GLASNIK MATEMATIČKI
Vol. 51(71)(2016), 109 – 115

FINITE p-GROUPS WHICH ARE NOT GENERATED BY THEIR NON-NORMAL SUBGROUPS

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Abstract. Here we classify finite non-Dedekindian p-groups which are not generated by their non-normal subgroups. (Theorem 1).

The purpose of this paper is to classify non-Dedekindian finite p-groups which are not generated by their non-normal subgroups. It is surprising that such p-groups must be of class 2 with a cyclic commutator subgroup.

We consider here only finite p-groups and our notation is standard (see [1]). We prove the following result.

Theorem 1. *Let G be a non-Dedekindian p-group and let G_0 be the subgroup generated by all nonnormal subgroups of G, where we assume $G_0 < G$. Then G is of class 2, G/G_0 is cyclic and for each $g \in G - G_0$, $\{1\} \neq \langle g \rangle \cap G_0 \trianglelefteq G$ and $G/(\langle g \rangle \cap G_0)$ is abelian so that G' is cyclic.*

Proof. Since our group G has at least p (non-normal) conjugate cyclic subgroups, it follows that the subgroup G_0 is noncyclic. Let $x \in G - G_0$. Then $\langle x \rangle \trianglelefteq G$, by hypothesis, and so G' centralizes $\langle x \rangle$. It follows from $\langle G - G_0 \rangle = G$ that $G' \leq \mathrm{Z}(G)$ and so $\mathrm{cl}(G) = 2$.

Let $g \in G - G_0$. Then $Z = \langle g \rangle \triangleleft G$. Write $Z_0 = Z \cap G_0$; then Z_0, being the intersection of two G-invariant subgroups, is G-invariant. We claim that G/Z_0 is Dedekindian. Indeed, let X/Z_0 be any proper subgroup in G/Z_0. We have to show that $X \triangleleft G$. If $X \not\leq G_0$, then $X \trianglelefteq G$. Now assume that $X < G_0$ (the subgroup G_0 is G-invariant). Then $XZ = ZX$ is normal in G

2010 *Mathematics Subject Classification.* 20D15.

Key words and phrases. Finite p-groups, Dedekindian p-groups, non-normal subgroups.

since $XZ \not\leq G_0$. By the product formula, one has

$$|XZG_0| = |ZG_0| = \frac{|Z||G_0|}{|Z_0|}.$$

On the other hand,

$$|XZG_0| = \frac{|XZ||G_0|}{|XZ \cap G_0|} = \frac{|X||Z|}{|Z_0|} \cdot \frac{|G_0|}{|XZ \cap Z_0|} = |XZG_0| \cdot \frac{|X|}{|XZ \cap G_0|}$$

which implies $X = XZ \cap G_0 \triangleleft G$, and we are done. We have proved that G/Z_0 is Dedekindian. In particular, $Z_0 \neq \{1\}$ since G is non-Dedekindian, by hypothesis. If $p > 2$, then G/Z_0 is abelian and so $G' \leq Z_0$ and G' is cyclic. If $p = 2$, then G/Z_0 is either abelian or Hamiltonian (= nonabelian Dedekindian).

It follows from the above that $\Omega_1(G) \leq G_0$.

Now assume that $p > 2$. Amongst all elements in the set $G - G_0$, we choose an element a of the smallest possible order. Then $a^p \in G_0$ and $G' \leq \langle a^p \rangle$ (see the previous paragraph). We set $|G'| = p^d$, $d \geq 1$. Suppose that G/G_0 is not cyclic. Then there is $b \in G - (G_0\langle a \rangle)$ such that $b^p \in G_0$. We have $\langle a \rangle \cap \langle b \rangle \geq G'$ and $\mathrm{o}(b) \geq \mathrm{o}(a)$ by the minimality of $\mathrm{o}(a)$. Set

$$|\langle a \rangle / (\langle a \rangle \cap \langle b \rangle)| = p^s, \text{ where } s \geq 1 \text{ and } \mathrm{o}(a) \geq p^{d+s}.$$

Hence there is $b' \in \langle b \rangle - \langle a \rangle$ such that $a^{p^s} = (b')^{-p^s}$. In that case, since $\mathrm{cl}(G) = 2$, one obtains

$$(ab')^{p^s} = a^{p^s}(b')^{p^s}[b', a]^{\binom{p^s}{2}} = [b', a]^{\binom{p^s}{2}},$$

where $s \geq 1, \mathrm{o}(a) \geq p^{d+s}$ and $\langle [b', a]^{\binom{p^s}{2}} \rangle < G'$ so that $\mathrm{o}([b', a]^{\binom{p^s}{2}}) < p^d$. It follows that

$$\mathrm{o}(ab') < p^{d+s} \text{ and so } \mathrm{o}(ab') < \mathrm{o}(a).$$

If $b' \in \langle b^p \rangle \leq G_0$, then $ab' \in G - G_0$. If $\langle b' \rangle = \langle b \rangle$, then $ab' \in G - (G_0\langle a \rangle)$ and so again $ab' \in G - G_0$. But this contradicts the minimality of $\mathrm{o}(a)$. We have proved that in case $p > 2$, G/G_0 is cyclic.

Suppose $p = 2$ and G/G_0 is nonabelian. Then for each $g \in G - G_0$, $G/(\langle g \rangle \cap G_0)$ is Hamiltonian (i.e., Dedekindian nonabelian). Let Q/G_0 be a subgroup of G/G_0 which is isomorphic to Q_8 and let R/G_0 be a unique subgroup of order 2 in Q/G_0. Then for each $x \in Q - R$, $x^2 \in R - G_0$. Let $a, b \in Q - R$ be such that $\langle a, b \rangle$ covers $Q/R \cong \mathrm{E}_4$. Note that $\langle a \rangle \trianglelefteq G$, $\langle b \rangle \trianglelefteq G$ and since $\langle a \rangle \cap G_0 \neq \{1\}$ and $\langle b \rangle \cap G_0 \neq \{1\}$, we get $\mathrm{o}(a) = 2^s$, $s \geq 3$, and $\mathrm{o}(b) \geq 2^3$. Because

$$[a, b] \in R - G_0 \text{ and } [a, b] \in \langle a \rangle \cap \langle b \rangle,$$

we have

$$\langle a \rangle \cap \langle b \rangle = \langle a^2 \rangle = \langle b^2 \rangle = \langle [a, b] \rangle.$$

But then $C = \langle a, b\rangle$ is a 2-group of maximal class and order 2^{s+1}, $s \geq 3$, and in this case $\langle a\rangle$ is a unique cyclic subgroup of order 2^s in C, contrary to the fact that $\mathrm{o}(b) = 2^s$. We have proved that in case $p = 2$, G/G_0 must be abelian and so $G' \leq G_0$.

Suppose that G' is noncyclic. By the above, $p = 2$ and for each $g \in G - G_0$, $\{1\} \neq \langle g\rangle \cap G_0 \trianglelefteq G$, where $G/(\langle g\rangle \cap G_0)$ is Hamiltonian (=nonabelian Dedekindian). Set $D = \langle g\rangle \cap G_0$ and $R/D = (G/D)' \cong \mathrm{C}_2$, where $R = G'D$. We know that $G' \leq G_0$ (since G/G_0 is abelian) and so $R \leq G_0$ and G/R is elementary abelian. In particular, $G/G_0 \neq \{1\}$ is elementary abelian and $\langle g^2\rangle = D$. Note that all quaternion subgroups in a Hamiltonian 2-group X generate X. Hence there is a quaternion subgroup $K/D \cong \mathrm{Q}_8$ in the Hamiltonian group G/D such that $K \not\leq G_0$. We have $K > R$ and $K/R \cong \mathrm{E}_4$ so that for each $x \in K - R$, $x^2 \in R - D$. We may choose some elements $a, b \in K - G_0$ such that $Q = \langle a, b\rangle$ covers K/R and so Q also covers K/D. Note that $\langle a\rangle \trianglelefteq G$, $\langle b\rangle \trianglelefteq G$ and $[a, b] \in R - D$. Also,

$$[a, b] \in \langle a\rangle \cap \langle b\rangle \text{ and so } \langle [a, b]\rangle = \langle a^2\rangle = \langle b^2\rangle = \langle a\rangle \cap \langle b\rangle.$$

This gives $|Q : Q'| = 4$ and so (by a well known result of O. Taussky) Q is a 2-group of maximal class with two distinct cyclic subgroups $\langle a\rangle$ and $\langle b\rangle$ of index 2. By inspection of 2-groups of maximal class (and noting that G is of class 2), we get $\mathrm{o}(a) = \mathrm{o}(b) = 4$ and $Q \cong \mathrm{Q}_8$ with $Q' = \langle a^2\rangle = \langle b^2\rangle$. Hence $K = Q \times D$ since $Q \trianglelefteq G$ and Q covers $K/D \cong \mathrm{Q}_8$. Also, $\langle g\rangle \trianglelefteq G$ and $Q \cap \langle g\rangle = \{1\}$ and so Q centralizes $\langle g\rangle$. The factor-group $G/\langle a^2\rangle$ is Hamiltonian and so

$$\mathrm{o}(g) = 4,\ D = \langle g^2\rangle \cong \mathrm{C}_2 \text{ and } G' = \langle a^2, g^2\rangle \cong \mathrm{E}_4$$

since G' covers $\langle a^2, g^2\rangle/\langle a^2\rangle$ and G' is noncyclic. For each $x \in G$,

$$x^4 \in \langle a^2\rangle \cap \langle g^2\rangle = \{1\} \text{ and so } \exp(G) = 4.$$

Let $K_1/\langle a^2\rangle \cong \mathrm{Q}_8$ with $K_1 \not\leq G_0$. Then choose $a_1, b_1 \in K_1 - G_0$ such that $\langle a_1, b_1\rangle$ covers $K_1/\langle a^2\rangle$. We get

$$Q_1 = \langle a_1, b_1\rangle \cong \mathrm{Q}_8 \text{ with } Q \cap Q_1 = \{1\} \text{ and } Q_1' = \langle a_1^2\rangle = \langle b_1^2\rangle,$$

$$\text{so } \langle Q, Q_1\rangle = Q \times Q_1.$$

Set $a^2 = t$, $a_1^2 = t_1$ and let $x \in Q - \langle t\rangle$, $x_1 \in Q_1 - \langle t_1\rangle$ so that xx_1 is one of 36 elements of order 4 with $(xx_1)^2 = x^2x_1^2 = tt_1$. We claim that $\langle xx_1\rangle$ is not normal in $Q \times Q_1$ and so $xx_1 \in G_0$. Indeed, let $y \in Q - \langle x\rangle$ so that

$$(xx_1)^y = x^{-1}x_1 = (xx_1)t,\ \text{where } (xx_1)t \notin \langle xx_1\rangle.$$

But all these 36 elements of order 4 generate $Q \times Q_1$ (of order 64) and so $Q \times Q_1 \leq G_0$, a contradiction. We have proved that also in case $p = 2$, G' is cyclic.

In the following five paragraphs we assume that G/G_0 is noncyclic. By the above, $p = 2$ and G/G_0 is abelian.

Assume that there are $a_1, a_2 \in G - G_0$ such that $\langle a_1 \rangle \cap \langle a_2 \rangle = \{1\}$. We know that $G/(\langle a_1 \rangle \cap G_0)$ and $G/(\langle a_2 \rangle \cap G_0)$ are Dedekindian and $[a_1, a_2] \in \langle a_1 \rangle \cap \langle a_2 \rangle = \{1\}$ and so $\langle a_1, a_2 \rangle$ is abelian. If both $G/(\langle a_1 \rangle \cap G_0)$ and $G/(\langle a_2 \rangle \cap G_0)$ are abelian, then

$$G' \leq (\langle a_1 \rangle \cap G_0) \cap (\langle a_2 \rangle \cap G_0) = \{1\},$$

a contradiction. Assume for a moment that both $G/(\langle a_1 \rangle \cap G_0)$ and $G/(\langle a_2 \rangle \cap G_0)$ are Hamiltonian. Then for each $x \in G$,

$$x^4 \in (\langle a_1 \rangle \cap G_0) \cap (\langle a_2 \rangle \cap G_0) = \{1\} \text{ and so } \exp(G) = 4.$$

In particular,

$$\mathrm{o}(a_1) = \mathrm{o}(a_2) = 4, \ \langle a_1^2, a_2^2 \rangle \cong \mathrm{E}_4 \text{ with } \langle a_1^2, a_2^2 \rangle \leq \mathrm{Z}(G).$$

We have

$$G' \leq \langle a_1^2, a_2^2 \rangle, \ G' \text{ covers } \langle a_1^2, a_2^2 \rangle / \langle a_1^2 \rangle \text{ and } \langle a_1^2, a_2^2 \rangle / \langle a_2^2 \rangle \text{ and } G' \text{ is cyclic}$$

and so $G' = \langle a_1^2 a_2^2 \rangle$. For each

$$x \in G, \quad [a_2, x] \in \langle a_2 \rangle \cap G' = \{1\} \text{ and so } a_2 \leq \mathrm{Z}(G).$$

But then in the Hamiltonian 2-group $G/\langle a_1^2 \rangle$ the element $(\langle a_2 \rangle \langle a_1^2 \rangle)/\langle a_1^2 \rangle \cong \mathrm{C}_4$ of order 4 lies in its center, a contradiction. We have proved that if $a_1, a_2 \in G - G_0$ are such that $\langle a_1 \rangle \cap \langle a_2 \rangle = \{1\}$, then one of $G/(\langle a_1 \rangle \cap G_0)$ and $G/(\langle a_2 \rangle \cap G_0)$ is abelian and the other one is Hamiltonian.

Assume in addition that $(G_0 \langle a_1, a_2 \rangle)/G_0$ is noncyclic. Set $\Omega_1(\langle a_1 \rangle) = \langle t_1 \rangle$ and $\Omega_1(\langle a_2 \rangle) = \langle t_2 \rangle$ so that $\langle t_1, t_2 \rangle \cong \mathrm{E}_4$ and $\langle t_1, t_2 \rangle \leq \mathrm{Z}(G)$. Without loss of generality we may suppose that $G/(\langle a_1 \rangle \cap G_0)$ is abelian and $G/(\langle a_2 \rangle \cap G_0)$ is Hamiltonian. Since G/G_0 is elementary abelian, we get

$$\mathrm{o}(a_1) = 4, \ G' = \langle a_1^2 \rangle \cong \mathrm{C}_2 \text{ and } 1 \neq a_2^2 \in G_0.$$

It follows that $(G_0 \langle a_1, a_2 \rangle)/G_0 \cong \mathrm{E}_4$. Let a_2' be an element of order 4 in $\langle a_2 \rangle$ so that

$$(a_1 a_2')^2 = a_1^2 (a_2')^2 = t_1 t_2 \text{ and } a_1 a_2' \in G - G_0.$$

But then $\langle a_1 \rangle$, $\langle a_2 \rangle$, $\langle a_1 a_2' \rangle$ are three cyclic subgroups in G which are not contained in G_0 and they have pairwise a trivial intersection. By the previous paragraph, this is not possible. We have proved that whenever $a_1, a_2 \in G - G_0$ are such that $(\langle a_1, a_2 \rangle G_0)/G_0$ is noncyclic, then $\langle a_1 \rangle \cap \langle a_2 \rangle \neq \{1\}$.

Let E/G_0 be a four-subgroup in the noncyclic abelian group G/G_0. Amongst all elements in $E - G_0$ choose an element a of the smallest possible order 2^s. We have $s \geq 2$ since $a^2 \neq 1$. Set $F = G_0 \langle a \rangle$ and let b be any element in $E - F$ so that $\mathrm{o}(b) \geq 2^s$. By the above, $D = \langle a \rangle \cap \langle b \rangle \neq \{1\}$. Let b' be an element of order 2^s in $\langle b \rangle$ such that

$$a^{2^n} = (b')^{-2^n}, \text{ where } |\langle a \rangle : D| = |\langle b' \rangle : D| = 2^n, \ n \geq 1,$$
$$\text{and } D = \langle a^{2^n} \rangle = \langle (b')^{2^n} \rangle.$$

We compute

$$(ab')^{2^n} = a^{2^n}(b')^{2^n}[b',a]^{\binom{2^n}{2}} = [b',a]^{2^{n-1}(2^n-1)},$$
$$\text{where } ab' \in E - G_0 \text{ and } [b',a] \in D.$$

Since a was an element of the smallest possible order in the set of all elements in $E - G_0$, we get

$$n = 1,\ a^2 \in D,\ \text{and } \langle [b',a]\rangle = D \neq \{1\}.$$

On the other hand,

$$[b,a]^2 = [b,a^2] = 1 \text{ and so } [b^2,a] = [b,a]^2 = 1.$$

Hence, if $b' \in \langle b^2\rangle$ (in case $\mathrm{o}(b) > \mathrm{o}(a) = 2^s$), we get $[b',a] = 1$ and so $D = \{1\}$, a contradiction. It follows that

$$\mathrm{o}(a) = \mathrm{o}(b) = 2^s \text{ and } \langle [b,a]\rangle = D \cong \mathrm{C}_2. \text{ where } D = \langle a^2\rangle = \langle b^2\rangle.$$

Hence

$$s = 2,\ \mathrm{o}(a) = \mathrm{o}(b) = 4,\ \text{and } Q = \langle a,b\rangle \cong \mathrm{Q}_8.$$

We have proved that all elements in $E-F$ are of order 4 and each such element has the same square a^2. We know that G' is cyclic, $G' \leq G_0$, $G' \leq \mathrm{Z}(G)$ and $G' \geq \langle a^2\rangle = \langle a,b\rangle'$. Suppose that $G' > \langle a^2\rangle$ and let $x \in G' - \langle a^2\rangle$ be such that $x^2 = a^2$, where $[x,a] = 1$. But then xa is an involution in $E - G_0$, a contradiction. Hence $G' = \langle a^2\rangle \cong \mathrm{C}_2$. Since all elements in $E - F$ are of order 4 and they generate E and $E' = \langle a^2\rangle \cong \mathrm{C}_2$, we get $\exp(G) = 4$. In particular, all elements in $F - G_0$ are of order 4 and let $y \in F - G_0$. Then y is also of the smallest possible order 4 in $E - G_0$. By repeating the above argument with the element y (instead of a),we get that for each $b \in E - F$, $b^2 = y^2$ and so $y^2 = a^2$. We have proved that for each $x \in E - G_0$, $x^2 = a^2$. For any $x,y \in G$,

$$[x^2,y] = [x,y]^2 = 1 \text{ since } G' = \langle a^2\rangle \cong \mathrm{C}_2.$$

Hence $\mho_1(G) \leq \mathrm{Z}(G)$.

Let c be an element of order 4 in G_0. Then

$$ac \in E - G_0 \text{ and so } a^2 = (ac)^2 = a^2c^2[c,a]$$
$$\text{implying } c^2 = [a,c] \in \langle a^2\rangle \text{ and } c^2 = a^2.$$

But then $\langle c\rangle \trianglelefteq G$ and so there is $b \in E - G_0$ which centralizes $\langle c\rangle$. It follows that bc is an involution in $E - G_0$, a contradiction. We have proved that G_0 is elementary abelian. If $G_0 \not\leq \mathrm{Z}(E)$, then there are $t \in G_0 - \langle a^2\rangle$ and $x \in E - G_0$ such that $[t,x] = a^2 = x^2$. But then $\langle t,x\rangle \cong \mathrm{D}_8$ and so there are involutions in $\langle t,x\rangle - G_0$, a contradiction. We have proved that E is Hamiltonian and so $E \neq G$ because G is not Dedekindian.

Let $v \in G - E$ be such that $v^2 \in E$. Since $\mho_1(G) \leq \mathrm{Z}(G)$, we get $1 \neq v^2 \in \mathrm{Z}(E) = G_0$. Then, by the above, $\langle v\rangle \cap \langle a\rangle \neq \{1\}$ and so $v^2 = a^2$.

Let $a, b \in E - G_0$ be such that $\langle a, b\rangle$ covers E/G_0. Because there are no involutions in $G - G_0$, we have

$$[v, a] = [v, b] = [a, b] = a^2 \text{ and } [v, ab] = [v, a] = [v, b] = a^2a^2 = a^4 = 1.$$

But then $(ab)^2 = v^2 = a^2$ implies that $(ab)v$ is an involution in $G - G_0$, a final contradiction. We have proved that also in case $p = 2$, G/G_0 is cyclic.

Suppose that $p = 2$ and there is $g \in G - G_0$ such that $G/(\langle g\rangle \cap G_0)$ is Hamiltonian. We set $D = \langle g\rangle \cap G_0 \neq \{1\}$ and note that $G' \leq G_0$ implies that G/G_0 is elementary abelian. But G/G_0 is also cyclic and so $|G : G_0| = 2$. We get $g^2 \in G_0$ and so $D = \langle g^2\rangle \neq \{1\}$. Since the Hamiltonian group G/D is generated by its quaternion subgroups, there is a quaternion subgroup K/D in G/D such that $K \not\leq G_0$. Let $a, b \in K - G_0$ be such that $Q = \langle a, b\rangle$ covers K/D, where $ab \in G_0$. Let R/D be a unique subgroup of order 2 in K/D so that $R \leq G_0$ and G' covers R/D. We have

$$a^2 \in R - D,\ b^2 \in R - D,\ (ab)^2 \in R - D, \text{ and } [a, b] \in R - D.$$

On the other hand,

$$[a, b] \in \langle a\rangle \cap \langle b\rangle \text{ and so } \langle a\rangle \cap \langle b\rangle = \langle a^2\rangle = \langle b^2\rangle = \langle [a, b]\rangle.$$

Since $Q/Q' \cong \mathrm{E}_4$, Q is of maximal class (by O. Taussky) and since Q has two distinct cyclic subgroups $\langle a\rangle$ and $\langle b\rangle$ of index 2, we get

$$Q \cong \mathrm{Q}_8,\ \mathrm{o}(a) = \mathrm{o}(b) = 4,\ \langle [a, b]\rangle \cong \mathrm{C}_2,\ Q \cap \langle g^2\rangle = \{1\}$$
$$\text{and so } \langle Q, \langle g\rangle\rangle = Q \times \langle g\rangle.$$

Also,

$$G' \leq R \text{ and } G' \geq \langle [a, b]\rangle \cong \mathrm{C}_2,$$

and so the fact that G' is cyclic implies $G' \cap \langle g^2\rangle = \{1\}$. It follows

$$G' = \langle [a, b]\rangle = \langle a^2\rangle \cong \mathrm{C}_2$$

and for any $x, y \in G$,

$$[x^2, y] = [x, y]^2 = 1 \text{ implying } \mho_1 G) \leq \mathrm{Z}(G).$$

Since $G' \cap \langle g\rangle = \{1\}$, we have $\langle g\rangle \leq \mathrm{Z}(G)$ and so $G = G_0 * \langle g\rangle$ gives that G_0 is nonabelian. We have $ab \in G_0$ and so $abg \notin G_0$ which implies $\langle abg\rangle \trianglelefteq G$. We compute

$$(abg)^2 = (ab)^2g^2 = a^2g^2.$$

If $g^4 \neq 1$, then

$$(abg)^4 = g^4 \neq 1 \text{ and so } G' = \langle a^2\rangle \not\leq \langle abg\rangle.$$

If $g^4 = 1$,then a^2g^2 is an involution distinct from a^2 and so again $G' \not\leq \langle abg\rangle$. It follows that in any case $G' \not\leq \langle abg\rangle$ and so $\langle abg\rangle \leq \mathrm{Z}(G)$. But then

$$ab \in \mathrm{Z}(G) \text{ giving } \mathrm{C}_4 \cong \langle ab\rangle \leq \mathrm{Z}(Q),$$

a contradiction. We have proved that for each $g \in G - G_0$, $G/(\langle g\rangle \cap G_0)$ is abelian. Our theorem is proved. ∎

References

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Received: 8.10.2015.