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STABILITY OF CRITICAL POINTS OF QUADRATIC HOMOGENEOUS DYNAMICAL SYSTEMS

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Abstract. In this work, we give sufficient conditions ensuring the instability of a critical point of a homogeneous quadratic system in $\mathbb{R}^n$ using the multiplication of the corresponding non-associative algebra. This result generalizes a theorem of Zalar and Mencinger (see [5]). We also state a classification theorem giving the stability or the instability of any stationary point of a quadratic homogeneous system in $\mathbb{R}^2$. As expected, the second theorem in [5] is part of this classification.

1. Introduction

Stationary points stability analysis is an important topic in nonlinear dynamics and especially in polynomial differential equations. A quadratic differential equation in $\mathbb{R}^n$ is a system of the form

$$\dot{X} = C + T(X) + Q(X) \tag{1.1}$$

where C is a fixed element of $\mathbb{R}^n$, $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear and $Q : \mathbb{R}^n \to \mathbb{R}^n$ a homogeneous mapping of degree 2 i.e $Q(\alpha X) = \alpha^2 Q(X)$ for all $\alpha \in \mathbb{R}$ and $X \in \mathbb{R}^n$.

It is well known that we can associate to any given homogeneous quadratic mapping Q defined on $\mathbb{R}^n$ a bilinear mapping $\beta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\beta(X, Y) = \frac{1}{2}[Q(X + Y) - Q(X) - Q(Y)].$$

Considered as a multiplication, the bilinear form β gives $\mathbb{R}^n$ the structure of a commutative non-associative algebra (non-associative means not necessarily associative).

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It is also possible to transform any system given under the form (1.1) in $\mathbb{R}^n$ into a quadratic homogeneous system in $\mathbb{R}^{n+1}$ and we may gain informations for the non-homogeneous system as a restriction of a homogeneous one to a convenient hyperplane (see [2]). For this reason, we concentrate on the study of stability of quadratic homogeneous differential equations.

In his seminal work [3], L. Markus emphasized the link between the differential equation $\dot{x} = \beta(x,x) = x^2$ and the algebra $\mathcal{A}$ whose multiplication table is given by the mapping β. It turns out that stationary points of the differential equation correspond to nilpotents of the algebra if any. Kinyon and Sagle [2] obtained additional substantial results. More recently, Mencinger and Zalar [5] studied the stability of stationary points of the system $\dot{x} = x^2$ and stated two main theorems. The first is a generalization of Corollary 3.8 in [2]. This result is based on the Peirce decomposition when the coefficient λ is real number not necessarily equal to $\frac{1}{2}$ and it gives a sufficient condition of instability of a critical point while the second is an example showing that the condition of the first theorem is not necessary. In this work, we give a result that generalizes the first theorem in [5] in the sense that the nilpotent n belongs to an eigenspace corresponding to some not necessarily idempotent vector. Finally, in the second part, we state a classification theorem giving the stability of any stationary point of a quadratic homogeneous system in $\mathbb{R}^2$. As predicted, it tuns out that the second result in [5] is part of this classification.

We hope this work to be an additional step toward a partial classification of stability of stationary points in dimension 3.

2. Stability of critical points

Let $\mathcal{A}$ be a finite dimensional real commutative non-associative algebra and $\dot{x} = x^2$ the corresponding differential equation. It is well known that the origin, which is always a stationary point, is never asymptotically stable (see [2]). We recall also that critical points of the quadratic equation $\dot{x} = x^2$ correspond to nilpotents of order two . In other words, n is a stationary point if and only if $n^2 = 0$. Let n be a nonzero critical point, λ a real constant and u a vector of $\mathcal{A}$. We define

$$A_\lambda(u) = \{x \in \mathcal{A} : ux = \lambda x\}.$$

When $\mathcal{A}$ has a nonzero idempotent u i.e a vector u verifying $u^2 = u$ and a nonzero nilpotent n of order two such that n belongs to $A_\lambda(u)$, Zalar and Mencinger proved in [5, Theorem 2.1, p. 21] that n is not stable.

We will prove that the result of Zalar and Mencinger remains true when u is not necessarily an idempotent.

Theorem 2.1. *Let n be a nonzero nilpotent of order two and $u \in \mathcal{A}$ such that $u^2 = \delta u + \gamma n$ for γ, $\delta \in \mathbb{R}$. If $n \in A_\lambda(u)$ then n is not stable.*

PROOF. Let $\epsilon > 0$ and $x(t)$ be the solution of $\dot{x} = x^2$ satisfying the initial condition $x(0) = \epsilon u + n$, we can suppose $\delta \geq 0$ (we replace u by $-u$). We have

$$x(t) = f(t)u + g(t)n,$$

where f and g are real functions. We have

$$\begin{aligned}\frac{df}{dt}u + \frac{dg}{dt}n &= (fu + gn)^2 = f^2u^2 + 2fgun \\ &= f^2(\delta u + \gamma n) + 2\lambda fgn = \delta f^2 u + (\gamma f^2 + 2\lambda fg)n\end{aligned} \tag{2.1}$$

which implies

$$\begin{aligned}\frac{df}{dt} &= \delta f^2, \\ \frac{dg}{dt} &= \gamma f^2 + 2\lambda fg,\end{aligned} \tag{2.2}$$

with

$$\begin{aligned}f(0) &= \epsilon, \\ g(0) &= 1.\end{aligned}$$

If $\delta \neq 0$, the first component of the solution is

$$f(t) = \frac{\epsilon}{1 - \epsilon\delta t},$$

which is defined on the interval $]-\infty, \frac{1}{\epsilon\delta}[$ and blows up in finite time. Hence n is not stable.

If $\delta = 0$, we easily obtain

$$\begin{aligned}f(t) &= \epsilon, \\ g(t) &= (1 + \frac{\epsilon\gamma}{2\lambda})e^{2\lambda\epsilon t} - \frac{\epsilon\gamma}{2\lambda}, \lambda \neq 0.\end{aligned} \tag{2.3}$$

We distinguish two sub-cases, when $\lambda > 0$ we have $\lim_{t\to\infty} |g(t)| = +\infty$. When $\lambda < 0$, we choose the initial condition $\epsilon < 0$ and we obtain $\lim_{t\to\infty} |g(t)| = +\infty$. Finally, if $\delta = \lambda = 0$, we have $g(t) = \gamma\epsilon^2 t + 1$ which means that the critical point n is not stable. □

When n does not belong to $A_\lambda(u)$ for any $u \in \mathcal{A}$, one might wonder whether it is possible to prove a more general result concerning the stability or the instability of the critical point. Any real commutative non-associative algebra $\mathcal{A}$ contains either a nilpotent or an idempotent element (see [1]). If $\mathcal{A}$ does not contain any nilpotent element, the only critical point of the corresponding differential equation is zero and since there exists at least one idempotent p, the line through p is an invariant line, crossing zero, moving away from it in one direction and approaching zero in the other direction. Thus zero is not stable. Therefore, we will suppose from now on that the algebra $\mathcal{A}$ contains at least one nilpotent n, also that the algebra is of dimension two and we have the following multiplication table:

	u	n
u	$\delta u + \gamma n$	$\beta u + \alpha n$
n	$\beta u + \alpha n$	0

In this basis, the equation $\dot{x} = x^2$ writes

$$\begin{aligned} \frac{df}{dt}u + \frac{dg}{dt}n &= f^2u^2 + 2fgun \\ &= f^2(\gamma n + \delta u) + 2(\alpha n + \beta u)fg \\ &= (\delta f^2 + 2\beta fg)u + (\gamma f^2 + 2\alpha fg)n, \end{aligned} \tag{2.4}$$

which gives

$$\begin{aligned} \frac{df}{dt} &= \delta f^2 + 2\beta fg, \\ \frac{dg}{dt} &= \gamma f^2 + 2\alpha fg. \end{aligned} \tag{2.5}$$

We are interested in the study of the stability of critical points of $\dot{x} = x^2$ according to the values of α, β, γ and δ. We recall that if n is a nilpotent element then an is also a critical point for any real constant a. We will prove the following classification theorem

THEOREM 2.2. *Consider the system (2.5), we have*

i. *Suppose $\alpha\beta \neq 0$, if $\beta < 0$ (respectively $\beta > 0$) any critical point an is stable for a positive and unstable for a negative (respectively unstable for a positive and stable for a negative).*

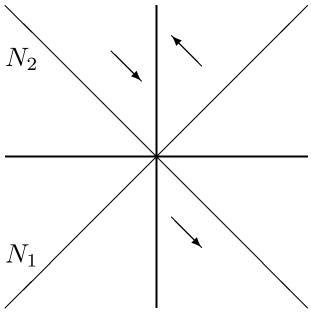
ii. *Suppose $\alpha\beta = 0$,*
   - ii.a *If $\beta = 0$ ($\alpha \in \mathbb{R}$), for any real number a, the critical point an is unstable.*
   - ii.b *If $\alpha = 0$ then if $\beta < 0$ (respectively $\beta > 0$) any critical point an is stable for a positive and unstable for a negative (respectively unstable for a positive and stable for a negative).*

PROOF. Suppose that $\alpha\beta \neq 0$, we define the two nullclines (N_1) and (N_2) as

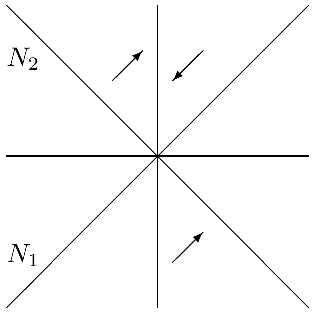
$$\begin{aligned} (N_1) : g + \frac{\delta}{2\beta}f = 0, \\ (N_2) : g + \frac{\gamma}{2\alpha}f = 0. \end{aligned}$$

When $\beta < 0$, we have eight different possibilities corresponding to the eight figures below. We have represented by arrows the direction of change of f and g near critical points (Figure 1). All critical points under the form an

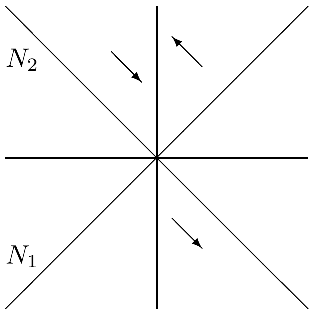
are on the $y - axis$



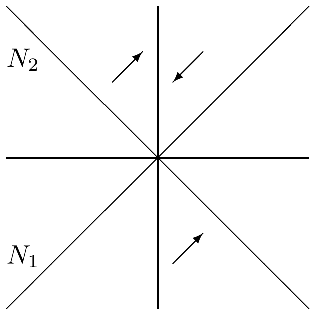
$\alpha > 0,\ \gamma > 0,\ \delta > 0$



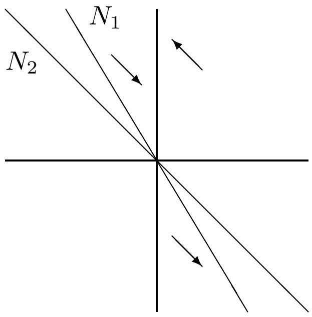
$\alpha < 0,\ \gamma < 0,\ \delta > 0$



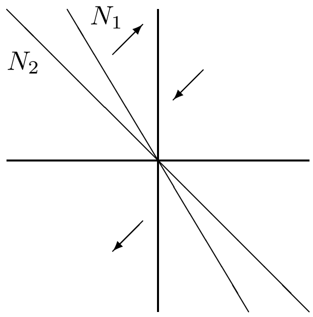
$\alpha > 0,\ \gamma < 0,\ \delta < 0$



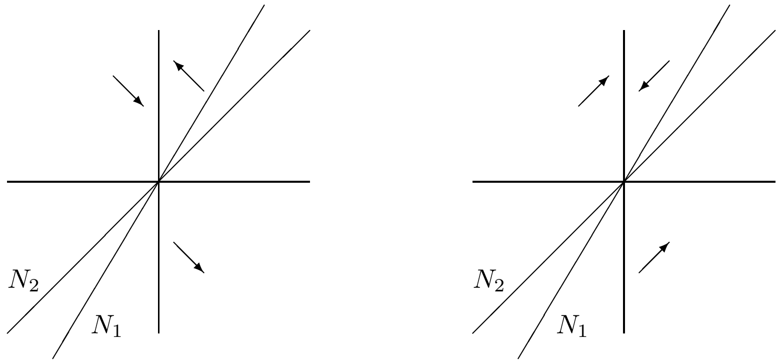
$\alpha < 0,\ \gamma > 0,\ \delta > 0$



$\alpha > 0,\ \gamma > 0,\ \delta < 0$



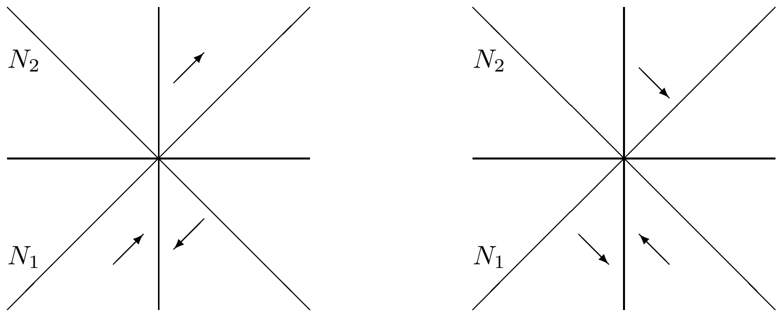
$\alpha < 0,\ \gamma < 0,\ \delta < 0$



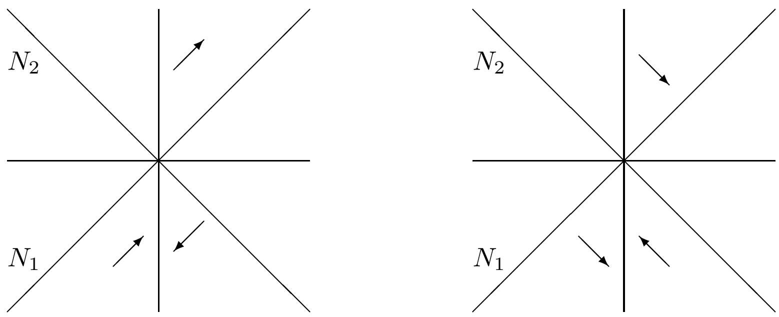
$\alpha > 0$, $\gamma < 0$, $\delta > 0$ $\qquad$ $\alpha < 0$, $\gamma > 0$, $\delta > 0$

Figure 1 : $\beta < 0$

We easily have $\lim_{t\to\infty} \frac{\dot{g}(t)}{\dot{f}(t)} = \frac{\alpha}{\beta}$ and this means that the slope of the trajectories tends to a finite limit when $t \to +\infty$. Thus if a point is close enough to the critical point an, its ω-limit set is a critical point belonging to the $y - axis$ and close enough to the point an. Consequently, an is stable. The same reasoning can be used for $\beta > 0$ (Figure 2).



$\alpha > 0$, $\gamma > 0$, $\delta > 0$ $\qquad$ $\alpha < 0$, $\gamma < 0$, $\delta > 0$



$\alpha > 0$, $\gamma < 0$, $\delta < 0$ $\qquad$ $\alpha < 0$, $\gamma > 0$, $\delta > 0$

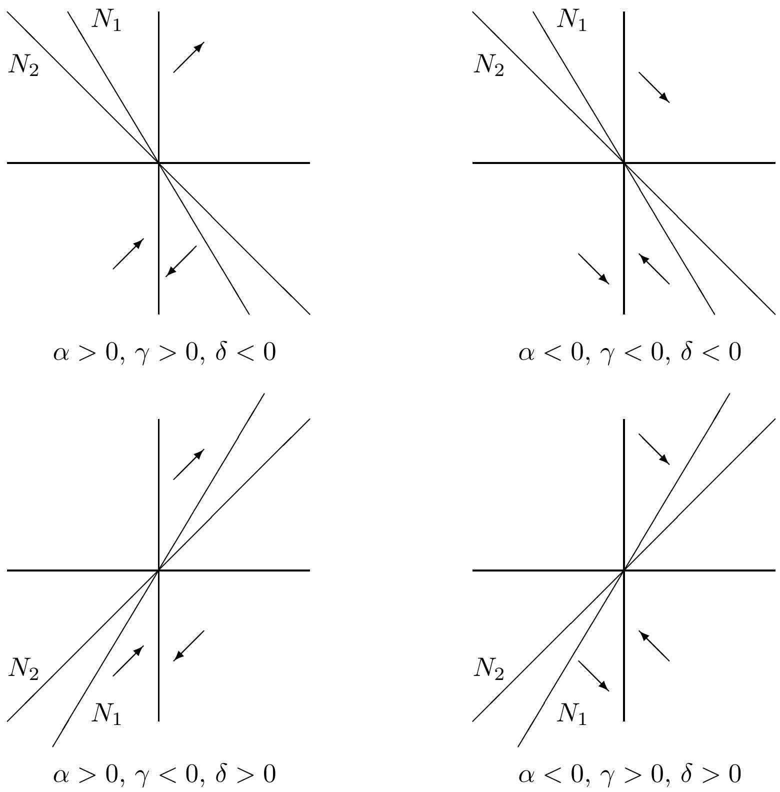


Figure 2 : $\beta > 0$

Suppose now that $\alpha\beta = 0$.

For $\beta = 0$, the result has been stated in Theorem 2.1.

If $\alpha = 0$, the system (2.5) becomes

$$\frac{df}{dt} = \delta f^2 + 2\beta f g \qquad (2.6)$$
$$\frac{dg}{dt} = \gamma f^2.$$

If we suppose $\delta = 0$, we have the following configuration of nullclines

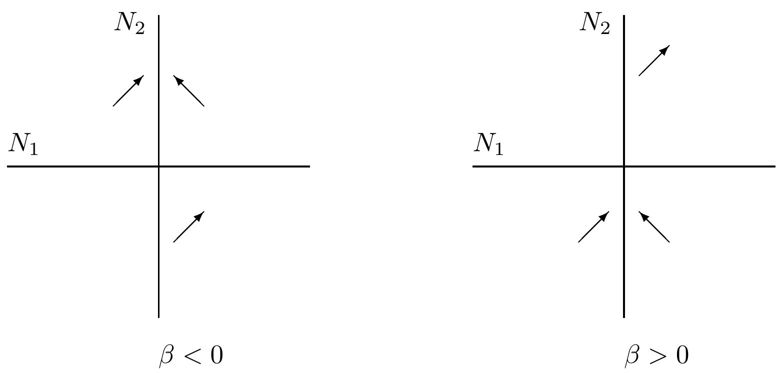


Figure 3 : $\delta = 0$

For $\delta \neq 0$

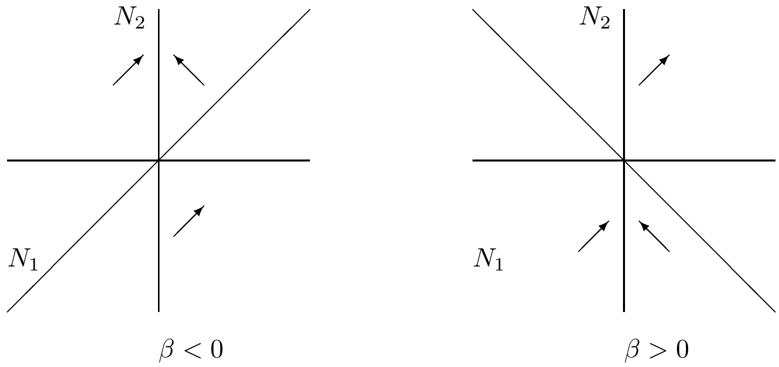


Figure 4 : $\delta \neq 0$

We notice that $\lim_{t\to\infty} \frac{\dot{g}(t)}{\dot{f}(t)} = 0$ $(\beta \neq 0)$ and thus we obtain the same conclusion as stated in i. ∎

REMARK 2.3. In [5], Zalar and Mencinger considered the algebra $\mathcal{A}$ given by the multiplication table

	n	u
n	0	$-u$
u	$-u$	$-n$

for which they prove that the nilpotent n is stable (see Theorem 3.7, p. 26). This result is a special case of ii.b, since $\alpha = 0$ and $\beta = -1 < 0$, any critical point an for $a > 0$ is stable.

Instability results in $\mathbb{R}^2$ may extend to instability results in $\mathbb{R}^n$ when there exist a convenient invariant plane including a nilpotent element and we have the following theorem.

THEOREM 2.4. *Let* $\dot{x} = x^2$ *in* $\mathbb{R}^n$. *Suppose there exist a nilpotent* n_0 *and a vector* u *such that*

$$\begin{cases} n_0 u = \alpha n_0 + \beta u, \\ u^2 = \gamma n_0 + \delta u. \end{cases}$$

If $\beta \geq 0$ *(respectively* $\beta < 0$*) then all critical points* an_0 *are unstable for* $a > 0$ *(respectively* $a < 0$*).*

References

[1] J. L. Kaplan and J. A. Yorke, *Nonassociative real algebras and quadratic differential equations,* Nonlinear Anal. **3** (1977), 49–51.
[2] M. K. Kinyon and A. A. Sagle, *Quadratic dynamical systems and algebras,* J. Differential Equations **117** (1995), 67–126.
[3] L. Markus, *Quadratic differential equations and non-associative algebras,* 1960 Contributions to the theory of nonlinear oscillations, Vol. V, Princeton Univ. Press, Princeton, 1960, 185–213.
[4] S. Walcher, Algebras and differential equations, Hadronic Press, Palm Harbor, 1991.
[5] B. Zalar and M. Mencinger, *On stability of critical points of quadratic differential equations in nonassociative algebras*, Glas. Mat. Ser. III **38(58)** (2003), 19–27.

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