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Several explicit formulae for Bernoulli polynomials

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Abstract. We prove several explicit formulae for the n-th Bernoulli polynomial $B_n(x)$, in which $B_n(x)$ is equal to an affine combination of the polynomials $(x-1)^n$, $(x-2)^n, \ldots, (x-k-1)^n$, where k is any fixed positive integer greater then or equal to n.

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1. Introduction

Over the years, Bernoulli numbers B_n and polynomials $B_n(x)$ have proven to be important mathematical objects. Since they appeared on the scene (17th century), they have been of interest to many mathematicians and they have appeared in many different fields of mathematics (see the interesting document of Mazur [9]). Bernoulli polynomials are usually defined by means of the generating function [1, p. 48]

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad (|t|<2\pi).$$

We can also find Bernoulli polynomials defined by [8, p. 367]:

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k}.$$

We mention the explicit formula [6, Vol. 8, (2.6)]

$$B_n(x)=\sum_{i=0}^{n}\frac{1}{i+1}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(x+j)^n, \tag{1}$$

since this is the known formula for Bernoulli polynomials that will be used in this paper.

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A large number of explicit formulas for Bernoulli numbers and polynomials are known nowadays [2, 4, 15]. We refer the reader to the article of Gould [7] and mention, in particular, formula (3) of that article, namely

$$B_n = \frac{1}{n+1}\sum_{i=1}^{n}\sum_{j=1}^{i}(-1)^{i+j}\frac{\binom{n+1}{j}}{\binom{n}{i}}(i-j)^n, \tag{2}$$

which is "an old result" rediscovered in 1959. (Note that we can write (2) with the indices i, j beginning at 0. See also [6, Vol. 8, (1.9)] and [3, (24.6.2)].)

The first main result we will prove in this paper is

Proposition 1. *For $0 \leq n \leq k$ we have*

$$(k+1)(-1)^n\sum_{i=0}^{n}\frac{1}{i+1}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(j+1)^n = \sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{i+j}\frac{\binom{k+1}{j}}{\binom{k}{i}}(j-i-1)^n. \tag{3}$$

By writing (3) as

$$(-1)^n\sum_{i=0}^{n}\frac{1}{i+1}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(j+1)^n = \frac{1}{k+1}\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{i+j}\frac{\binom{k+1}{j}}{\binom{k}{i}}(j-i-1)^n, \tag{4}$$

we can see that (3) is an identity for the Bernoulli number B_n. In fact, the left-hand side of (4) is, according to (1), the Bernoulli number $B_n = (-1)^n B_n(1)$. That is, formula (4) says that B_n is equal to the expression of the right-hand side of (4), *for any $k \geq n$.*

The second main result of this paper says that formula (3) is the particular case $x = 0$ of an identity involving Bernoulli polynomials.

Proposition 2. *For $0 \leq n \leq k$ we have*

$$B_n(x) = \frac{1}{k+1}\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{i+j}\frac{\binom{k+1}{j}}{\binom{k}{i}}(x-i+j-1)^n. \tag{5}$$

By using the known property of Bernoulli polynomials $(-1)^n B_n(x) = B_n(1-x)$, identity (5) can be written as

$$B_n(x) = \frac{1}{k+1}\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{i+j}\frac{\binom{k+1}{j}}{\binom{k}{i}}(x+i-j)^n, \tag{6}$$

and then identity (2) is the particular case of (5) in which $x = 0$ and $k = n$.

In the spirit of Gould's article [7] mentioned before, it is difficult to claim originality for (5). However, according to some of the non-trivial consequences of this formula (we already began to explore them [11]), which are not widely known, we believe that it must have some ingredient of originality.

In Section 2, we state and prove four lemmas that will be used in Section 3, where we present the proofs of Propositions 1 and 2. In Section 4, we give some concrete examples of (5).

Throughout the paper, k denotes a non-negative integer. We will be using, without further comments, standard results about interchanging indices in double sums (for example, those contained in Gould's book [6, Vol. 1]).

2. Some lemmas

Besides the basic facts involving binomial coefficients (e.g., the basic combinatorial identities), we will need some more specialized results of this kind. More specifically, in the proofs of the lemmas of this section, we will use five combinatorial identities from Gould's book [5]. For the reader's convenience, we quote them here:

$$\sum_{k=0}^{n}(-1)^k\binom{x}{k} = (-1)^n\binom{x-1}{n}. \tag{7}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^j = \begin{cases}0, & \text{if } 0\le j<n;\\ (-1)^n n!, & \text{if } j=n.\end{cases} \tag{8}$$

$$\sum_{k=1}^{n}\binom{x+k}{k}\frac{1}{x+k} = \frac{1}{x}\left(\binom{x+n}{n}-1\right). \tag{9}$$

$$\sum_{k=0}^{n}\frac{(-1)^k}{\binom{x}{k}} = \frac{x+1}{x+2}\left(1+\frac{(-1)^n}{\binom{x+1}{n+1}}\right). \tag{10}$$

$$\sum_{k=j}^{n}(-1)^k\binom{n}{k}\binom{k}{j} = \begin{cases}0, & \text{if } j\neq n;\\ (-1)^n, & \text{if } j=n.\end{cases} \tag{11}$$

(In Gould's book [5], these are identities (1.5), (1.13), (1.50), (2.1) and (3.119), respectively.)

Lemma 1. *For $k\ge 0$ we have*

$$\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{i+j}\frac{\binom{k+1}{j}}{\binom{k}{i}} = k+1. \tag{12}$$

Proof. Observe that

$$\begin{aligned}\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{i+j}\frac{\binom{k+1}{j}}{\binom{k}{i}} &= \sum_{j=0}^{k}\sum_{i=0}^{j}(-1)^{i}\frac{\binom{k+1}{k-j}}{\binom{k}{i+k-j}}\\ &= \sum_{j=0}^{k}\sum_{i=0}^{j}(-1)^{i}\frac{\binom{k+1}{j+1}}{\binom{k}{j-i}}\\ &= \sum_{j=0}^{k}(-1)^j\binom{k+1}{j+1}\sum_{i=0}^{j}(-1)^i\frac{1}{\binom{k}{i}}.\end{aligned}$$

By using identity (10) we obtain

$$\begin{aligned}\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{i+j}\frac{\binom{k+1}{j}}{\binom{k}{i}} &= \sum_{j=0}^{k}(-1)^{j}\binom{k+1}{j+1}\frac{k+1}{k+2}\left(1+\frac{(-1)^{j}}{\binom{k+1}{j+1}}\right)\\ &= \frac{k+1}{k+2}\left(\sum_{j=0}^{k}(-1)^{j}\binom{k+1}{j+1}+k+1\right)\\ &= k+1\,,\end{aligned}$$

as desired. □

Lemma 2. *For $0 \le i \le k$ and real t with $t \neq 0$, we have*

(a)

$$1-t^{k+2}\sum_{j=0}^{i+1}\binom{k+2}{j}(t^{-1}-1)^{j} = (-1)^{i}(t-1)^{2+i}\sum_{j=0}^{k-i}\binom{j+i+1}{i+1}t^{j}. \tag{13}$$

(b)

$$1-t^{k+1}\sum_{j=0}^{i}\binom{k+1}{j}(t^{-1}-1)^{j} = (1-t)^{1+i}\sum_{j=0}^{k-i}\binom{j+i}{i}t^{j}. \tag{14}$$

Proof. (a): We can write formula (13) as

$$(1-t)^{2+i}\sum_{j=0}^{k-i}\binom{j+i+1}{i+1}t^{j}+\sum_{j=0}^{i+1}\binom{k+2}{j}t^{k+2-j}\,(1-t)^{j} = 1. \tag{15}$$

Let us write $P(t)$ for the $(k+2)$-th degree polynomial of the left-hand side in (15). Observe that $P\,(0) = 1$, so we need to prove that the remaining coefficients of $P(t)$ are equal to 0. We have

$$\begin{aligned}P(t) =& \sum_{j=0}^{k-i}\sum_{r=0}^{i+2}\binom{k+1-j}{i+1}\binom{i+2}{r}(-1)^{r}t^{k+r-i-j}\\ &+\sum_{j=0}^{i+1}\sum_{r=0}^{j}\binom{k+2}{j}\binom{j}{r}(-1)^{r}t^{k+2+r-j}.\end{aligned} \tag{16}$$

Let us introduce a new index s in the sums of the left-hand side of (16), namely $s = k+r-i-j$ for the first sum, and $s = k+2+r-j$ for the second one. In both cases we have $0 \le s \le k+2$, and we can write (16) as

$$\begin{aligned}P(t) = (-1)^{k}\sum_{s=0}^{k+2}&\left(\sum_{j=0}^{k-i}\binom{k+1-j}{i+1}\binom{i+2}{s+i+j-k}(-1)^{i+s+j}\right.\\ &\left.+\sum_{j=0}^{i+1}\binom{k+2}{j}\binom{j}{s+j-k-2}(-1)^{s+j}\right)t^{s}\end{aligned}$$

or

$$P(t) = (-1)^k \sum_{s=0}^{k+2} \left(\sum_{j=0}^{k-i} \binom{i+1+j}{j} \binom{i+2}{s-j} (-1)^{s+j} \right.$$
$$\left. + \sum_{j=0}^{i+1} \binom{k+2}{j} \binom{j}{k+2-s} (-1)^{s+j} \right) t^s.$$

We need to show that for $0 \le i \le k$ and $1 \le s \le k+2$ we have

$$\sum_{j=0}^{k-i} \binom{i+1+j}{j} \binom{i+2}{s-j} (-1)^j + (-1)^k \sum_{j=0}^{i+1} \binom{k+2}{j} \binom{j}{k+2-s} (-1)^j = 0. \tag{17}$$

According to (11), for $1 \le s \le k+2$ we have

$$\sum_{j=0}^{k+2} \binom{k+2}{j} \binom{j}{k+2-s} (-1)^j = 0\,,$$

so we can write formula (17) as

$$\sum_{j=0}^{k-i} \binom{i+1+j}{j} \binom{i+2}{s-j} (-1)^j - (-1)^k \sum_{j=i+2}^{k+2} \binom{k+2}{j} \binom{j}{k+2-s} (-1)^j = 0$$

or

$$\sum_{j=0}^{k-i} \binom{i+1+j}{j} \binom{i+2}{s-j} (-1)^j - \sum_{j=0}^{k-i} \binom{k+2}{j} \binom{k+2-j}{s-j} (-1)^j = 0. \tag{18}$$

Observe that

$$\sum_{j=0}^{k-i} \binom{k+2}{j} \binom{k+2-j}{s-j} (-1)^j = (-1)^{k+i} \binom{k+2}{s} \binom{s-1}{k-i}, \tag{19}$$

(identity (7), together with some trivial simplifications). We have also that

$$\sum_{j=0}^{k-i} \binom{i+1+j}{j} \binom{i+2}{s-j} (-1)^j = (-1)^{k+i} \binom{k+2}{s} \binom{s-1}{k-i}\,, \tag{20}$$

which can be demonstrated by means of an induction argument on k (left to the reader). Identities (19) and (20) prove (18), and then the proof of (13) is complete.

(b): The argument that proves (14) is similar to that presented in (a), and we show only some steps of this argument. We write (14) as

$$(1-t)^{1+i} \sum_{j=0}^{k-i} \binom{j+i}{i} t^j + t^{k+1} \sum_{j=0}^{i} \binom{k+1}{j} (t^{-1}-1)^j = 1$$

or

$$\sum_{j=0}^{k-i}\sum_{l=0}^{1+i}\binom{1+i}{l}\binom{j+i}{i}(-1)^l t^{l+j}+\sum_{j=0}^{i}\sum_{l=0}^{j}\binom{k+1}{j}\binom{j}{l}(-1)^l\, t^{k+1-j+l}=1. \quad (21)$$

If $P(t)$ is the left-hand side polynomial in (21), we can see at once that $P(0)=1$, so we will prove that the coefficient of t^s in the polynomial $P(t)$ is equal to 0 for $s\geq 1$. We set $s=l+j$ in the first sum of the left-hand side of (21), and we set $s=k+1-j+l$ in the second one. Then we write (21) as

$$\begin{aligned}&\sum_{s=0}^{k+1}\sum_{j=0}^{i}\binom{k+1-i}{s-j}\binom{k+j-i}{j}(-1)^{s+j}\,t^s\\&\quad+(-1)^k\sum_{s=i+1}^{k+1}\sum_{j=0}^{k-i}\binom{k+1}{j}\binom{j}{k+1-s}(-1)^{s+j+1}t^s=1\,,\end{aligned}$$

so we need to show that for $0\leq i\leq k$ and $1\leq s\leq k+1$ we have

$$\sum_{j=0}^{i}\binom{k+1-i}{s-j}\binom{k+j-i}{j}(-1)^j-(-1)^k\sum_{j=0}^{k-i}\binom{k+1}{j}\binom{j}{k+1-s}(-1)^j=0. \quad (22)$$

In fact, one can check that the identity

$$(-1)^k\sum_{j=0}^{k-i}\binom{k+1}{j}\binom{j}{k+1-s}(-1)^j=(-1)^i\binom{k+1}{s}\binom{s-1}{i} \quad (23)$$

is equivalent to $\sum_{j=0}^{k-i}\binom{s}{j+i+1}(-1)^j=\binom{s-1}{i}$, which in turn can be demonstrated easily. On the other hand, we also have

$$\sum_{j=0}^{i}\binom{k+1-i}{s-j}\binom{k+j-i}{j}(-1)^j=(-1)^i\binom{k+1}{s}\binom{s-1}{i}, \quad (24)$$

which can be written as

$$\sum_{j=0}^{k-i}\binom{i+1}{s-j}\binom{i+j}{j}(-1)^j=(-1)^{k+i}\binom{k+1}{s}\binom{s-1}{k-i}. \quad (25)$$

Identity (25) can be demonstrated with an induction argument on k (left to the reader). Identities (23) and (24) prove (22), and then the proof of (14) is complete. □

Lemma 3. *For $0\leq i\leq k$ we have*

(*a*)

$$\sum_{j=i}^{k}\frac{\binom{k+2}{j-i}}{\binom{k}{j}}=\frac{k+1}{i+1}\binom{k+1}{i+1}. \quad (26)$$

(b)

$$\sum_{j=i}^{k} \frac{\binom{k+1}{j-i}}{\binom{k}{j}} = (k+1)\sum_{j=0}^{k-i} \frac{1}{i+j+1}\binom{j+i}{i}. \tag{27}$$

Proof. We use that

$$\binom{n}{m}^{-1} = (n+1)\int_0^1 t^m(1-t)^{n-m}dt\,, \tag{28}$$

which is a popular tool used in problems where reciprocals of binomial coefficients are involved [12, 13, 14, 16]. The integral of the right-hand side of (28) is $B(m+1, n-m+1)$, the Beta function evaluated at $(m+1, n-m+1)$.

(a): We use (28) to write the left-hand side of (26) as

$$\sum_{j=i}^{k} \frac{\binom{k+2}{j-i}}{\binom{k}{j}} = \sum_{j=i}^{k}\binom{k+2}{j-i}(k+1)\int_0^1 t^j(1-t)^{k-j}dt.$$

Then we have to show that

$$\int_0^1 (1-t)^k \sum_{j=i}^{k}\binom{k+2}{j-i}\left(\frac{t}{1-t}\right)^j dt = \frac{1}{i+1}\binom{k+1}{i+1}$$

or

$$\int_0^1 (1-t)^k \left(\frac{t}{1-t}\right)^i \sum_{j=0}^{k-i}\binom{k+2}{j}\left(\frac{t}{1-t}\right)^j dt = \frac{1}{i+1}\binom{k+1}{i+1}. \tag{29}$$

Observe that

$$\begin{aligned}\sum_{j=0}^{k-i}\binom{k+2}{j}\left(\frac{t}{1-t}\right)^j &= \sum_{j=0}^{k+2}\binom{k+2}{j}\left(\frac{t}{1-t}\right)^j - \sum_{j=k-i+1}^{k+2}\binom{k+2}{j}\left(\frac{t}{1-t}\right)^j \\ &= \left(\frac{t}{1-t}+1\right)^{k+2} - \sum_{j=0}^{i+1}\binom{k+2}{j}\left(\frac{t}{1-t}\right)^{k+2-j} \\ &= \left(\frac{1}{1-t}\right)^{k+2}\left(1 - t^{k+2}\sum_{j=0}^{i+1}\binom{k+2}{j}\left(\frac{1-t}{t}\right)^j\right).\end{aligned} \tag{30}$$

Use (30) to write the left-hand side of (29) as

$$\begin{aligned}&\int_0^1 (1-t)^k \left(\frac{t}{1-t}\right)^i \sum_{j=0}^{k-i}\binom{k+2}{j}\left(\frac{t}{1-t}\right)^j dt \\ &= \int_0^1 \left(\frac{1}{1-t}\right)^2\left(\frac{t}{1-t}\right)^i\left(1 - t^{k+2}\sum_{j=0}^{i+1}\binom{k+2}{j}(t^{-1}-1)^j\right)dt.\end{aligned} \tag{31}$$

According to (13), from (31) we have that

$$\begin{aligned}
&\int_0^1 (1-t)^k \left(\frac{t}{1-t}\right)^i \sum_{j=0}^{k-i} \binom{k+2}{j} \left(\frac{t}{1-t}\right)^j dt \\
&\quad = \int_0^1 \left(\frac{1}{1-t}\right)^2 \left(\frac{t}{1-t}\right)^i (-1)^i (t-1)^{2+i} \sum_{j=0}^{k-i} \binom{j+i+1}{i+1} t^j dt.
\end{aligned}$$

That is, we have that

$$\begin{aligned}
\int_0^1 (1-t)^k \left(\frac{t}{1-t}\right)^i \sum_{j=0}^{k-i} \binom{k+2}{j} \left(\frac{t}{1-t}\right)^j dt &= \sum_{j=0}^{k-i} \binom{j+i+1}{i+1} \int_0^1 t^{i+j} dt \\
&= \sum_{j=0}^{k-i} \frac{1}{i+j+1} \binom{j+i+1}{i+1} \\
&= \frac{1}{i+1} \binom{k+1}{i+1},
\end{aligned}$$

as desired. (In the last step, we used (9).)

(b): The left-hand side of (27) can be written (according to (28)) as

$$\sum_{j=i}^{k} \frac{\binom{k+1}{j-i}}{\binom{k}{j}} = (k+1) \sum_{j=i}^{k} \binom{k+1}{j-i} \int_0^1 t^j (1-t)^{k-j} dt\,,$$

and then we have to show that

$$\int_0^1 (1-t)^k \sum_{j=i}^{k} \binom{k+1}{j-i} \left(\frac{t}{1-t}\right)^j dt = \sum_{j=0}^{k-i} \frac{1}{i+j+1} \binom{j+i}{i}$$

or

$$\int_0^1 (1-t)^k \left(\frac{t}{1-t}\right)^i \sum_{j=0}^{k-i} \binom{k+1}{j} \left(\frac{t}{1-t}\right)^j dt = \sum_{j=0}^{k-i} \frac{1}{i+j+1} \binom{j+i}{i}. \tag{32}$$

As we did with (30), we can see that

$$\sum_{j=0}^{k-i} \binom{k+1}{j} \left(\frac{t}{1-t}\right)^j = \left(\frac{1}{1-t}\right)^{k+1} \left(1 - t^{k+1} \sum_{j=0}^{i} \binom{k+1}{j} \left(\frac{1-t}{t}\right)^j\right). \tag{33}$$

Then, beginning with the left-hand side of (32), using (33) and (14), we have that

$$\begin{aligned}
&\int_0^1 (1-t)^k \left(\frac{t}{1-t}\right)^i \sum_{j=0}^{k-i} \binom{k+1}{j} \left(\frac{t}{1-t}\right)^j dt \\
&\quad = \int_0^1 \left(\frac{1}{1-t}\right) \left(\frac{t}{1-t}\right)^i \left(1 - t^{k+1} \sum_{j=0}^{i} \binom{k+1}{j} (t^{-1}-1)^j\right) dt \\
&\quad = \int_0^1 \left(\frac{1}{1-t}\right) \left(\frac{t}{1-t}\right)^i (1-t)^{1+i} \sum_{j=0}^{k-i} \binom{j+i}{i} t^j dt \\
&\quad = \sum_{j=0}^{k-i} \binom{j+i}{i} \int_0^1 t^{i+j} dt \\
&\quad = \sum_{j=0}^{k-i} \frac{1}{i+j+1} \binom{j+i}{i},
\end{aligned}$$

as desired. □

Lemma 4. *For $2 \le m \le k$ we have*

$$\sum_{i=0}^{k} \sum_{j=0}^{i} (-1)^{i+j} \frac{\binom{k+2}{j}}{\binom{k}{i}} (j-i-1)^m = 0. \tag{34}$$

Proof. We write the left-hand side of (34) as

$$\sum_{i=0}^{k} \sum_{j=0}^{i} (-1)^{i+j} \frac{\binom{k+2}{j}}{\binom{k}{i}} (j-i-1)^m = (-1)^m \sum_{i=0}^{k} (-1)^i (i+1)^m \sum_{j=i}^{k} \frac{\binom{k+2}{j-i}}{\binom{k}{j}}.$$

Thus, by using (26) (and the fact that $m \ge 2$) we have that

$$\begin{aligned}
&\sum_{i=0}^{k} \sum_{j=0}^{i} (-1)^{i+j} \frac{\binom{k+2}{j}}{\binom{k}{i}} (j-i-1)^m \\
&\quad = (-1)^m \sum_{i=0}^{k} (-1)^i (i+1)^m \frac{k+1}{i+1} \binom{k+1}{k-i} \\
&\quad = (-1)^{m+k} (k+1) \sum_{i=0}^{k} (-1)^i \frac{(k+1-i)^m}{k+1-i} \binom{k+1}{i} \\
&\quad = (-1)^{m+k} (k+1) \sum_{i=0}^{k+1} \binom{k+1}{i} (-1)^i (k+1-i)^{m-1} \\
&\quad = (-1)^{m+k} (k+1) \sum_{l=0}^{m-1} (-1)^l \binom{m-1}{l} (k+1)^{m-1-l} \sum_{i=0}^{k+1} \binom{k+1}{i} (-1)^i i^l. \qquad (35)
\end{aligned}$$

Identity (8) tells us that for $0 \leq l \leq k$ we have

$$\sum_{i=0}^{k+1} \binom{k+1}{i} (-1)^i i^l = 0. \tag{36}$$

In particular, for $0 \leq l \leq m-1$, identity (36) is valid, and then (35) is equal to 0, finishing the proof of (34). $\square$

3. Proofs of Propositions 1 and 2

Now we are ready to prove the main results of the article.

Proof of Proposition 1. Beginning with the right-hand side of (3), we can write

$$\sum_{i=0}^{k} \sum_{j=0}^{i} (-1)^{i+j} \frac{\binom{k+1}{j}}{\binom{k}{i}} (j-i-1)^n = (-1)^n \sum_{i=0}^{k} (-1)^i (i+1)^n \sum_{j=i}^{k} \frac{\binom{k+1}{j-i}}{\binom{k}{j}}.$$

According to (27), we have

$$\begin{aligned}\sum_{i=0}^{k} \sum_{j=0}^{i} (-1)^{i+j} \frac{\binom{k+1}{j}}{\binom{k}{i}} (j-i-1)^n &= (k+1)(-1)^n \sum_{i=0}^{k} (-1)^i (i+1)^n \\ &\quad \times \sum_{j=0}^{k-i} \frac{1}{i+j+1} \binom{j+i}{i}\end{aligned}$$

or

$$\sum_{i=0}^{k} \sum_{j=0}^{i} (-1)^{i+j} \frac{\binom{k+1}{j}}{\binom{k}{i}} (j-i-1)^n = (k+1)(-1)^n \sum_{i=0}^{k} (-1)^i (i+1)^n \sum_{j=i}^{k} \frac{1}{j+1} \binom{j}{i}.$$

Some further simplifications give us

$$\begin{aligned}\sum_{i=0}^{k} \sum_{j=0}^{i} (-1)^{i+j} \frac{\binom{k+1}{j}}{\binom{k}{i}} (j-i-1)^n &= (k+1)(-1)^n \sum_{j=0}^{k} (-1)^j (j+1)^n \sum_{i=j}^{k} \frac{1}{i+1} \binom{i}{j} \\ &= (k+1)(-1)^n \sum_{i=0}^{k} \frac{1}{i+1} \sum_{j=0}^{k} (-1)^j \binom{i}{j} (j+1)^n.\end{aligned}$$

Finally, we claim that for $0 \leq n \leq k$ we have

$$\sum_{i=0}^{k} \frac{1}{i+1} \sum_{j=0}^{k} (-1)^j \binom{i}{j} (j+1)^n = \sum_{i=0}^{n} \frac{1}{i+1} \sum_{j=0}^{i} (-1)^j \binom{i}{j} (j+1)^n. \tag{37}$$

In fact, beginning with the left-hand side of (37) we have that

$$\sum_{i=0}^{k}\frac{1}{i+1}\sum_{j=0}^{k}(-1)^j\binom{i}{j}(j+1)^n = \sum_{i=0}^{n}\frac{1}{i+1}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(j+1)^n + \sum_{i=n+1}^{k}\frac{1}{i+1}\sum_{j=0}^{i}(-1)^j\binom{i}{j}(j+1)^n.$$

But for $i \geq n+1$ we have $\sum_{j=0}^{i}(-1)^j\binom{i}{j}(j+1)^n = 0$ (identity (8)), from where our claim (37) follows, and the proof is complete. $\square$

Proof of Proposition 2. Let us denote the right-hand side polynomial in (5) as $\mathbb{B}_n(x)$. That is,

$$\mathbb{B}_n(x) = \frac{1}{k+1}\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{i+j}\frac{\binom{k+1}{j}}{\binom{k}{i}}(x-i+j-1)^n. \tag{38}$$

The proof is based on the following characterization of Bernoulli polynomials [10, pp. 17–18]: *the only polynomial* $\mathbb{B}_n(x)$ *such that* $\mathbb{B}_n(x+1)-\mathbb{B}_n(x) = nx^{n-1}$, $\mathbb{B}_n(0) = B_n$, *is the n-th Bernoulli polynomial* $B_n(x)$. Proposition 1 tells us that $\mathbb{B}_n(0) = B_n$. Thus, we need to prove that the polynomial (38) satisfies the difference equation $\mathbb{B}_n(x+1)-\mathbb{B}_n(x) = nx^{n-1}$. That is, we need to prove that

$$\frac{1}{k+1}\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{i+j}\frac{\binom{k+1}{j}}{\binom{k}{i}}\left((x-i+j)^n-(x-i+j-1)^n\right) = nx^{n-1}.$$

Observe that

$$\begin{aligned}&\frac{1}{k+1}\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{i+j}\frac{\binom{k+1}{j}}{\binom{k}{i}}\left((x-i+j)^n-(x-i+j-1)^n\right)\\ &\quad=\sum_{r=0}^{n}\binom{n}{r}\frac{1}{k+1}\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{i+j}\frac{\binom{k+1}{j}}{\binom{k}{i}}\left((j-i)^{n-r}-(j-i-1)^{n-r}\right)x^r\\ &\quad=nx^{n-1}+\sum_{r=0}^{n-2}\binom{n}{r}\frac{1}{k+1}\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{i+j}\frac{\binom{k+1}{j}}{\binom{k}{i}}\left((j-i)^{n-r}-(j-i-1)^{n-r}\right)x^r\end{aligned}$$

(in the last step, we used (12)). Thus, our proof ends if we show that for $0 \leq r \leq n-2$ we have

$$\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{i+j}\frac{\binom{k+1}{j}}{\binom{k}{i}}\left((j-i)^{n-r}-(j-i-1)^{n-r}\right) = 0. \tag{39}$$

We will prove that for $2 \leq m \leq k$ we have the identity

$$\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{i+j}\frac{\binom{k+1}{j}}{\binom{k}{i}}(j-i)^m = \sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{i+j}\frac{\binom{k+1}{j}}{\binom{k}{i}}(j-i-1)^m, \tag{40}$$

which plainly implies that we have (39) for $0 \le r \le n-2$, and which in turn gives us the desired end of the proof.

We can write (40) as

$$\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{i+j+1}\frac{\binom{k+1}{j-1}}{\binom{k}{i}}(j-i-1)^m = \sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{i+j}\frac{\binom{k+1}{j}}{\binom{k}{i}}(j-i-1)^m\,, \tag{41}$$

and we can easily see that (41) is equivalent to

$$\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{i+j}\frac{\binom{k+2}{j}}{\binom{k}{i}}(j-i-1)^m = 0. \tag{42}$$

But in Lemma 4 we proved that (42) is true for $2 \le m \le k$. Then our proof is complete. □

4. Examples

In Proposition 2, we demonstrated that the Bernoulli polynomial $B_n(x)$ can be written as a linear combination of $(k+1)$ polynomials $(x-1)^n$,$(x-2)^n$, …, $(x-k-1)^n$, where k is any positive integer $\ge n$. An additional observation is that Lemma 1 says that this linear combination is in fact an *affine* combination: if

$$B_n(x) = \sum_{t=0}^{k} c_t(x-1-t)^n,$$

the coefficients $c_0, \ldots, c_k$ satisfy that

$$\sum_{t=0}^{k} c_t = 1.$$

In this section, we give some concrete examples.

With $k = 1$ formula (5) looks as

$$B_n(x) = \frac{3}{2}(x-1)^n - \frac{1}{2}(x-2)^n\,, \tag{43}$$

which works for $n = 0, 1$, namely

$$\begin{aligned} B_0(x) &= \frac{3}{2}(x-1)^0 - \frac{1}{2}(x-2)^0 = 1,\\ B_1(x) &= \frac{3}{2}(x-1) - \frac{1}{2}(x-2) = x - \frac{1}{2}. \end{aligned}$$

(For $n > 1$, the right-hand side of (43) does not give us $B_n(x)$. For example, if $n = 2$, the right-hand side of (43) is $\frac{3}{2}(x-1)^2 - \frac{1}{2}(x-2)^2 = x^2 - x - \frac{1}{2}$, which is not $B_2(x) = x^2 - x + \frac{1}{6}$.)

With $k=2$ formula (5) is

$$B_n(x)=\frac{11}{6}(x-1)^n-\frac{7}{6}(x-2)^n+\frac{1}{3}(x-3)^n\,,$$

which works for $n=0,1,2$, namely

$$\begin{aligned}
B_0(x)&=\frac{11}{6}(x-1)^0-\frac{7}{6}(x-2)^0+\frac{1}{3}(x-3)^0=1,\\
B_1(x)&=\frac{11}{6}(x-1)-\frac{7}{6}(x-2)+\frac{1}{3}(x-3)=x-\frac{1}{2},\\
B_2(x)&=\frac{11}{6}(x-1)^2-\frac{7}{6}(x-2)^2+\frac{1}{3}(x-3)^2=x^2-x+\frac{1}{6}.
\end{aligned}$$

From a different point of view, for each non-negative integer n, expression (5) gives us infinitely many formulas to write the n-th Bernoulli polynomial $B_n(x)$, namely, formulas of the right-hand side of (5) with $k\ge n$. For example, the first 3 formulas for the Bernoulli polynomial $B_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{2}x$ are the following:

$$\begin{aligned}
B_3(x)&=\frac{25}{12}(x-1)^3-\frac{23}{12}(x-2)^3+\frac{13}{12}(x-3)^3-\frac{1}{4}(x-4)^3,\\
&=\frac{137}{60}(x-1)^3-\frac{163}{60}(x-2)^3+\frac{137}{60}(x-3)^3-\frac{21}{20}(x-4)^3+\frac{1}{5}(x-5)^3,\\
&=\frac{49}{20}(x-1)^3-\frac{71}{20}(x-2)^3+\frac{79}{20}(x-3)^3\\
&\quad-\frac{163}{60}(x-4)^3+\frac{31}{30}(x-5)^3-\frac{1}{6}(x-6)^3
\end{aligned}$$

(corresponding to $k=3,4,5$). In particular, the Bernoulli polynomial $B_0(x)=1$ can be written as

$$B_0(x)=\frac{1}{k+1}\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{i+j}\frac{\binom{k+1}{j}}{\binom{k}{i}}\,,$$

where k is any non-negative integer. (This affirmation is precisely (12).) Similarly, the Bernoulli polynomial $B_1(x)=x-\frac{1}{2}$ can be written as

$$B_1(x)=\frac{1}{k+1}\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{i+j}\frac{\binom{k+1}{j}}{\binom{k}{i}}(x-i+j-1)\,,$$

where k is any positive integer.

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