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THE LANGLANDS QUOTIENT THEOREM FOR FINITE CENTRAL EXTENSIONS OF p-ADIC GROUPS II: INTERTWINING OPERATORS AND DUALITY

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Abstract. In this paper, we prove that the Langlands quotient may be realized as the image of a standard intertwining operator in the context of finite central extensions of connected, reductive p-adic groups. We also verify that the duality of Aubert and Schneider-Stuhler holds in this context.

1. Introduction

This paper is a continuation of [3].

The Langlands quotient theorem is proved in [3] for finite central extensions of connected reductive p-adic groups. However, the proof is algebraic in nature and does not bring standard intertwining operators into the picture. As the connection with standard intertwining operators is important in some contexts, we address that here. In particular, we show that the Langlands quotient may be realized as the image of a standard intertwining operator. To do so, we use the results of [10] on standard intertwining operators for finite central extensions and properties of the Langlands quotient for such groups established in [3].

Although not directly related to the Langlands classification, another useful tool in representation theory–and which is often used in conjunction with the Langlands classification–is the duality of Aubert and Schneider, Stuhler (see [1,12]; also [4]). We include a sketch of the proof here, but note that it is essentially the same as that of [1]. We also take the opportunity to remove an

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unneeded hypothesis from [3]–namely, the assumption that the characteristic of the underlying field is zero (see Remark 2.1).

2. Notation and preliminaries

In this section, we briefly review background material and introduce notation needed in the remainder of the paper. The reader is referred to [3] for more details.

Let F be a nonarchimedean local field. Note that we make no assumption on the characteristic of F (see Remark 2.1 below).

Let G be the group of F-points of a connected reductive group defined over F. We call $(\tilde{G}, \rho)$ a finite central extension of G if the following hold:

1. $\rho : \tilde{G} \longrightarrow G$ is a surjective homomorphism of topological groups.
2. $C = \ker(\rho)$ is a finite subgroup of $Z(\tilde{G})$, where $Z(H)$ denotes the center of H.
3. ρ is a topological covering (as described in [11]). In particular, there is an open neighborhood O of the identity in G and a homeomorphism $j : \rho^{-1}(O) \longrightarrow O \times C$ such that $pr_1 \circ j = \rho$ on $\rho^{-1}(O)$.

We introduce some terminology for finite central extensions. If $(\tilde{G}, \rho)$ is a finite central extension of G, a section of ρ is a continuous map $\mu : G \to \tilde{G}$ such that $\rho \circ \mu = id_G$. A lifting of a subgroup H of G is a continuous homomorphism $s : H \to \tilde{G}$ such that $\rho \circ s = id_H$. Obviously, if G lifts to $\tilde{G}$–in other words, if the sequence $1 \longrightarrow C \longrightarrow \tilde{G} \xrightarrow{\rho} G \longrightarrow 1$ splits–then $\tilde{G} \cong G \times C$. As in [3], we retain the following convention: if H be a subgroup of G, the preimage of H in $\tilde{G}$ will be denoted by $\tilde{H}$ and a lifting of H (if it exists) will be denoted by $\hat{H}$. Hence, $\tilde{H} = \rho^{-1}(H)$ and $\hat{H} \cong H$.

The group $\tilde{G}$ has a neighborhood basis of the identity consisting of compact open subgroups, and we can define smooth and admissible representations in the standard way. We denote by $R(\tilde{G})$ the Grothendieck group of the category of smooth finite length representations of $\tilde{G}$. Let π_1 and π_2 be smooth finite length representations of $\tilde{G}$. If π_1 and π_2 have the same irreducible components appearing with the same multiplicities, we write $\pi_1 = \pi_2$. If π_1 and π_2 are actually equivalent as representations, we write $\pi_1 \cong \pi_2$.

Fix a maximal split torus A in G. We denote by $W = W(G, A)$ the Weyl group of G with respect to A. Let $\Phi = \Phi(G, A)$ be the set of roots. Fix a minimal parabolic subgroup $P_\emptyset$ containing A. The choice of $P_\emptyset$ determines the set of simple roots S and the set of positive roots $\Phi^+ \subset \Phi$. For $I \subset S$, we denote by P_I the standard parabolic subgroup of G determined by I and by L_I the standard Levi subgroup of P_I.

Let $P = MU$ be a parabolic subgroup of G. We call

$$\tilde{P} = \rho^{-1}(P)$$

a parabolic subgroup of $\tilde{G}$. Let $\tilde{M} = \rho^{-1}(M)$ and $\hat{U}$ the canonical lifting of U to $\tilde{G}$ described in the first appendix to [11]. Then

$$\tilde{P} = \tilde{M}\hat{U}$$

serves as the Levi factorization. Set $\tilde{A}_M = \rho^{-1}(A_M)$, where A_M is the split component of the center of M. The normalized parabolic induction $\mathrm{Ind}_{\tilde{P}}^{\tilde{G}}$ and the Jacquet functor $r_{\tilde{P}}^{\tilde{G}}$ are defined as usual. In the case when $\tilde{P} = \tilde{M}\hat{U}$ is a standard parabolic subgroup, we also denote these two functors by $i_{\tilde{M}}^{\tilde{G}}$ and $r_{\tilde{M}}^{\tilde{G}}$, respectively. We use the same notation for the corresponding homomorphisms $i_{\tilde{M}}^{\tilde{G}} : R(\tilde{M}) \to R(\tilde{G})$ and $r_{\tilde{M}}^{\tilde{G}} : R(\tilde{G}) \to R(\tilde{M})$ on the Grothendieck groups.

Remark 2.1. In [3], it was assumed that the characteristic of F was zero. This is not needed, however, for reasons detailed below.

Recall that G acts on $\tilde{G}$ by conjugation: $g \in G$ acts on $\tilde{x} \in \tilde{G}$ by

$$\tilde{x} \mapsto \tilde{g}\tilde{x}\tilde{g}^{-1},$$

where $\tilde{g}$ is any element of $\rho^{-1}(g)$. Let $P = MU$ be a parabolic subgroup of G. Let $s_U : U \to \tilde{G}$ be the canonical lifting described in [11, Appendix I]. Then s_U is the unique P-equivariant homomorphism of U into $\tilde{G}$. Take $a \in A_M$ and let $\tilde{a}$ be any element of $\rho^{-1}(a)$. Define $s_{\tilde{a}} : U \to \tilde{G}$ by

$$s_{\tilde{a}}(x) = \tilde{a}^{-1} s_U(axa^{-1})\tilde{a}.$$

Since s_U is P-equivariant, we have $s_{\tilde{a}} = s_U$.

The assumption of characteristic zero was made to that ensure the lifting from U to $\tilde{G}$ was unique. This was used in several proofs to conclude $s_{\tilde{a}} = s_U$. However, as observed above, one can directly obtain $s_{\tilde{a}} = s_U$, which is what is really needed. Thus the assumption of characteristic zero may be removed.

3. Standard intertwining operators and the Langlands classification

In this section, we show that the Langlands quotient may be realized as the image of a standard intertwining operator.

For $P = MU$ a standard parabolic subgroup of G, we let $X(M)$ denote the set of rational characters of M. Let q be the number of elements in the residue field of F. There is a homomorphism (cf. [8]) $H_M : M \to \mathfrak{a}_M = \mathrm{Hom}(X(M), \mathbb{R})$ such that $q^{\langle \nu, H_M(m)\rangle} = |\nu(m)|$ for all $m \in M$, $\nu \in X(M)$. Given $\nu \in \mathfrak{a}_M^*$, let us write

$$\exp \nu = q^{\langle \nu, H_M(\cdot)\rangle}$$

for the corresponding character of M. As in [3, Note 2.4], there is then an associated unramified character of $\tilde{M}$; for clarity, we denote this character $\widetilde{\exp}\,\nu$.

We now recall the Langlands classification for $\tilde{G}$ in the quotient setting (see [3, Remark 4.2]). A set of Langlands data for $\tilde{G}$ is a triple $(\tilde{P}, \nu, \tau)$ with the following properties:

(1) $\tilde{P} = \tilde{M}\hat{U}$ is a standard parabolic subgroup of $\tilde{G}$,
(2) $\nu \in (\mathfrak{a}_{\tilde{M}})^*_+ = \{x \in \mathfrak{a}^*_M \mid \langle x, \alpha \rangle > 0, \text{ for all } \alpha \in S(P, A_M)\}$, and
(3) τ is (the equivalence class of) an irreducible tempered representation of $\tilde{M}$,

where $S(P, A_M)$ denotes the set of simple roots for the pair (P, A_M).

We now state the Langlands classification in the quotient setting.

Theorem 3.1 (The Langlands classification). *Suppose $(\tilde{P}, \nu, \tau)$ is a set of Langlands data for $\tilde{G}$. Then the induced representation $i^{\tilde{G}}_{M}(\widetilde{\exp}\,\nu \otimes \tau)$ has a unique irreducible quotient, which we denote by $L(\tilde{P}, \nu, \tau)$. Conversely, if π is an irreducible admissible representation of $\tilde{G}$, then there exists a unique $(\tilde{P}, \nu, \tau)$ as above such that $\pi \cong L(\tilde{P}, \nu, \tau)$.*

Let $\tilde{P} = \tilde{M}\hat{U}$ and $\tilde{P}' = \tilde{M}\hat{U}'$ be two parabolic subgroups of $\tilde{G}$ with the same Levi factor $\tilde{M}$. Let (π, V) be an admissible representation of $\tilde{M}$. For $f \in V_{\mathrm{Ind}^{\tilde{G}}_{\tilde{P}}\pi}$ and $\tilde{x} \in \tilde{G}$, we formally define

$$(J_{\tilde{P}'|\tilde{P}}(\pi)f)(\tilde{x}) = \int_{(\hat{U}\cap\hat{U}')\backslash\hat{U}'} f(u'\tilde{x})du'.$$

If the integral converges absolutely for all $f \in V_{\mathrm{Ind}^{\tilde{G}}_{\tilde{P}}\pi}$ and $\tilde{x} \in \tilde{G}$, then $J_{\tilde{P}'|\tilde{P}}(\pi)$ defines an intertwining operator $\mathrm{Ind}^{\tilde{G}}_{\tilde{P}}\pi \to \mathrm{Ind}^{\tilde{G}}_{\tilde{P}'}\pi$.

Let $\tilde{P} = \tilde{M}\hat{U}$ be a standard parabolic subgroup of $\tilde{G}$. Let τ be an irreducible tempered representation of $\tilde{M}$ and

$$\nu \in (\mathfrak{a}_{\tilde{M}})^*_+.$$

We consider the representation

$$\Pi = \mathrm{Ind}^{\tilde{G}}_{\tilde{P}} \left(\widetilde{\exp}\,\nu \otimes \tau\right).$$

Then the contragredient $\tilde{\Pi}$ satisfies

$$\tilde{\Pi} \cong \mathrm{Ind}^{\tilde{G}}_{\tilde{P}} \left(\widetilde{\exp}\,(-\nu) \otimes \tilde{\tau}\right).$$

The triple $(\tilde{P}, -\nu, \tilde{\tau})$ is a set of Langlands data (in the subrepresentation version of Langlands' classification). It follows that $\tilde{\Pi}$ has a unique irreducible subrepresentation, and consequently, Π has a unique irreducible quotient. Denote this quotient by π.

Recall that two parabolic subgroups of G are called opposite if their intersection is a Levi subgroup of each of them. Let P^- denote the opposite parabolic subgroup of P and $\bar{P}$ be the unique standard parabolic subgroup conjugate to P^-. The integral defining

$$J_{\tilde{P}^-|\tilde{P}}(\widetilde{\exp}\,\nu \otimes \tau)$$

is absolutely convergent ([10, Théorème 2.4.1]), so it defines an intertwining operator

$$\Pi \to \Pi' = \mathrm{Ind}_{\tilde{P}^-}^{\tilde{G}}\ (\widetilde{\exp}\,\nu \otimes \tau)\,.$$

We claim that the image of $J_{\tilde{P}^-|\tilde{P}}(\widetilde{\exp}\,\nu \otimes \tau)$ is irreducible. Let w_ℓ denote the longest element in the Weyl group of G (or a representative thereof). Since $w_\ell(\Pi') \cong \Pi'$, we get an intertwining operator

$$\mathcal{J} : \Pi \to \Pi'' = \mathrm{Ind}_{\tilde{\bar{P}}}^{\tilde{G}}\ (\widetilde{\exp}\,w_\ell(\nu) \otimes w_\ell(\tau))\,.$$

Again, we have a set of Langlands data $(\tilde{\bar{P}}, w_\ell(\nu), w_\ell(\tau))$. It follows that Π'' has a unique irreducible subrepresentation; denote it by π''. Let $\pi' \cong \pi''$ be the corresponding subquotient in Π'. We claim $\pi \cong \pi'$.

From [3, Remark 4.5], we know

$$\widetilde{\exp}\,(-\nu) \otimes \tilde{\tau} \le r_{\tilde{P}}^{\tilde{G}}(\tilde{\pi})$$

and $\tilde{\pi}$ is the only irreducible subquotient of $\tilde{\Pi}$ having $\widetilde{\exp}\,(-\nu) \otimes \tilde{\tau}$ in its Jacquet module with respect to $\tilde{P}$. Then, using [7] and [3],

$$\widetilde{\exp}\,\nu \otimes \tau \le r_{\tilde{P}^-}^{\tilde{G}}(\pi).$$

On the other hand,

$$\widetilde{\exp}\,w_\ell(\nu) \otimes w_\ell(\tau) \le r_{\tilde{\bar{P}}}^{\tilde{G}}(\pi'')$$

and again we have uniqueness. Conjugation with w_ℓ gives

$$\widetilde{\exp}\,\nu \otimes \tau \le r_{\tilde{P}^-}^{\tilde{G}}(\pi').$$

Now, uniqueness implies $\pi \cong \pi'$. Finally, simple properties of intertwining operators tell us that π' is the image of $J_{\tilde{P}^-|\tilde{P}}(\widetilde{\exp}\,\nu \otimes \tau)$.

We summarize:

THEOREM 3.2. *The Langlands subrepresentation $L(P, \nu, \tau)$ may be realized as the image of the standard intertwining operator $J_{\tilde{P}^-|\tilde{P}}(\widetilde{\exp}\,\nu \otimes \tau)$.*

REMARK 3.3. As in [9, Lemma 1.1], the above discussion may be used to relate the data for the subrepresentation version of Langlands' classification with that for the quotient version, as well as relating the data for $\tilde{\pi}$ with that for π.

4. Duality

In this section, we define the duality operator $D_{\tilde{G}}$ and study its properties.

Following [1], for $I, J \subset S$, define

$$\mathcal{D}(I,J) = \{w \in W \mid w^{-1} \cdot I \subset \Phi^+,\, w \cdot J \subset \Phi^+\}.$$

We denote by w_J the longest element of $\mathcal{D}(J, \emptyset)$.

The following covers the basic properties (1.1)–(1.4) of [1], which are the key properties needed to prove the basic properties of duality:

Proposition 4.1. *Let $\tilde{M}, \tilde{L}$ be the standard Levi factors of $\tilde{G}$ corresponding to $I, J \subset S$.*

(1) *If $\tilde{L} < \tilde{M}$, then $i_{\tilde{M}}^{\tilde{G}} \circ i_{\tilde{L}}^{\tilde{M}} = i_{\tilde{L}}^{\tilde{G}}$ and $r_{\tilde{L}}^{\tilde{M}} \circ r_{\tilde{M}}^{\tilde{G}} = r_{\tilde{L}}^{\tilde{G}}$.*

(2) *We have*

$$r_{\tilde{M}}^{\tilde{G}} \circ i_{\tilde{L}}^{\tilde{G}} = \sum_{w \in \mathcal{D}(I,J)} i_{\tilde{M}'}^{\tilde{M}} \circ w \circ r_{\tilde{L}'}^{\tilde{L}},$$

where $\tilde{L}' = \tilde{L} \cap w^{-1}(\tilde{M})$ and $\tilde{M}' = \tilde{M} \cap w(\tilde{L})$.

(3) *If $\tilde{M} = w\tilde{L}w^{-1}$ for an element $w \in W$, then*

$$i_{\tilde{M}}^{\tilde{G}} \circ w = i_{\tilde{L}}^{\tilde{G}}.$$

(4) *If $\tilde{M} = {w_J}^{-1}\tilde{L}w_J$, then*

$$\tilde{\ } \circ r_{\tilde{L}}^{\tilde{G}} = w_J \circ r_{\tilde{M}}^{\tilde{G}} \circ \tilde{\ },$$

where $\tilde{\ }$ denotes contragredient.

Proof. The first of these is simply [6, Proposition 1.9], which holds in the generality needed here. The second is done in [3, Proposition 3.3] and is essentially a corollary of [6, Theorem 5.2].

The proof of (3) in [5] relies on three results: the linear independence of characters, the Langlands classification, and the geometric lemma of [6] (or [7, Theorem 6.5]). The linear independence of characters is general, and holds for the groups we are considering (cf. [13, Lemma 1.13.1]); the Langlands classification for the groups under consideration is done in [3]. The geometric lemma is just (2) above. With these observations, the proof from [5] extends to cover the groups under consideration.

(4) In [3], we explained how parts of Casselman's work [7] for G can be applied to $\tilde{G}$. More specifically, in [3, Section 2] we proved the structure results for $\tilde{G}$ which are a basis for Casselman's proof in sections 4.1 and 4.2 of [7]. Then [7, Corollary 4.2.5] holds for $\tilde{G}$. Conjugation by w_J gives (4). $\square$

We define the duality operator $D_{\tilde{G}}$ on the Grothendieck group $R(\tilde{G})$ as in [1, 12]:

$$D_{\tilde{G}} = \sum_{I \subset S} (-1)^{|I|} i_{\tilde{L}_I}^{\tilde{G}} \circ r_{\tilde{L}_I}^{\tilde{G}}.$$

The following is [1, Théorème 1.7]:

THEOREM 4.2. *The duality operator $D_{\tilde{G}}$ has the following properties:*

(1) $D_{\tilde{G}} \circ \tilde{} = \tilde{} \circ D_{\tilde{G}}$,

(2) *for $J \subset S$, one has*

$$D_{\tilde{G}} \circ i^{\tilde{G}}_{\tilde{L}_J} = i^{\tilde{G}}_{\tilde{L}_J} \circ D_{\tilde{L}_J}, \qquad r^{\tilde{G}}_{\tilde{L}_J} \circ D_{\tilde{G}} = Ad(w_J) \circ D_{\tilde{L}_{J'}} \circ r^{\tilde{G}}_{\tilde{L}_{J'}},$$

where w_J is as above and $J' = w_J^{-1}(J)$,

(3) $D^2_{\tilde{G}} = id$ *(i.e., $D_{\tilde{G}}$ is an involution),*

(4) *if π is supercuspidal, $D_{\tilde{G}}(\pi) = (-1)^{|S|}\pi$.*

PROOF. (1) follows immediately from Proposition 4.1 (4) and the definition of D_G. Both parts of (2) follows from Proposition 4.1 (1)–(3) via the same calculations as in [1]. Note that the result of Solomon used for the first part of (2) is essentially a combinatorial identity on the Weyl group, so applies equally well to finite central extensions. (3) follows from (2) and induction/restriction in stages (Proposition 4.1 (1)) via the same calculation as in [1]. (4) is immediate from the definition. ∎

It remains to verify that irreducibility is preserved (up to $\pm$) by duality. The proof is essentially a sketch of that from [1]. We start by reviewing the notation from [1].

Let E be a $\tilde{G}$-module. For $I \subset S$, we denote by $E(U_I)$ the subspace of E spanned by the elements $ux - x$, where $x \in E$ and $u \in \tilde{U}_I$. Set $E_{U_I} = E/E(U_I) = r^{\tilde{G}}_{\tilde{L}_I}(E)$. Define

$$E_I = (i^{\tilde{G}}_{\tilde{L}_I} \circ r^{\tilde{G}}_{\tilde{L}_I})(E).$$

Let $I \subset J \subset S$. The natural projection from E_{U_J} to E_{U_I} induces a map $\phi^J_I : E_J \to E_I$. If $J = I \cup \{\alpha\}$, we define

$$\tilde{E}_J = E_J \otimes_{\mathbb{C}} \Lambda^{|S-J|}(\mathbb{C}^{S-J}) \quad \text{and} \quad \tilde{\phi}^J_I = \phi^J_I \otimes_{\mathbb{C}} \varepsilon^J_I,$$

where $\varepsilon^J_I : \Lambda^{|S-J|}(\mathbb{C}^{S-J}) \to \Lambda^{|S-I|}(\mathbb{C}^{S-I})$ is the map given by $\omega \mapsto \omega \wedge \alpha$. Define

$$\tilde{E}_{J,I} = r^{\tilde{G}}_{\tilde{L}_I}(\tilde{E}_J).$$

THEOREM 4.3. *The duality operator $D_{\tilde{G}}$ takes irreducible representations to irreducible representations, up to sign.*

PROOF. The proof follows that of [1]. We outline the argument, indicating any changes needed for the case of finite central extensions. We remark that the principal changes are (1) the use of [6] in place of [7] (as the results of [6] are done in the generality of groups of totally disconnected type, rather than just reductive p-adic groups), and (2) a minor correction to Aubert's proof.

Let $I \subset S$. Let E be an irreducible representation of $\tilde{G}$ such that E has supercuspidal support on (the associate class of) $\tilde{L}_I$.

As in Aubert, d_k is defined as follows: If $|J| = k$, we let d_k be defined on $\tilde{E}_J$ by

$$\bigoplus_{\substack{I \subset J \\ |I|+1=|J|}} \tilde{\phi}_I^J.$$

It is a straightforward matter to show that

$$0 \longrightarrow E \xrightarrow{d_{|S|}} \bigoplus_{|J|=|S|-1} \tilde{E}_J \xrightarrow{d_{|S|-1}} \bigoplus_{|J|=|S|-2} \tilde{E}_J \xrightarrow{d_{|S|-2}} \cdots \xrightarrow{d_1} \tilde{E}_\emptyset \longrightarrow 0$$

is a complex. As in [2], the key is to show that it is in fact an exact sequence.

Suppose $w \in \mathcal{D}(J, I)$. It is a straightforward argument to check that the map

$$\begin{aligned} \left(i_{\tilde{L}_J}^{\tilde{P}_J w \tilde{P}_I} \circ r_{\tilde{L}_J}^{\tilde{G}}\right)(E) &\longrightarrow \left(i_{w^{-1}\tilde{L}_J w \cap \tilde{L}_I}^{\tilde{P}_I} \circ Ad(w^{-1}) \circ r_{\tilde{L}_J \cap w\tilde{L}_I w^{-1}}^{\tilde{G}}\right)(E) \\ f &\longmapsto \Phi_f, \end{aligned}$$

with Φ_f defined by $\Phi_f(\tilde{p}) = f(w\tilde{p})$ for all $\tilde{p} \in \tilde{P}_I$, is an isomorphism of $\tilde{P}_I$-modules. Taking the Jacquet modules, we may then obtain

(4.1)
$$\begin{aligned} \left[\left(i_{\tilde{L}_J}^{\tilde{P}_J w \tilde{P}_I} \circ r_{\tilde{L}_J}^{\tilde{G}}\right)(E)\right]_{U_I} &\cong \left[\left(i_{w^{-1}\tilde{L}_J w \cap \tilde{L}_I}^{\tilde{P}_I} \circ Ad(w^{-1}) \circ r_{\tilde{L}_J \cap w\tilde{L}_I w^{-1}}^{\tilde{G}}\right)(E)\right]_{U_I} \\ &\cong \left(i_{\tilde{L}_I \cap w^{-1}\tilde{L}_J w}^{\tilde{L}_I} \circ Ad(w^{-1}) \circ r_{w\tilde{L}_I w^{-1} \cap \tilde{L}_J}^{\tilde{L}_J}\right)(r_{\tilde{L}_J}^{\tilde{G}}(E)). \end{aligned}$$

If this is nonzero, we must have $w\tilde{L}_I w^{-1} \cap \tilde{L}_J$ conjugate to $\tilde{L}_I$ (since $E \in Alg(\{I\})$), hence $w\tilde{L}_I w^{-1} \subset \tilde{L}_J$. In this case, we may conclude that (4.1) reduces to the following:

$$\left[\left(i_{\tilde{L}_J}^{\tilde{P}_J w \tilde{P}_I} \circ r_{\tilde{L}_J}^{\tilde{G}}\right)(E)\right]_{U_I} \cong (w^{-1}) \circ r_{w\tilde{L}_I w^{-1}}^{\tilde{G}}(E)$$

as $\tilde{L}_I$-modules.

Choose $W = \theta_1 \supset \theta_2 \supset \cdots \supset \theta_{t+1} = \emptyset$ with $\theta_i - \theta_{i+1} = w_i$ so that $Y_j = \tilde{P}_\emptyset \theta_{t+1-i} \tilde{P}_\emptyset$ satisfies the hypotheses of Theorem 5.2 [6] (for $\tilde{P}_\emptyset$ double-cosets). We note that the ϕ_K^J–and hence d_{k,U_I}–respect the filtration by θ_i. As in Aubert's proof, if

$$\begin{aligned} 0 \longrightarrow E_{U_I}^{\theta_i}/E_{U_I}^{\theta_{i+1}} \longrightarrow \bigoplus_{|J|=|S|-1} \tilde{E}_{J,I}^{\theta_i}/\tilde{E}_{J,I}^{\theta_{i+1}} \longrightarrow \bigoplus_{|J|=|S|-2} \tilde{E}_{J,I}^{\theta_i}/\tilde{E}_{J,I}^{\theta_{i+1}} \longrightarrow \\ \cdots \longrightarrow \bigoplus_{|J|=|I|} \tilde{E}_{J,I}^{\theta_i}/\tilde{E}_{J,I}^{\theta_{i+1}} \end{aligned}$$

is exact for all $i = 1, \dots, t$ (where $\tilde{E}^{\theta_i}_{J,I}$ denotes the image of the elements of $\tilde{E}_{J,I}$ supported on $\tilde{P}_\emptyset \theta_i \tilde{P}_\emptyset$, then

$$0 \longrightarrow E_{U_I} \stackrel{d_{|S|,I}}{\longrightarrow} \bigoplus_{|J|=|S|-1} \tilde{E}_{J,I} \stackrel{d_{|S|-1,I}}{\longrightarrow} \bigoplus_{|J|=|S|-2} \tilde{E}_{J,I} \stackrel{d_{|S|-2,I}}{\longrightarrow} \cdots \stackrel{d_{|I|,I}}{\longrightarrow} \bigoplus_{|J|=|I|} \tilde{E}_{J,I}$$

is also exact.

We now make a change–a minor correction, actually–to Aubert's setup. We keep Θ the same as for Aubert: Θ consists of subsets $\theta \subset W$ having the property that if $w \in \theta$, then $w' \in \theta$ for any w' having $\ell(w') > \ell(w)$. Suppose I, J fixed as above. We define $\theta' \subset \theta$ to be the largest subset which is left-invariant under multiplication by W_J and right-invariant under multiplication by W_I. In particular, a function in $E^{\theta}_{J,I}$ is actually supported on $E^{\theta'}_{J,I}$.

Fix i and let $w = w_i$. Suppose $\tilde{E}^{\theta_i}_{J,I}/\tilde{E}^{\theta_{i+1}}_{J,I} \neq 0$. Then, $\theta'_i \neq \theta'_{i+1}$. We claim that $w \in \mathcal{D}(J,I)$ (or equivalently, $w^{-1} \in \mathcal{D}(I,J)$). Since $\{w\} = \theta_i - \theta_{i+1}$, we have $W_J w W_I \subset \theta'_i \subset \theta_i$ (using $\theta'_i \neq \theta_i$). Now, $W_J w W_I - \{w\} \subset \theta_{i+1}$, so everything in $W_J w W_I$ must have length at least $\ell(w)$. As $\mathcal{D}(J,I)$ consists of minimal length double-coset representatives, the claim follows. Further, we claim that $w W_I w^{-1} \subset W_J$ and

$$\tilde{E}^{\theta_i}_{J,I}/\tilde{E}^{\theta_{i+1}}_{J,I} \cong Ad(w^{-1}) \circ r^{\tilde{G}}_{w\tilde{L}_I w^{-1}}(E)$$

as $\tilde{L}_I$-modules (conversely, if $w \in \mathcal{D}(J,I)$ and $wW_Iw^{-1} \subset W_J$, then $\tilde{E}^{\theta_i}_{J,I}/\tilde{E}^{\theta_{i+1}}_{J,I} \cong w^{-1} \circ r^{\tilde{G}}_{w\tilde{M}_I w^{-1}}(E) \neq 0$).

As noted earlier, $\tilde{E}^{\theta_i}_{J,I} = \tilde{E}^{\theta'_i}_{J,I}$ and $\tilde{E}^{\theta_{i+1}}_{J,I} = \tilde{E}^{\theta'_{i+1}}_{J,I}$. In the notation of section 5 [6], $E^{\theta'}_{J,I}$ plays the role of F_Y (more precisely, F_Y applied to $r^{\tilde{G}}_{\tilde{M}_J}(E)$). Then $\tilde{E}^{\theta_i}_{J,I}/\tilde{E}^{\theta_{i+1}}_{J,I}$ appears in the role of F_Z (again, applied to $r^{\tilde{G}}_{\tilde{M}_J}(E)$). Let v be a representative of w^{-1} in $\tilde{G}$ (note that (3) on p. 460 in [6] wants $\bar{w} \in \tilde{G}$ such that $\tilde{P}_J \bar{w}^{-1} \subset Z$; our v plays this role). By Theorem 5.2 [6],

$$\tilde{E}^{\theta_i}_{J,I}/\tilde{E}^{\theta_{i+1}}_{J,I} \cong \Phi_Z(r^{\tilde{G}}_{\tilde{M}_J}(E)) \cong i^{\tilde{M}_I}_{\tilde{M}_I \cap v(\tilde{M}_J)} \circ v \circ r^{\tilde{M}_J}_{\tilde{M}_J \cap v^{-1}(\tilde{M}_I)}(r^{\tilde{G}}_{\tilde{M}_J}(E)).$$

Since $E \in Alg(\{I\})$, we must have $\tilde{M}_I \subset v(\tilde{M}_J)$, so $v^{-1}\tilde{M}_I v \subset \tilde{M}_J \Rightarrow wW_Iw^{-1} \subset W_J$, as claimed. Therefore, we have

$$\tilde{E}^{\theta_i}_{J,I}/\tilde{E}^{\theta_{i+1}}_{J,I} \cong i^{\tilde{M}_I}_{\tilde{M}_I} \circ w^{-1} \circ r^{\tilde{M}_J}_{w(\tilde{M}_I)}(r^{\tilde{G}}_{\tilde{M}_J}(E)) \cong w^{-1} \circ r^{\tilde{G}}_{w\tilde{M}_I w^{-1}}(E),$$

also as claimed.

As in [1], it now suffices to show the exactness of

$$\cdots \longrightarrow \bigoplus_{\substack{|J| = k \\ w_i \in \mathcal{D}(J,I) \\ w_i W_I w_i^{-1} \subset W_J}} w_i^{-1} \circ r^{\tilde{G}}_{w_i \tilde{M}_I w_i^{-1}}(E) \longrightarrow \cdots$$

Note that $w_i \in \mathcal{D}(J,I)$ with $w_i W_I w_i^{-1} \subset W_J$ if and only if $w_i \in \mathcal{D}(J,\emptyset)$ with $w_i(I) \subset J$. If we let $S^{w_i} = \{\alpha \,|\, w_i^{-1} \cdot \alpha > 0\}$, we see that $w_i \in \mathcal{D}(J,\emptyset)$ is equivalent to $J \subset S^{w_i}$. Thus, we are reduced to showing the exactness of

$$\cdots \longrightarrow \bigoplus_{\substack{|J|=k \\ w(I)\subset J\subset S^w}} w^{-1} \circ r^{\tilde{G}}_{w\tilde{M}_I w^{-1}}(E) \xrightarrow{\delta_k} \cdots \tag{4.2}$$

The next step is to identify δ_k. Consider the following part of a commutative diagram:

$$\begin{array}{ccc}
\tilde{E}^{\theta_j}_{J,I}/\tilde{E}^{\theta_{j+1}}_{J,I} & \xrightarrow{\tilde{d}^I_J} & \bigoplus_{\alpha\in J} \tilde{E}^{\theta_j}_{J-\{\alpha\},I}/\tilde{E}^{\theta_{j+1}}_{J-\{\alpha\},I} \\
F^{J,I}_j \downarrow & & \downarrow \sum_{\alpha\in J} F^{J-\{\alpha\},I}_j \\
\mathrm{Ad}(w_j)(r^{\tilde{G}}_{w_j\tilde{M}_I w_j^{-1}}(E)) & \xrightarrow{\delta} & \bigoplus \mathrm{Ad}(w_j)(r^{\tilde{G}}_{w_j\tilde{M}_I w_j^{-1}}(E)).
\end{array}$$

Here, $F^{J,I}_j$ is inherited from $F_j = F_{\theta_j}/F_{\theta_{j+1}}$ (as in section 5 [6]) (note that up to sign, the image of $F^{J,I}_j$ is $\Phi^{J,I}_j(E_J) \cong \mathrm{Ad}(w_j)(r^{\tilde{G}}_{w_j\tilde{M}_I w_j^{-1}}(E))$). We would like to show that up to sign, δ is just the identity.

To make the connection between our setup and that of [6] clear, we pause to note that for [6], $P = \tilde{P}_J$, $Q = \tilde{P}_I$, and $w = w_j^{-1}$. Recall that the equivalence $F_Z \cong \Phi_Z$ is defined by an intertwining operator A constructed from an intertwining operator $\bar{A}$ with certain properties. In our case,

$$\bar{A}f = \int_{\hat{U}_I\cap w_j^{-1}(\tilde{P}_J)\backslash\hat{U}_I} r_{w(\tilde{M}_I),\tilde{M}_J}(f(w_j\hat{u}_I\tilde{m}_I))\,d\mu(\hat{u}_I).$$

Therefore, if $f + E^{\theta_{j+1}}_{J,I} \in E^{\theta_j}_{J,I}/E^{\theta_{j+1}}_{J,I}$,

$$F^{J,I}_j(f + E^{\theta_{j+1}}_{J,I}) = \int_{\hat{U}_I\cap w_j^{-1}(\tilde{P}_J)\backslash\hat{U}_I} r_{w(\tilde{M}_I),\tilde{M}_J}(f(w_j\hat{u}_I\tilde{m}_I))\,d\mu(\hat{u}_I).$$

On the other hand,

$$\begin{aligned}
F^{J-\{\alpha\},I}_j \circ \phi^J_{J-\{\alpha\}}(f + E^{\theta_{j+1}}_{J,I}) &= F^{J-\{\alpha\},I}_j(r^J_{J-\{\alpha\}}(f) + E^{\theta_{j+1}}_{J-\{\alpha\},I}) \\
&= \int_{\hat{U}_I\cap w_j^{-1}(\tilde{P}_{J-\{\alpha\}})\backslash\hat{U}_I} r_{w(\tilde{M}_I),\tilde{M}_{J-\{\alpha\}}}(r^J_{J-\{\alpha\}}(f)(w_j\hat{u}_I\tilde{m}_I))\,d\mu(\hat{u}_I).
\end{aligned}$$

Observe that $r_{w(\tilde{M}_I),\tilde{M}_{J-\{\alpha\}}} \circ r^J_{J-\{\alpha\}} = r_{w(\tilde{M}_I),\tilde{M}_J}$. As

$$w \in \mathcal{D}(J,I),\ \alpha \in J \Rightarrow U_I \cap w_j^{-1}(\tilde{P}_{J-\{\alpha\}}) = U_I \cap w_j^{-1}(\tilde{P}_J),$$

these integrals are the same. Therefore, up to sign, δ is the identity.

The rest of the proof is now the same as in [1]. □

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