \varphi(\langle x, y\rangle)$, for all $x, y \in V$. A φ-morphism $\Phi : V \to B(H, K)$, where $\varphi : A \to B(H)$ is a representation of A, is called a representation of V. We can define non-degenerate representations and unitarily equivalent representations for Hilbert modules over locally C*-algebras like a Hilbert C*-modules case.

Suppose A is a locally C*-algebra, V is a Hilbert A-module and $\varphi : A \to B(H)$ is a representation of A on some Hilbert space H. Suppose $p \in S(A)$ and φ_p is a representation of A_p associated to φ; then there exist a Hilbert space K and a representation $\Phi_p : V_p \to B(H, K)$ which is a φ_p-morphism. For details we refer to the proof of [13, Theorem 3.1]. It is easy to see that the map $\Phi : V \to B(H, K)$, $\Phi(v) = \Phi_p(\sigma_p^V(v))$ is a φ-morphism, i.e., it is a representation of V.

Lemma 4. *Let V be a Hilbert module over locally C*-algebra A and $\Phi : V \to B(H, K)$ a representation of V. If $p \in S(A)$ and φ_p is a representation of A_p associated to φ, then the map $\Phi_p : V_p \to B(H, K)$, $\Phi_p(\sigma_p^V(v)) = \Phi(v)$ is a φ_p-morphism. Specifically, Φ_p is a representation of V_p and Φ is non-degenerate if and only if Φ_p is. In this case, we say that Φ_p is a representation of V_p associated to Φ.*

Proof. Let $v, v' \in V$ and $\overline{p}_V(v - v') = 0$. Since $\|\varphi(a)\| \leq p(a)$ for all $a \in A$, we have $\langle \Phi(v - v'), \Phi(v - v')\rangle = \varphi(\langle v - v', v - v'\rangle) = 0$, which shows Φ_p is well-defined. We also have

$$\begin{aligned}\langle \Phi_p(\sigma_p^V(v)), \Phi_p(\sigma_p^V(v'))\rangle = \langle \Phi(v), \Phi(v')\rangle = \varphi(\langle v, v'\rangle) &= \varphi_p \circ \pi_p(\langle v, v'\rangle) \\ &= \varphi_p(\langle \sigma_p^V(v), \sigma_p^V(v')\rangle).\end{aligned}$$

Then, by definition of Φ_p, the representation Φ is non-degenerate if and only if Φ_p is non-degenerate. □

Let V and W be two full Hilbert modules over locally C*-algebras A and B, respectively. Let E be a Hilbert B-module, $\Psi : A \to L_B(E)$ a non-degenerate continuous $*$-morphism and $\Phi : W \to B(H, K)$ a non-degenerate representation of W. We construct a non-degenerate representation from W to V via E as follows.

Construction 2. *We define a sesquilinear form $\langle .,. \rangle$ on the vector space $E \otimes_{alg} H$ by $\langle x \otimes h, y \otimes k\rangle = \langle h, \varphi(\langle x, y\rangle)k\rangle_H$ and make the Hilbert space ${}_EH$ as in Construction 1. The map ${}^A_E\varphi : A \to B({}_EH)$ defined by*

$${}^A_E\varphi(a)(x \otimes h) = \Psi(a)x \otimes h, \quad a \in A, \ x \in E, \ h \in H,$$

is a representation of A. The representation $({}_EH, {}^A_E\varphi)$ is called the Rieffel induced representation from B to A via E, cf. [11]. Since A acts as an adjointable operator on Hilbert B-module E, we can construct interior tensor product $V \otimes_\Psi E$ as a Hilbert B-module. Hence, we find the Hilbert spaces ${}_EH$ and ${}_{V\otimes_\Psi E}H$. Let $v \in V$; then the map $E \times H \to {}_{V\otimes_\Psi E}H$, $(x, h) \mapsto v \otimes x \otimes h$ is a bilinear form and so there is a unique linear transformation ${}_E\Phi(v) : E \otimes_{alg} H \to {}_{V\otimes_\Psi E}H$ which can be extended to a bounded linear operator ${}^V_E\Phi(v)$ from ${}_EH$ to ${}_{V\otimes_\Psi E}H$. To see this, suppose $q \in S(B)$, $x \in E$, $h \in H$ and (φ_q, H) is a representation of B_q associated to (φ, H). We have

$$\begin{aligned}
\langle {}_E\Phi(v)(x\otimes h)\ ,\ {}_E\Phi(v)(x\otimes h)\rangle &= \langle v\otimes x\otimes h, v\otimes x\otimes h\rangle\\
&= \langle h, \varphi(\langle v\otimes x, v\otimes x\rangle)h\rangle_H\\
&= \langle h, \varphi(\langle x, \Psi(\langle v,v\rangle)x\rangle)h\rangle_H\\
&= \langle h, \varphi_q\circ\pi_q(\langle\Psi(\langle v,v\rangle)^{1/2}x, \Psi(\langle v,v\rangle)^{1/2}x\rangle)h\rangle_H\\
&= \langle h, \varphi_q(\langle\sigma_q(\Psi(\langle v,v\rangle)^{1/2}x), \sigma_q(\Psi(\langle v,v\rangle)^{1/2}x)\rangle)h\rangle_H\\
&= \langle h, \varphi_q(\langle(\pi_q)_*(\Psi(\langle v,v\rangle)^{1/2})(\sigma_q(x)), (\pi_q)_*(\Psi(\langle v,v\rangle)^{1/2})(\sigma_q(x))\rangle)h\rangle_H\\
&\le \tilde{q}(\Psi\langle v,v\rangle)\langle h, \varphi_q(\langle\sigma_q(x),\sigma_q(x)\rangle)h\rangle_H\\
&= \tilde{q}(\Psi\langle v,v\rangle)\langle h, (\varphi_q\circ\pi_q)(\langle x,x\rangle)h\rangle_H\\
&= \tilde{q}(\Psi\langle v,v\rangle)\langle h, \varphi(\langle x,x\rangle)h\rangle_H\\
&= \tilde{q}(\Psi\langle v,v\rangle)\langle x\otimes h, x\otimes h\rangle.
\end{aligned}$$

The following equalities hold for every $v, v^{'}\in V$, $x, x^{'}\in E$ and $h, h^{'}\in H$

$$\begin{aligned}
\langle x\otimes h\ ,\ {}^V_E\Phi^*(v)\ {}^V_E\Phi(v^{'})(x^{'}\otimes h^{'})\rangle &= \langle {}^V_E\Phi(v)(x\otimes h)\ ,\ {}^V_E\Phi(v)(x^{'}\otimes h^{'})\rangle\\
&= \langle v\otimes x\otimes h\ ,\ v^{'}\otimes x^{'}\otimes h^{'}\rangle\\
&= \langle h, \varphi(\langle v\otimes x, v^{'}\otimes x^{'}\rangle)h\rangle_H\\
&= \langle h, \varphi(\langle x, \Psi(\langle v, v^{'}\rangle)x^{'}\rangle)h^{'}\rangle_H\\
&= \langle x\otimes h\ ,\ \Psi(\langle v, v^{'}\rangle)x^{'}\otimes h^{'}\rangle\\
&= \langle x\otimes h\ ,\ {}^A_E\varphi(\langle v, v^{'}\rangle)(x^{'}\otimes h^{'})\rangle,
\end{aligned}$$

which imply $\langle {}^V_E\Phi(v), {}^V_E\Phi(v^{'})\rangle = {}^V_E\Phi^*(v)\ {}^V_E\Phi(v^{'}) = {}^A_E\varphi(\langle v, v^{'}\rangle)$. That is, the map ${}^V_E\Phi : V\to B({}_EH, {}_{V\otimes_\Psi E}H)$ is a ${}^A_E\varphi$-morphism and so it is a representation of V. We now show that ${}^V_E\Phi$ is non-degenerate. To see this, recall that $\overline{\Psi(A)(E)} = E$ and $\overline{\langle V,V\rangle} = A$, which imply $\overline{\Psi(\langle V,V\rangle)(E)} = E$. Suppose $x, x^{'}\in E$ and $h\in H$, we have

$$\begin{aligned}
\|(x-x^{'})\otimes h\|^2 &= \langle h, \varphi(\langle x-x^{'}, x-x^{'}\rangle)h\rangle_H\\
&\le \|h\|^2\|\varphi(\langle x-x^{'}, x-x^{'}\rangle)\|\\
&\le \|h\|^2 q(\langle x-x^{'}, x-x^{'}\rangle) = \|h\|^2\bar{q}_E(x-x^{'}).
\end{aligned}$$

Given $\epsilon>0$, there exist $v_i, v_i^{'}\in V$ and $x_i\in E$ such that $\bar{q}_E(\sum_i\Psi(\langle v_i, v_i^{'}\rangle)x_i - x)<\epsilon$. In view of the above inequality, the term $\sum_i\Psi(\langle v_i, v_i^{'}\rangle)x_i\otimes h$ approximates $x\otimes h$ in ${}_EH$. But we have

$$\begin{aligned}
\sum_i\Psi(\langle v_i, v_i^{'}\rangle)x_i\otimes h &= \sum_i\ {}^A_E\varphi(\langle v_i, v_i^{'}\rangle)(x_i\otimes h)\\
&= \sum_i\ {}^V_E\Phi^*(v_i)\ {}^V_E\Phi(v_i^{'})(x_i\otimes h)\\
&= \sum_i\ {}^V_E\Phi^*(v_i)(\ v_i^{'}\otimes x_i\otimes h),
\end{aligned}$$

which implies ${}^V_E\Phi(V)^*({}_{V\otimes_\Psi E}H) = {}_EH$. The equality ${}^V_E\Phi(V)({}_EH) = {}_{V\otimes_\Psi E}H$ follows from the definition of ${}^V_E\Phi$, i.e., ${}^V_E\Phi$ is non-degenerate.

Definition 1. *The representation ${}^{V}_{E}\Phi$ in Construction 2 is a called Rieffel induced representation from W to V via E.*

Theorem 3. *Let V and W be two full Hilbert modules over locally C^*-algebras A and B, respectively. Let E be a Hilbert B-module, $\Psi : A \to L_B(E)$ a non-degenerate continuous $*$-morphism and $\Phi : W \to B(H, K)$ a non-degenerate representation. If $q \in S(B)$ and (φ_q, H) is a non-degenerate representation of B_q associated to (φ, H), then there is $p \in S(A)$ such that A_p acts non-degenerately on E_q and the representations ${}^{V}_{E}\Phi$ and ${}^{V_p}_{E_q}\Phi_q \circ \sigma^V_p$ of V are unitarily equivalent.*

Proof. Continuity of Ψ implies that there exists $p \in S(A)$ such that $\tilde{q}(\Psi(a)) \leq p(a)$ for each $a \in A$, which guarantees $\Psi_p : A_p \to L_{B_q}(E_q)$, $\Psi_p(\pi_p(a)) = (\pi_q)_*(\Psi(a))$ is a $*$-morphism of C*-algebras. Moreover, Ψ_p is non-degenerate since

$$\begin{aligned}\overline{\Psi_p(A_p)(E_p)} = \overline{\Psi_p(\pi_p(A))(\sigma^E_p(E))} &= \overline{(\pi_q)_*(\Psi(A)\sigma^E_q(E))} \\ &= \sigma^E_q\,\overline{(\Psi(A)(E))} \\ &= \sigma^E_q(E) = E_q.\end{aligned}$$

If Φ_q is a non-degenerate representation of W_q associated to Φ, then ${}^{V_p}_{E_q}\Phi_q : V_p \to B({}_{E_q}H\,,\ {}_{V_p\otimes_{\Psi_q}E_q}H)$ defined by ${}^{V_p}_{E_q}\Phi_q(\sigma^V_p(v))(\sigma^E_q(x)\otimes h) = \sigma^V_p(v)\otimes\sigma^E_q(x)\otimes h$ is a non-degenerate representation of V_p which is also a ${}^{A_p}_{E_q}\varphi_q$-morphism. Indeed, ${}^{V_p}_{E_q}\Phi_q$ is the Rieffel induced representation from W_q to V_p via E_q. Hence, ${}^{V_p}_{E_q}\Phi_q \circ \sigma^V_p$ is a non-degenerate representation of V and it is a ${}^{A_p}_{E_q}\varphi_q \circ \pi_p$-morphism. The representations $({}^{A}_{E}\varphi\ ,\ {}_{E}H)$ and $({}^{A_p}_{E_q}\varphi_q \circ \pi_p\ ,\ {}_{E_q}H)$ of A are unitarily equivalent by [11, proposition 3.4]. We define the linear map $U_1 : E \otimes_{alg} H \to E_q \otimes_{alg} H$, $U_1(x \otimes h) = \sigma^E_q(x) \otimes h$ which satisfies

$$\begin{aligned}\langle U_1(x\otimes h), U_1(x\otimes h)\rangle &= \langle \sigma^E_q(x)\otimes h, \sigma^E_q(x)\otimes h\rangle \\ &= \langle h, \varphi_q(\langle \sigma^E_q(x), \sigma^E_q(x)\rangle)h\rangle_H \\ &= \langle h, \varphi_q(\pi_q(\langle x, x\rangle))h\rangle_H \\ &= \langle h, \varphi(\langle x, x\rangle)h\rangle_H \\ &= \langle x\otimes h, x\otimes h\rangle,\end{aligned}$$

for all $x \in E$ and $h \in H$. Then U_1 can be extended to a bounded linear operator, which is again denoted by U_1 from ${}_EH$ onto ${}_{E_q}H$. It is easy to see that U_1 is a unitary operator. We define the linear map $U_2 : V \otimes_{alg} E \otimes_{alg} H \to V_p \otimes_{alg} E_q \otimes_{alg} H$ by $U_2(v \otimes x \otimes h) = \sigma^V_p(v) \otimes \sigma^E_q(x) \otimes h$. For every $v \in V$, $x \in E$ and $h \in H$ we have

$$\begin{aligned}\langle U_2(v\otimes x\otimes h), U_2(v\otimes x\otimes h)\rangle &= \langle \sigma^V_p(v)\otimes\sigma^E_q(x)\otimes h, \sigma^V_p(v)\otimes\sigma^E_q(x)\otimes h\rangle \\ &= \langle h, \varphi_q\Big(\langle \sigma^V_p(v)\otimes\sigma^E_q(x), \sigma^V_p(v)\otimes\sigma^E_q(x)\rangle\Big)h\rangle_H \\ &= \langle h, \varphi_q\Big(\langle \sigma^E_q(x), \Psi_p(\langle \sigma^V_p(v), \sigma^V_p(v)\rangle)\sigma^E_q(x)\rangle\Big)h\rangle_H \\ &= \langle h, \varphi_q\Big(\langle \sigma^E_q(x), \Psi_p(\pi_p(\langle v, v\rangle))\sigma^E_q(x)\rangle\Big)h\rangle_H\end{aligned}$$

$$\begin{aligned}\langle U_2(v\otimes x\otimes h), U_2(v\otimes x\otimes h)\rangle &= \langle h, \varphi_q\Big(\langle\sigma_q^E(x), (\pi_q)_*(\Psi(\langle v,v\rangle))\sigma_q^E(x)\rangle\Big)h\rangle_H\\ &= \langle h, \varphi_q\Big(\langle\sigma_q^E(x), \sigma_q^E(\Psi(\langle v,v\rangle)x)\rangle\Big)h\rangle_H\\ &= \langle h, \varphi_q\Big(\pi_q(\langle x, \Psi(\langle v,v\rangle)x)\Big)h\rangle_H\\ &= \langle h, \varphi(\langle x, \Psi(\langle v,v\rangle)x)h\rangle_H\\ &= \langle v\otimes x\otimes h, v\otimes x\otimes h\rangle,\end{aligned}$$

and so U_2 can be extended to a bounded linear operator U_2 from ${}_{V\otimes_\Psi E}H$ onto ${}_{V_p\otimes_{\Psi_q}E_q}H$. It is easy to see that U_2 is unitary. Moreover, $U_2\ {}^V_E\Phi(v) = ({}^{V_p}_{E_q}\Phi_q \circ \sigma_p^V)\, U_1(v)$ for all $v\in V$. Hence, the representations ${}^V_E\Phi$ and ${}^{V_p}_{E_q}\Phi_q \circ \sigma_p^V$ are unitarily equivalent. □

Theorem 4. *Let $\Phi_1 : W \to B(H_1, K_1)$ and $\Phi_2 : W \to B(H_2, K_2)$ be two non-degenerate representations of W. If Φ_1 and Φ_2 are unitarily equivalent, then ${}^V_E\Phi_1$ and ${}^V_E\Phi_2$ are unitarily equivalent, too.*

Proof. Let $q, q'\in S(B)$, $(\varphi_{1\,q}, H_1)$ be a representation of B_q associated to φ_1 and let $(\varphi_{2\,q'}, H_2)$ be a representation of $B_{q'}$ associated to φ_2. Consider $r\in S(B)$ such that $q, q' \leq r$. By Theorem 3, there exists $p\in S(A)$ such that A_p acts non-degenerately on E_r and the representation ${}^V_E\Phi_i$ is unitarily equivalent to ${}^{V_p}_{E_r}\Phi_{i\,r} \circ \sigma_p^V$ for $i = 1, 2$. Since $\Phi_{1\,r}$ and $\Phi_{2\,r}$ are unitarily equivalent representations of W_r, Lemma 3 implies that the representations ${}^{V_p}_{E_r}\Phi_{1\,r}$ and ${}^{V_p}_{E_r}\Phi_{2\,r}$ are unitarily equivalent. □

Corollary 2. *If $\Phi : W \to B(H, K)$ and $\oplus_{i\in I}\Phi_i : W \to B(\oplus_{i\in I}H_i, \oplus_{i\in I}K_i)$ are unitarily equivalent, then ${}^V_E\Phi$ and $\oplus_{i\in I}\ {}^V_E\Phi_i$ are unitarily equivalent.*

Proof. Let $q\in S(B)$ and $\Phi_q : W_q \to B(H, K)$ be a representation of W_q associated to Φ. For every $i\in I$, define $\Phi_{i\,q} : W_q \to B(H_i, K_i)$ by $\Phi_{i\,q}(\sigma_q^W(w)) = \Phi_i(w)$. If $\sigma_q^W(w) = 0$, then $\Phi_q(\sigma_q^W(w)) = 0$ and so $\Phi(w) = 0$. Since Φ and $\oplus_{i\in I}\Phi_i$ are unitarily equivalent, we conclude that $\oplus_{i\in I}\Phi_i(w) = 0$ and therefore, $\Phi_i(w) = 0$ for each $i\in I$. It proves that $\Phi_{i\,q}$ is well-defined for any $i\in I$. It is easy to see that Φ_q is unitarily equivalent to $\oplus_{i\in I}\Phi_{i\,q}$. By Theorem 3, there exists $p\in S(A)$ such that A_p acts non-degenerately on E_q and the representations ${}^V_E\Phi$ and ${}^{V_p}_{E_q}\Phi_q \circ \sigma_p^V$ of V are unitarily equivalent. The representations ${}^V_E\Phi_i$ and ${}^{V_p}_{E_q}\Phi_{i\,q} \circ \sigma_p^V$, $i\in I$ are unitarily equivalent, too. On the other hand, Corollary 1 implies that the representations ${}^{V_p}_{E_q}\Phi_q$ and $\oplus_{i\in I}\ {}^{V_p}_{E_q}\Phi_{i\,q}$ of V_p are unitarily equivalent. Consequently, the representations ${}^{V_p}_{E_q}\Phi_q \circ \sigma_p^V$ and $\oplus_{i\in I}({}^{V_p}_{E_q}\Phi_{i\,q} \circ \sigma_p^V)$ of V are unitarily equivalent. □

4. The imprimitivity theorem for Hilbert modules

In this section, we introduce the concept of Morita equivalence between Hilbert modules over locally C*-algebras and give a module version of the imprimitivity theorem.

Let A and B be locally C*-algebras. We say that A and B are *strongly Morita equivalent*, written $A \sim_M B$, if there is a full Hilbert A module E such that locally C*-algebras B and $K_A(E)$ are isomorphic. Joita [10, Proposition 4.4] showed that strong Morita equivalence is an equivalence relation in the set of all locally C*-algebras. The vector space $\tilde{E} := K_A(E, A)$ is a full Hilbert $K_A(E)$-module with the following action and inner product

$$(T, S) \to TS, \quad S \in K_A(E),\ T \in K_A(E, A),$$
$$\langle T, S\rangle = T^* S, \quad T, S \in K_A(E, A).$$

Since locally C*-algebras B and $K_A(E)$ are isomorphic, $\tilde{E}$ may be regarded as a Hilbert B-module. Moreover, the linear map α from A to $K_B(\tilde{E})$ defined by $\alpha(a)(\theta_{b,x}) = \theta_{ab,x}$ is an isomorphism of locally C*-algebras by [10, Lemma 4.2 and Remark 4.3]. It is easy to see that for each $p \in S(A)$, the linear map $U_p : (\tilde{E})_p \to \tilde{E}_p$ defined by $U_p(T + N_p^{\tilde{E}}) = (\pi_p)_*(T)$ is unitary and so the Hilbert $K_{A_p}(E_p)$-modules $(\tilde{E})_p$ and $\tilde{E}_p$ are the same.

Definition 2. *Suppose V and W are Hilbert modules over locally C*-algebras A and B, respectively. The Hilbert modules V and W are called Morita equivalent if $K_A(V)$ and $K_B(W)$ are strong Morita equivalent as locally C*-algebras. In this case, we write $V \sim_M W$.*

Lemma 5. *Let V be a full Hilbert module over locally C*-algebra A. Then $K_A(V)$ is strong Morita equivalent to $\overline{\langle V, V\rangle}$.*

Proof. The module $\tilde{V} = K_A(V, A)$ is a full Hilbert $K_A(V)$-module by [10, Corollary 3.3]. Then locally C*-algebras $K_{K_A(V)}(\tilde{V})$ and $K_A(A)$ are isomorphic by Lemma 4.2 in [10]. Since $\overline{\langle V, V\rangle} = A \simeq K_A(A)$, locally C*-algebras $K_A(V)$ and $\overline{\langle V, V\rangle}$ are strong Morita equivalent. □

Corollary 3. *Two full Hilbert modules over locally C*-algebras are Morita equivalent if and only if their underlying locally C*-algebras are strong Morita equivalent.*

Theorem 5. *Let V and W be two full Hilbert modules over locally C*-algebras A and B, respectively, such that $V \sim_M W$. If E is a Hilbert A-module which gives the strong Morita equivalence between A and B, and Φ is a non-degenerate representation of V, then Φ is unitarily equivalent to ${}^{V}_{\tilde{E}}({}^{W}_{E}\Phi)$.*

Proof. Let $p \in S(A)$ and Φ_p be a non-degenerate representation of V_p associated to Φ. Using [11, Lemma 4.1], there is $q \in S(B)$ such that $A_p \sim_M B_q$ and E_p gives the strong Morita equivalent between A_p and B_q. The representations φ_p and ${}^{A_p}_{\tilde{E}_p}({}^{B_q}_{E_p}\varphi_p)$ of A_p are unitarily equivalent by [15, Theorem 3.29]. Then the representations Φ_p and ${}^{V_p}_{\tilde{E}_p}({}^{W_q}_{E_p}\Phi_p)$ of V_p are unitarily equivalent by Lemma 2 and consequently, the representations ${}^{V_p}_{\tilde{E}_p}({}^{W_q}_{E_p}\Phi_p) \circ \sigma_p^V$ and $\Phi_p \circ \sigma_p^V = \Phi$ of V are unitarily equivalent. In view of Theorems 3 and 4, we have

- the representations ${}^{W}_{E}\Phi$ and ${}^{W_q}_{E_p}\Phi_p \circ \sigma_q^W$ of W are unitarily equivalent,

- the representations ${}^{V}_{\tilde{E}}({}^{W}_{E}\Phi)$ and ${}^{V}_{\tilde{E}}({}^{W_q}_{E_p}\Phi_p \circ \sigma_q^W)$ of V are unitarily equivalent, and
- the representations ${}^{V}_{\tilde{E}}({}^{W_q}_{E_p}\Phi_p \circ \sigma_q^W)$ and ${}^{V_p}_{\tilde{E_P}}({}^{W_q}_{E_p}\Phi_p \circ \sigma_q^W)_q \circ \sigma_p^V$ of V are unitarily equivalent.

The assertion now follows from the fact that $({}^{W_q}_{E_p}\Phi_p \circ \sigma_q^W)_q = {}^{W_q}_{E_p}\Phi_p$. □

We now reformulate the imprimitivity theorem within the framework of Hilbert modules as follows.

Theorem 6. *Let V and W be two Hilbert modules over locally C*-algebras A and B, respectively. If $V \sim_M W$, then there is a bijective correspondence between equivalence classes of non-degenerate representations of V and W.*

Proof. By replacing the underlying C*-algebras A and B, we may assume that V and W are full Hilbert modules over A and B, respectively. Let E be a Hilbert A-module which gives strong Morita equivalence between A and B. Then, by Theorems 4 and 5, the map $\Phi \mapsto {}^{W}_{E}\Phi$ from the set of all non-degenerate representations of V to the set of all non-degenerate representations of W induces a bijective correspondence between equivalence classes of non-degenerate representations of V and W. □

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