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ON THE ALMOST CONVERGENCE OF DOUBLE SEQUENCES

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Abstract. We find necessary and sufficient conditions for transformations of double sequences almost convergent in the sense of G. H. Hardy to double sequences convergent in the sense of F. Pringsheim. The results extend the work of F. Móricz and B. E. Rhoades on transformations of sequences almost convergent in the Pringsheim's sense.

The first definitions and investigations of the convergence of double sequences are usually atributted to F. Pringsheim, who studied such sequences and series more than hundred years ago (see [1, p. 78]). Pringsheim defined what we call the P *limit* and gave examples of convergence (P *convergence*) of double sequences with and without the usual convergence of rows and columns ([13, pp. 104–112]). G. H. Hardy ([6]) considered in more details the case of convergence of double sequences where, besides the existence of the P limit, rows and columns converge. F. Móricz discovered an alternative approach to the Hardy convergence, which significantly influenced the whole theory ([9]–[11]; cf. [2]). Moreover, following G. G. Lorentz ([7]), F. Móricz and B. E. Rhoades found necessary and sufficient (*N.S.*) conditions under which P almost convergent double sequences are transformed into P convergent double sequences ([12, Theorem 1, p. 285]).

In a previous paper [3] the author of this article found conditions under which double sequences almost convergent in the Hardy (H) sense are transformed into P convergent double sequences. The results were not completely satisfactory because they were obtained under a uniformity condition inherited from the usual H convergence (cf. [3, p. 252] and [14, p. 14]).

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In this paper we have *N.S.* conditions for transformations of double sequences that H almost converge without any further restriction. These conditions are the same as the Móricz-Rhoades conditions for transformations of P almost convergent double sequences. Uniformities of row or column convergence now give various particular cases. Moreover, we have results on transformations of double sequences that H almost converge only by columns or only by rows. For some results that can be reduced to results at [12] and [3] our approach here gives alternative proofs.

1. The Hardy convergence

Let us denote the set of all double sequences $x = (x_{ij})$, $i, j \in \mathbb{N}$, of complex (or real) numbers by $\mathbf{s}$. We consider (x_{ij}) as the function on the ij-coordinate plane: $(x_{ij})_{i\in\mathbb{N}}$ is the j-th *row*, and $(x_{ij})_{j\in\mathbb{N}}$ is the i-th *column* of x. Let $\mathbf{b}$ be the set of all bounded double sequences from $\mathbf{s}$. A double sequence x from $\mathbf{s}$ *converges* to L if, for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

$$|x_{ij} - L| < \varepsilon \quad \text{if} \quad i, j \geq N_\varepsilon. \tag{1.1}$$

After [1, p. 78] and [6, p. 88] this kind of convergence we call the convergence in the sense of Pringsheim (P *convergence*). The limit L is denoted by $\lim_{ij}$ and is called the P *limit* (see [13, p. 103]). The set of all convergent double sequences from $\mathbf{s}$ is the class $\mathbf{c}$. Bounded and convergent double sequences form the class $\mathbf{bc}$.

Starting from double series, both G. H. Hardy and later F. Móricz studied convergent double sequences with convergent rows and convergent columns. Such double sequences besides the P limit $L = \lim_{ij} x_{ij}$ have row limits $L'_j = \lim_i x_{ij}$ for every j and column limits $L''_i = \lim_j x_{ij}$ for every i. If (L'_j) and L exist we say that (x_{ij}) converges in the Hardy (H) sense *by rows*. If (L''_i) and L exist we have the H convergence *by columns*. The class of double sequences that H converges by rows and by columns form the class of H *convergent* double sequences. It is denoted by $\mathbf{rc}$ (*regularly* convergent, after Hardy [6, p. 88] and Hamilton [4, p. 30]). The P limit of double sequences from $\mathbf{rc}$ is also called the *principal* limit.

Double sequences from $\mathbf{rc}$ are bounded: $\mathbf{rc} \subseteq \mathbf{bc}$ ([4, p. 33]). Double sequences from $\mathbf{rc}$ with equal row and column limits we denote by $\mathbf{rcr}$. For these double sequences $L'_j = L''_i = L$. Subclasses of $\mathbf{c}$ of double sequences that converge to 0 are denoted by an $\mathbf{n}$ at the end: we have $\mathbf{bcn}$, $\mathbf{rcn}$ and $\mathbf{rcrn}$. The last class has all row limits and all column limits as well as the P limit equal to 0.

By the definition of $\mathbf{rc}$,

$$\lim_{ij} x_{ij} = \lim_i \lim_j x_{ij} = \lim_j \lim_i x_{ij}. \tag{1.2}$$

The convergence of rows to L'_j and the convergence of columns to L''_i is uniform with respect to j's, resp. to i's (see [14, Theorem 9, p. 14] and [4, Theorem 003, p. 34]). The convergence of rows and columns means that for every $\varepsilon > 0$ there exist N'_ε, $N''_\varepsilon \in \mathbb{N}$ such that for every j

$$|x_{ij} - L'_j| < \varepsilon \quad \text{if} \quad i \geq N'_\varepsilon, \tag{1.3}$$

and for every i

$$|x_{ij} - L''_i| < \varepsilon \quad \text{if} \quad j \geq N''_\varepsilon. \tag{1.4}$$

A double sequence x from **s** is *almost convergent* (*a-convergent*) to L if

$$\sigma^{mn}_{pq} = \frac{1}{pq} \sum_{i=m}^{m+p-1} \sum_{j=n}^{n+q-1} x_{ij}, \tag{1.5}$$

for p, $q \to \infty$ converges to L uniformly with respect to m and n. This means that for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that, for every m, $n \in \mathbb{N}$,

$$|\sigma^{mn}_{pq} - L| < \varepsilon \quad \text{if} \quad p, q \geq N_\varepsilon. \tag{1.6}$$

We denote L by $\mathrm{Lim}_{ij} x_{ij}$ and consider it as the *P a-limit* of x. The set of all a-convergent double sequences we denote by **ac**.

A single sequence (x_i) a-converges if $\frac{1}{p}\sum_{i=m}^{p+m-1} x_i$ converges when $p \to \infty$ uniformly with respect to $m \in \mathbb{N}$. This notion was introduced by G. G. Lorentz ([7]). It was extended to double sequences i. e. to the class **ac** by F. Móricz and B. E. Rhoades ([12]). Among others, they proved **bc** $\subset$ **ac** $\subset$ **b** ([12, pp. 283–4]).

The double sequence x *H a-converges by rows* if every row a-converges to $L'_j = \mathrm{Lim}_i x_{ij}$ and the P a-limit L of x exists (cf. Lemma 1.1). In such a case L is the *principal* a-limit, and L'_j are *row* a-limits. *Column* a-limits and the *H a-convergence by columns* are similarly defined. The double sequence x H a-converges if it is H a-convergent by rows *and* columns to the P a-limit. The class of H a-convergent sequences are denoted by **rac**. The **racn** is the subclass of **rac** with the P a-limit 0. The **racr** is a subclass of **rac** with equal L, L'_j and L''_i for every i, j. If $L = L'_j = L''_i = 0$ we have the class **racrn**.

Lemma 1.1 ([2, Theorem 1, p. 132]). *Let* (x_{ij}) *P a-converge to* $\mathrm{Lim}_{ij} x_{ij} = L$, *and, moreover, let every row* $(x_{ij})_{i\in\mathbb{N}}$ *a-converge to* $\mathrm{Lim}_i x_{ij} = L'_j$. *Then* (L'_j) *a-converges and*

$$\mathrm{Lim}_j \mathrm{Lim}_i x_{ij} = \mathrm{Lim}_{ij} x_{ij}. \tag{1.7}$$

Similarly, if the P a-limit and all column a-limits exist,

$$\mathrm{Lim}_i \mathrm{Lim}_j x_{ij} = \mathrm{Lim}_{ij} x_{ij}. \tag{1.8}$$

Corollary 1.2 ([2, p. 132]). *Let a-limits of rows, a-limits of columns and the principal a-limit of the double sequence x exist. Then*

$$\mathrm{Lim}_{ij} x_{ij} = \mathrm{Lim}_j \mathrm{Lim}_i x_{ij} = \mathrm{Lim}_i \mathrm{Lim}_j x_{ij}. \tag{1.9}$$

This corollary excludes the existence of the P a-limit if a-limits of L'_j and of L''_i exist and are unequal (for P limits cf. [13, p. 107]).

By row-column uniformities we have a partial inverse of Lemma 1.1.

LEMMA 1.3. *Let rows of $x = (x_{ij})$ a-converge to L'_j uniformly, and let the sequence (L'_j) a-converge. Then the P a-limit of x exists and x H a-converges by rows. Similarly for uniform a-convergence of columns.*

PROOF. By the uniform a-convergence of rows

$$\left|\frac{1}{p}\sum_{i=m}^{m+p-1} x_{ij} - L'_j\right| \le \frac{\varepsilon}{2} \quad \text{if} \quad p \ge N'_{\frac{\varepsilon}{2}} \tag{1.10}$$

for every $j \in \mathbb{N}$. By the a-convergence of row a-limits,

$$\left|\frac{1}{q}\sum_{j=n}^{n+q-1} L'_j - \operatorname{Lim}_j L'_j\right| \le \frac{\varepsilon}{2} \quad \text{if} \quad q \ge N'_{\frac{\varepsilon}{2}}. \tag{1.11}$$

Therefore,

$$\begin{aligned} &\left|\frac{1}{q}\sum_{j=n}^{n+q-1}\frac{1}{p}\sum_{i=m}^{m+p-1} x_{ij} - \operatorname{Lim}_j L'_j\right| \\ &\le \frac{1}{q}\sum_{j=n}^{n+q-1}\left|\frac{1}{p}\sum_{i=m}^{m+p-1} x_{ij} - L'_j\right| + \left|\frac{1}{q}\sum_{j=n}^{n+q-1} L'_j - \operatorname{Lim}_j L'_j\right| \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \tag{1.12}$$

This means that

$$\operatorname{Lim}_j \operatorname{Lim}_i x_{ij} = \operatorname{Lim}_{ij} x_{ij}. \tag{1.13}$$

□

2. TRANSFORMATION CONDITIONS

Let $A = \left[a_{ij}^{kl}\right]$, $i, j, k, l \in \mathbb{N}$, be a doubly infinite matrix of complex (or real) numbers. A double sequence $x = (x_{ij}) \in \mathbf{s}$ is transformed into a double sequence $Ax = y = (y^{kl}) \in \mathbf{s}$ by

$$y^{kl} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}^{kl} x_{ij} \tag{2.1}$$

if the double series (2.1) P converges for every $k, l \in \mathbb{N}$. It means that the partial sums $\sum_{i=1}^{r}\sum_{j=1}^{s}$ have the P limit if $r, s \to \infty$ ([13, (3) p. 113]). The partial sums for A are called *A-means* ([12, p. 284]). The matrix A is *bounded-regular* if every bounded and P convergent double sequence (x_{ij}) is transformed by the bounded set of A-means to the P convergent double sequence (y^{kl}) with the limit equal to the limit of (x_{ij}). The matrix A is *strongly regular* if every P a-convergent double sequence (x_{ij}) is transformed to the P convergent (y^{kl}) with the limit equal to the a-limit of (x_{ij}), and the A-means are also bounded (*y P converges boundedly*). The above definitions extends to double sequences the notion of regularity and of strong regularity of single sequences (see [7, pp. 171, 176]).

We consider bounded transformations of classes which are between **rcrn** and **b**. Necessary conditions that **rcrn** transforms to **b** include the existence of C such that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}^{kl}| \leq C < +\infty \tag{2.2}$$

for every $k,l \in \mathbb{N}$ ([4, 5., p. 42]). This implies that for every bounded y the series (2.1) which defines (y^{kl}) converges absolutely. Because our y are at least bounded, conditions on A always include (with the notation (2.4) below), $(v) \in \mathbf{b}$.

We use the notation

$$\begin{aligned} &\triangle_{10}a_{ij}^{kl} = a_{ij}^{kl} - a_{i+1,j}^{kl}, \qquad \triangle_{01}a_{ij}^{kl} = a_{ij}^{kl} - a_{i,j+1}^{kl}, \\ &\Box\, a_{ij}^{kl} = a_{ij}^{kl} - a_{i+1,j}^{kl} - a_{i,j+1}^{kl} + a_{i+1,j+1}^{kl}. \end{aligned} \tag{2.3}$$

Series are denoted by $\sum_i = \sum_{i=1}^{\infty}$, $\sum_j = \sum_{j=1}^{\infty}$. Also, we give notations for sequences derived from $\left[a_{ij}^{kl}\right]$. The list is a slightly enlarged list ([3, p. 254]). The only change is that instead of $(ii'_{\triangle})$ we write $(ii_{\triangle})$. Also, we have a_{ij} for a_{ij}^{kl}.

(2.4)

$$(i)\ (\forall\ i,j)\ a_{ij}$$

$$(ii)\ \begin{cases}(\forall j)\ \sum_i a_{ij}\\ (\forall i)\ \sum_j a_{ij}\end{cases} \qquad (ii_{\triangle})\ \begin{cases}(\forall j)\ \sum_i \triangle_{01}a_{ij}\\ (\forall i)\ \sum_j \triangle_{10}a_{ij}\end{cases}$$

$$(iii)\ \begin{cases}\sum_j\sum_i a_{ij}\\ \sum_i\sum_j a_{ij}\end{cases}$$

$$(iv)\ \begin{cases}(\forall j)\ \sum_i |a_{ij}|\\ (\forall i)\ \sum_j |a_{ij}|\end{cases} \qquad (iv_{\triangle})\ \begin{cases}(\forall j)\ \sum_i |\triangle_{10}a_{ij}|\\ (\forall i)\ \sum_j |\triangle_{01}a_{ij}|\end{cases} \quad (iv'_{\triangle})\ \begin{cases}(\forall j)\ \sum_i |\triangle_{01}a_{ij}|\\ (\forall i)\ \sum_j |\triangle_{10}a_{ij}|\end{cases} \qquad (iv_{\Box})\ \begin{cases}(\forall j)\ \sum_i |\Box\, a_{ij}|\\ (\forall i)\ \sum_j |\Box\, a_{ij}|\end{cases}$$

$$(v)\ \sum_i\sum_j |a_{ij}| \qquad (v_{\triangle})\ \begin{cases}\sum_i\sum_j |\triangle_{01}a_{ij}|\\ \sum_i\sum_j |\triangle_{10}a_{ij}|\end{cases} \qquad (v_{\Box})\ \sum_i\sum_j |\Box\, a_{ij}|$$

$$(vi)\ \begin{cases}\sum_j |\sum_i a_{ij}| \\ \sum_i |\sum_j a_{ij}|\end{cases} \qquad (vi_\triangle)\ \begin{cases}\sum_j |\sum_i \triangle_{01} a_{ij}| \\ \sum_i |\sum_j \triangle_{10} a_{ij}|\end{cases} \qquad (vi_\square)\ \begin{cases}\sum_j |\sum_i \square\, a_{ij}| \\ \sum_i |\sum_j \square\, a_{ij}|\end{cases}$$

For members of (2.4) which have { and two lines, the first and the second line are denoted by $(\ \ )_1$ and $(\ \ )_2$. E. g., $(vi_\triangle)_2 \in \mathbf{bcn}$ means $\sum_i |\sum_j \triangle_{10} a^{kl}_{ij}| \in \mathbf{bcn}$; $(vi_\triangle) \in \mathbf{bcn}$ means that $(vi_\triangle)_1$ and $(vi_\triangle)_2$ belong to **bcn**.

Lemma 2.1. (cf. [5, (17) p. 279]). *Let* $(v) \in \mathbf{b}$. *Then*

$$\sum_{i=1}^{\infty} |\triangle_{10} a^{kl}_{it}| \le \sum_{i=1}^{\infty}\sum_{j=t}^{\infty} |\square\, a^{kl}_{ij}|, \qquad \sum_{j=1}^{\infty} |\triangle_{01} a^{kl}_{tj}| \le \sum_{i=t}^{\infty}\sum_{j=1}^{\infty} |\square\, a^{kl}_{ij}|. \tag{2.5}$$

Proof. The $(v) \in \mathbf{b}$ implies $(v_\square) \in \mathbf{b}$ and

$$\sum_{i=1}^{\infty}\Big|\sum_{j=t}^{\infty} \square\, a^{kl}_{ij}\Big| = \sum_{i=1}^{\infty}\Big|\sum_{j=t}^{\infty}(\triangle_{10} a^{kl}_{ij} - \triangle_{10} a^{kl}_{i,j+1})\Big| = \sum_{i=1}^{\infty} |\triangle_{10} a^{kl}_{it}|. \tag{2.6}$$

Therefore,

$$(\forall t)\ \sum_{i=1}^{\infty} |\triangle_{10} a^{kl}_{it}| \le \sum_{i=1}^{\infty}\sum_{j=t}^{\infty} |\square\, a^{kl}_{ij}|\,. \tag{2.7}$$

Similarly for $\triangle_{01}$. ∎

By this Lemma, if $(v) \in \mathbf{b}$,

$$(v_\square) \in \mathbf{bcn} \Rightarrow \begin{cases}(iv_\triangle)_1 \in \mathbf{bcn} \\ (iv_\triangle)_2 \in \mathbf{bcn}\end{cases}. \tag{2.8}$$

Lemma 2.2 (cf. [8, p. 806, (8) and (3)]). *Let* $(v) \in \mathbf{b}$. *Then*

$$\left|\sum_{i=1}^{\infty} a^{kl}_{ij}\right| \le \sum_{j=1}^{\infty}\left|\sum_{i=1}^{\infty} \triangle_{01} a^{kl}_{ij}\right|, \qquad \left|\sum_{j=1}^{\infty} a^{kl}_{ij}\right| \le \sum_{i=1}^{\infty}\left|\sum_{j=1}^{\infty} \triangle_{10} a^{kl}_{ij}\right|. \tag{2.9}$$

Proof. With $a_{ij} = a^{kl}_{ij}$ and C from (2.2),

$$\sum_{j=1}^{\infty}\Big|\sum_{i=1}^{\infty}(a_{ij} - a_{i,j+1})\Big| \le 2\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}| \le 2C. \tag{2.10}$$

Let

$$\sum_{j=1}^{\infty}\Big|\sum_{i=1}^{\infty}(a_{ij} - a_{i,j+1})\Big| = D. \tag{2.11}$$

By

$$\Big|\sum_{i=1}^{\infty} a_{i1} - \sum_{i=1}^{\infty} a_{i2}\Big| + \cdots + \Big|\sum_{i=1}^{\infty} a_{it} - \sum_{i=1}^{\infty} a_{i,t+1}\Big| \le D \tag{2.12}$$

we have

$$\begin{aligned}&\Big|\big(\sum_{i=1}^{\infty}a_{i1}-\sum_{i=1}^{\infty}a_{i2}\big)+\cdots+\big(\sum_{i=1}^{\infty}a_{it}-\sum_{i=1}^{\infty}a_{i,t+1}\big)\Big|\\ &\quad=\left|\sum_{i=1}^{\infty}a_{i1}-\sum_{i=1}^{\infty}a_{i,t+1}\right|\le D.\end{aligned}\tag{2.13}$$

If $t\to\infty$, $|\sum_i a_{i1}|\le D$. Starting with $\sum_{j=s}^{\infty}$ we have $|\sum_i a_{is}|\le D$ for $s=2,3,\dots$. Therefore

$$(\forall j)\ \Big|\sum_{i=1}^{\infty}a_{ij}^{kl}\Big|\le\sum_{j=1}^{\infty}\Big|\sum_{i=1}^{\infty}\triangle_{01}a_{ij}^{kl}\Big|\tag{2.14}$$

and similarly for $\triangle_{10}$. By the notation (2.4), if $(v)\in\mathbf{b}$,

$$(vi_{\triangle})_1\in\mathbf{bcn}\Rightarrow(ii)_1\in\mathbf{bcn},\qquad(vi_{\triangle})_2\in\mathbf{bcn}\Rightarrow(ii)_2\in\mathbf{bcn}\tag{2.15}$$

(cf. also Remark 2.8). □

LEMMA 2.3. *Let* $(v)\in\mathbf{b}$. *Then*

$$\begin{aligned}&(\forall i,j)\quad|a_{ij}^{kl}|\le\sum_{j=1}^{\infty}\left|\triangle_{01}a_{ij}^{kl}\right|,\\ &(\forall i,j)\quad|a_{ij}^{kl}|\le\sum_{i=1}^{\infty}\left|\triangle_{10}a_{ij}^{kl}\right|\end{aligned}\tag{2.16}$$

and therefore

$$\begin{aligned}&(iv_{\triangle})_2\in\mathbf{bcn}\Rightarrow(i)\in\mathbf{bcn},\\ &(iv_{\triangle})_1\in\mathbf{bcn}\Rightarrow(i)\in\mathbf{bcn}.\end{aligned}\tag{2.17}$$

PROOF. By the proof of Lemma 2.2 with a_{ij}^{kl} instead of $\sum_i a_{ij}^{kl}$ for every i we have the first inequality. The second inequality is obtained in the same way. □

The next corollary follows from Lemma 2.1 and Lemma 2.3.

COROLLARY 2.4. *Let* $(v)\in\mathbf{b}$. *Then*

$$(v_{\square})\in\mathbf{bcn}\Rightarrow(i)\in\mathbf{bcn}.\tag{2.18}$$

By Lemma 2.3

$$(\forall j)\quad\sum_{i=1}^{\infty}|a_{ij}^{kl}|\le\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|\triangle_{01}a_{ij}^{kl}\right|,\qquad(\forall i)\quad\sum_{j=1}^{\infty}|a_{ij}^{kl}|\le\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|\triangle_{10}a_{ij}^{kl}\right|,\tag{2.19}$$

which gives

Corollary 2.5. *Let* $(v) \in \mathbf{b}$. *Then*

$$(v_\triangle)_1 \in \mathbf{bcn} \Rightarrow (iv)_1 \in \mathbf{bcn}, \qquad (v_\triangle)_2 \in \mathbf{bcn} \Rightarrow (iv)_2 \in \mathbf{bcn}. \tag{2.20}$$

Remark 2.6. By Corollary 2.5, $(v) \in \mathbf{b}$ and the Móricz-Rhoades $(v_\triangle) \in \mathbf{bcn}$ are *N.S.* for transformations of double sequences from **racrn** to double P convergent null-sequences. These conditions include (i), $(iv) \in \mathbf{bcn}$ (cf. [12, pp. 286–287]).

Lemma 2.7. *Let* $(v) \in \mathbf{b}$. *Then*

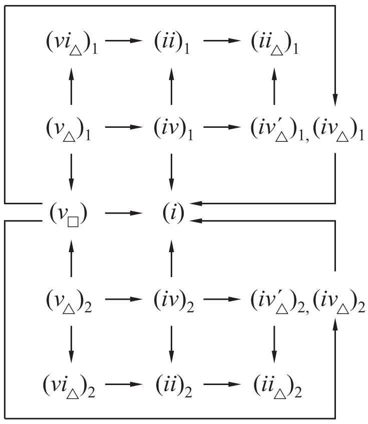
$$\begin{aligned} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\Box a_{ij}^{kl}| &\le 2\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \left|\triangle_{01} a_{ij}^{kl}\right|, \\ \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\Box a_{ij}^{kl}| &\le 2\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \left|\triangle_{10} a_{ij}^{kl}\right|. \end{aligned} \tag{2.21}$$

Therefore

$$\begin{aligned} (v_\triangle)_1 \in \mathbf{bcn} &\Rightarrow \quad (v_\Box) \in \mathbf{bcn}, \\ (v_\triangle)_2 \in \mathbf{bcn} &\Rightarrow \quad (v_\Box) \in \mathbf{bcn}. \end{aligned} \tag{2.22}$$

The results of lemmas are summarized in (2.23). Arrows give implications if members of (2.23) belong to **bcn** and if to every initial condition the $(v) \in \mathbf{b}$ is added. E. g. $(v_\triangle)_2 \to (vi_\triangle)_2$ means that $\sum_i \sum_j |\triangle_{10} a_{ij}^{kl}| \in \mathbf{bcn}$ and $\sum_i \sum_j |a_{ij}^{kl}| \in \mathbf{b}$ imply $\sum_i |\sum_j \triangle_{10} a_{ij}^{kl}| \in \mathbf{bcn}$.

(2.23)



Remark 2.8. The schema (2.23) neglects uniformities that in some cases follow from lemmas above. For example, by (2.20) we have that $(v_\triangle)_1 \in \mathbf{bcn}$ implies $(iv)_1 \in \mathbf{bcn}$ for every j, but also that the convergence by (k, l) is uniform with respect to j. Similarly for estimates by (v), $(vi_\triangle)$, $(v_\Box)$, etc.

REMARK 2.9. Instead of $(v) \in \mathbf{b}$, many papers start with $(v_\square) \in \mathbf{b}$ (and the notation $\triangle_{11}$ for our $\square$). Hardy ([6, 4.(2), p. 89–90]) has a *bounded variation* for bounded $(v_\square)$ and $(iv_\triangle)$. Mears ([8, p. 805]) defines *absolutely convergent* sequences by $(v_\square) \in \mathbf{b}$, with classes $a\mathbf{c}$, $a\mathbf{rc}$, etc. Hamilton ([5, p. 276]) has bounded $\sum_{i,j=1}^{\infty} |\square\, a_{ij}|$ jointly with other conditions.

3. Almost convergence by rows and columns

The a-convergence by rows and columns can be described by σ_{pq}^{mn} because of

$$\frac{1}{p}\sum_{i=m}^{m+p-1} x_{ij} = \sigma_{p1}^{mj}, \qquad \frac{1}{q}\sum_{j=n}^{n+q-1} x_{ij} = \sigma_{1q}^{in}. \tag{3.1}$$

The j-th row a-converges to L'_j if, for every $\varepsilon > 0$ there exists $N'_\varepsilon(j)$ such that, for $p \geq N'_\varepsilon(j)$ and every m,

$$|\sigma_{p1}^{mj} - L'_j| \leq \varepsilon. \tag{3.2}$$

The i-th column a-converges to L''_i if, for every $\varepsilon > 0$ there exists $N''_\varepsilon(i)$ such that, for $q \geq N''_\varepsilon(i)$ and every n,

$$|\sigma_{1q}^{in} - L''_i| \leq \varepsilon. \tag{3.3}$$

If x H a-converges by rows, L'_j a-converges to the P a-limit L, and for every $\varepsilon > 0$, there exists N'_ε such that, for $q \geq N'_\varepsilon$,

$$\left|\frac{1}{q}\sum_{j=n}^{n+q-1} L'_j - L\right| \leq \varepsilon. \tag{3.4}$$

If x H a-converges by columns there exists N''_ε such that, for $p \geq N''_\varepsilon$,

$$\left|\frac{1}{p}\sum_{i=m}^{m+p-1} L''_i - L\right| \leq \varepsilon. \tag{3.5}$$

LEMMA 3.1. *Let x H a-converges to L. Then, with N_ε from (1.6),*

$$N'_\varepsilon, N''_\varepsilon \leq N_\varepsilon. \tag{3.6}$$

PROOF. Because of (1.6),

$$p, q \geq N_\varepsilon \Rightarrow \left|\frac{1}{pq}\sum_{i=m}^{m+p-1}\sum_{j=n}^{n+q-1} x_{ij} - L\right| \leq \varepsilon. \tag{3.7}$$

Increasing p we obtain (3.4), and increasing q we obtain (3.5). As a result, we have (3.6). ∎

For $x \in$ **racrn**, which has at least one $x_{ij} \neq 0$, it follows that $N'_\varepsilon = N''_\varepsilon = 0$ and $N_\varepsilon > 0$. If x has equal rows convergent to 0 and at least one $x_{ij} \neq 0$, we have $N'_\varepsilon = N_\varepsilon > 0$, $N''_\varepsilon = 0$.

In what follows we use abbreviations

$$\begin{aligned} \Sigma^{kl}_{RC} &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a^{kl}_{mn}\frac{1}{pq}\sum_{i=m}^{m+p-1}\sum_{j=n}^{n+q-1} x_{ij} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a^{kl}_{mn}\sigma^{mn}_{pq}, \\ \Sigma^{kl}_{R} &= \sum_{m=1}^{\infty}\sum_{j=1}^{\infty} a^{kl}_{mj}\frac{1}{p}\sum_{i=m}^{m+p-1} x_{ij} = \sum_{m=1}^{\infty}\sum_{j=1}^{\infty} a^{kl}_{mj}\sigma^{mj}_{p1}, \\ \Sigma^{kl}_{C} &= \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} a^{kl}_{in}\frac{1}{q}\sum_{j=n}^{n+q-1} x_{ij} = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} a^{kl}_{in}\sigma^{in}_{1q}. \end{aligned} \tag{3.8}$$

With L as a P a-limit of x,

$$\begin{aligned} y^{kl} - L =& \big(\Sigma^{kl}_{RC} - y^{kl}\big) - \big(\Sigma^{kl}_{R} - y^{kl}\big) - \big(\Sigma^{kl}_{C} - y^{kl}\big) \\ &- \big(\Sigma^{kl}_{RC} - L\big) + \big(\Sigma^{kl}_{R} - L\big) + \big(\Sigma^{kl}_{C} - L\big). \end{aligned} \tag{3.9}$$

For $L = 0$ as well as for $L \neq 0$,

$$y^{kl} + \Sigma^{kl}_{RC} - \Sigma^{kl}_{R} - \Sigma^{kl}_{C} = \big(\Sigma^{kl}_{RC} - y^{kl}\big) - \big(\Sigma^{kl}_{R} - y^{kl}\big) - \big(\Sigma^{kl}_{C} - y^{kl}\big). \tag{3.10}$$

The subclass of **rac** with uniformly a-convergent rows and columns is denoted by **rac un**. This means that (3.2) holds uniformly with respect of j, as well as (3.3) with respect of i. The class **racrn un** has uniform a-convergent rows and columns to 0. These classes are in [3] denoted by **arc**, resp. by **arcrn**.

LEMMA 3.2. *Let x be bounded. For every $p, q \in \mathbb{N}$, $\varepsilon > 0$ and $k, l \to \infty$ sufficient conditions on $A = [a^{kl}_{ij}]$ such that*

$$Y^{kl} = \big(\Sigma^{kl}_{RC} - y^{kl}\big) - \big(\Sigma^{kl}_{R} - y^{kl}\big) - \big(\Sigma^{kl}_{C} - y^{kl}\big) \tag{3.11}$$

is by absolute value less of ε are

$$(v) \in \mathbf{b}; \qquad (v_\square) \in \mathbf{bcn}. \tag{3.12}$$

PROOF. We change the order of summation of the left-hand-side of (3.10). Instead of a^{kl}_{ij} we write a_{ij}. If the sum runs over an index that does not appear among indexes, the argument of the sum is constant. For example, $\sum_{m=i-p+1}^{i} a_{ij} = pa_{ij}$.

$$y^{kl} = \sum_{i=1}^{p-1}\sum_{j=1}^{q-1} x_{ij}a_{ij} + \frac{1}{p}\sum_{i=p}^{\infty}\sum_{j=1}^{q-1} x_{ij}\sum_{m=i-p+1}^{i} a_{ij} \tag{3.13}$$

$$+\frac{1}{q}\sum_{i=1}^{p-1}\sum_{j=q}^{\infty}x_{ij}\sum_{n=j-q+1}^{j}a_{ij}$$

$$+\frac{1}{pq}\sum_{i=p}^{\infty}\sum_{j=q}^{\infty}x_{ij}\sum_{m=i-p+1}^{i}\sum_{n=j-q+1}^{j}a_{ij},$$

$$(3.14)\quad \varSigma_{R}^{kl} = \frac{1}{p}\sum_{i=1}^{p-1}\sum_{j=1}^{q-1}x_{ij}\sum_{m=1}^{i}a_{mj}+\frac{1}{p}\sum_{i=p}^{\infty}\sum_{j=1}^{q-1}x_{ij}\sum_{m=i-p+1}^{i}a_{mj}$$

$$+\frac{1}{pq}\sum_{i=1}^{p-1}\sum_{j=q}^{\infty}x_{ij}\sum_{m=1}^{i}\sum_{n=j-q+1}^{j}a_{mj}$$

$$+\frac{1}{pq}\sum_{i=p}^{\infty}\sum_{j=q}^{\infty}x_{ij}\sum_{m=i-p+1}^{i}\sum_{n=j-q+1}^{j}a_{mj},$$

$$(3.15)\quad \varSigma_{C}^{kl} = \frac{1}{q}\sum_{i=1}^{p-1}\sum_{j=1}^{q-1}x_{ij}\sum_{n=1}^{j}a_{in}+\frac{1}{pq}\sum_{i=p}^{\infty}\sum_{j=1}^{q-1}x_{ij}\sum_{m=i-p+1}^{i}\sum_{n=1}^{j}a_{in}$$

$$+\frac{1}{q}\sum_{i=1}^{p-1}\sum_{j=q}^{\infty}x_{ij}\sum_{n=j-q+1}^{j}a_{in}$$

$$+\frac{1}{pq}\sum_{i=p}^{\infty}\sum_{j=q}^{\infty}x_{ij}\sum_{m=i-p+1}^{i}\sum_{n=j-q+1}^{j}a_{in},$$

$$(3.16)\quad \varSigma_{RC}^{kl} = \frac{1}{pq}\sum_{i=1}^{p-1}\sum_{j=1}^{q-1}x_{ij}\sum_{m=1}^{i}\sum_{n=1}^{j}a_{mn}+\frac{1}{pq}\sum_{i=p}^{\infty}\sum_{j=1}^{q-1}x_{ij}\sum_{m=i-p+1}^{i}\sum_{n=1}^{j}a_{mn}$$

$$+\frac{1}{pq}\sum_{i=1}^{p-1}\sum_{j=q}^{\infty}x_{ij}\sum_{m=1}^{i}\sum_{n=j-q+1}^{j}a_{mn}$$

$$+\frac{1}{pq}\sum_{i=p}^{\infty}\sum_{j=q}^{\infty}x_{ij}\sum_{m=i-p+1}^{i}\sum_{n=j-q+1}^{j}a_{mn}.$$

Grouping the corresponding terms of (3.13)–(3.16) we get

$$
\begin{aligned}
& y^{kl} - \sum{}_R^{kl} - \sum{}_C^{kl} + \sum{}_{RC}^{kl} \\
&= \sum_{i=1}^{p-1}\sum_{j=1}^{q-1} x_{ij}a_{ij} - \frac{1}{p}\sum_{i=1}^{p-1}\sum_{j=1}^{q-1} x_{ij} \sum_{m=1}^{i} a_{mj} \\
&\quad - \frac{1}{q}\sum_{i=1}^{p-1}\sum_{j=1}^{q-1} x_{ij}\sum_{n=1}^{j} a_{in} + \frac{1}{pq}\sum_{i=1}^{p-1}\sum_{j=1}^{q-1} x_{ij}\sum_{m=1}^{i}\sum_{n=1}^{j} a_{mn} \\
&\quad + \sum_{i=p}^{\infty}\sum_{j=1}^{q-1} x_{ij}\Big[\frac{1}{p}\sum_{m=i-p+1}^{i}(a_{ij}-a_{mj}) - \frac{1}{pq}\sum_{m=i-p+1}^{i}\sum_{n=1}^{j}(a_{in}-a_{mn})\Big] \\
&\quad + \sum_{i=1}^{p-1}\sum_{j=q}^{\infty} x_{ij}\Big[\frac{1}{q}\sum_{n=j-q+1}^{j}(a_{ij}-a_{in}) - \frac{1}{pq}\sum_{m=1}^{i}\sum_{n=j-q+1}^{j}(a_{mj}-a_{mn})\Big] \\
&\quad + \sum_{i=p}^{\infty}\sum_{j=q}^{\infty} x_{ij}\Big[\frac{1}{pq}\sum_{m=i-p+1}^{i}\sum_{n=j-q+1}^{j}(a_{ij}-a_{mj}-a_{in}+a_{mn})\Big].
\end{aligned} \tag{3.17}
$$

Differences of terms of A are finite sums of $a_{\mu\nu}$, $\triangle_{10}a_{\mu\nu}$, $\triangle_{01}a_{\mu\nu}$ and $\Box\, a_{\mu\nu}$ by

$$
\begin{aligned}
a_{mn} - a_{in} &= \sum_{\mu=m}^{i-1} \triangle_{10}a_{\mu n}, \qquad & a_{mn} - a_{mj} &= \sum_{\nu=n}^{j-1} \triangle_{01}a_{m\nu}, \\
a_{mj} - a_{ij} &= \sum_{\mu=m}^{i-1} \triangle_{10}a_{\mu j}, \qquad & a_{in} - a_{ij} &= \sum_{\nu=n}^{j-1} \triangle_{01}a_{i\nu}, \\
a_{mn} - a_{in} - a_{mj} + a_{ij} &= \sum_{\mu=m}^{i-1}\sum_{\nu=n}^{j-1} \Box\, a_{\mu\nu}.
\end{aligned} \tag{3.18}
$$

Therefore, increasing k and l for every $\varepsilon > 0$ the estimate

$$
\Big| y^{kl} - \sum{}_R^{kl} - \sum{}_C^{kl} + \sum{}_{RC}^{kl} \Big| \le \varepsilon \tag{3.19}
$$

is possible reducing the left-hand-side to the linear combination of

$$
|a_{ij}^{kl}|, \quad \sum_{i=1}^{\infty} |\triangle_{10}a_{ij}^{kl}|, \quad \sum_{j=1}^{\infty} |\triangle_{01}a_{ij}^{kl}|, \quad \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\Box\, a_{ij}^{kl}|. \tag{3.20}
$$

By Corollary 2.4 and Lemma 2.1 $(v) \in \mathbf{b}$, $(v_\Box) \in \mathbf{bcn}$ imply (i), $(iv_\triangle) \in \mathbf{bcn}$. Estimates by using $\sum_{ij} |\Box\, a_{ij}|$ are given in [3]. For a fixed ν

$$
\frac{1}{pq}\sum_{m=i-p+1}^{i-2}\sum_{n=j-q+1}^{j-1}\sum_{\mu=m}^{i-1}\sum_{\nu=n}^{j-1} |\Box\, a_{\mu\nu}| \tag{3.21}
$$

is a sum over a triangle with vertices

$$(i-p+1, i-p+1), \quad (i-1, i-p+1), \quad (i-1, i-1). \tag{3.22}$$

It is dominated by the sum over the rectangle with vertices (3.22) plus $(i-p+1, i-1)$:

$$\sum_{m=i-p+1}^{i-1} \sum_{\mu=1-p+1}^{i-1} |\Box\, a_{\mu\nu}| = (p-1) \sum_{\mu=1-p+1}^{i-1} |\Box\, a_{\mu\nu}|. \tag{3.23}$$

The same operation on the plane (ν, n) extends (3.23) to

$$(p-1)(q-1) \sum_{\mu=1-p+1}^{i-1} \sum_{\nu=j-q+1}^{j-1} |\Box\, a_{\mu\nu}|. \tag{3.24}$$

As the result, the corresponding term of (3.17) is dominated by

$$\begin{aligned} &\frac{(p-1)(q-1)}{pq} \sup_{ij} |x_{ij}| \sum_{i=p}^{\infty} \sum_{j=q}^{\infty} \sum_{\mu=i-p+1}^{i-1} \sum_{\nu=j-q+1}^{j-1} |\Box\, a_{\mu\nu}| \\ &\quad \le \frac{(p-1)^2(q-1)^2}{pq} \sup_{ij} |x_{ij}| \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\Box\, a_{ij}| \end{aligned} \tag{3.25}$$

as it is in [3, p. 259, (30)]. ∎

Remark 3.3. The proof of [3, Theorem 1, p. 256] has few obvious misprints. E.g., at p. 257⌊1(24) the bracket must be $(a_{mj} - a_{ij})$; p. 258⌈4(25) instead of $\frac{1}{p}$ must be $\frac{1}{q}$; p. 261⌈1 instead of $\in$ must be $\notin$; ⌈9(33) after $a_{ij}^{k_r, l_r}$ must stay x_{ij}, the sum ⌈10 must be $2\sum_{i=1}^{\infty}\sum_{j=1}^{2n_{r-1}}$, and instead of $=$ must be $\ge$. Similarly, at p. 263, (40)(d) must start with $\lim_{k,l} \sum_j |\triangle \alpha_j^{kl}|$; the words *rows* at ⌈22 and *columns* at ⌈24 must be canceled. Instead of [12]⌈15, [11]⌈16 at p. 255 must be [13], [12].

Lemma 3.4. *Assume that (x_{ij}) a-converges to 0. Then, sufficient conditions for $\left|\sum_R^{kl}\right| \le \varepsilon$ for a given $\varepsilon > 0$ as $p, q, k, l \to \infty$ are*

$$(v) \in \mathbf{b}; \qquad (v_\triangle)_1 \in \mathbf{bcn}. \tag{3.26}$$

Similarly, a sufficient condition for $\left|\sum_C^{kl}\right| \le \varepsilon$ as $p, q, k, l \to \infty$ in case that (x_{ij}) a-converges to 0 is

$$(v) \in \mathbf{b}; \qquad (v_\triangle)_2 \in \mathbf{bcn}. \tag{3.27}$$

Proof. We start with

$$\sum\nolimits_R^{kl} = \sum\nolimits_{RC}^{kl} - \left(\sum\nolimits_{RC}^{kl} - \sum\nolimits_R^{kl}\right). \tag{3.28}$$

The part of (3.28) in the brackets is

$$\Sigma^{kl}_{RC} - \Sigma^{kl}_{R} = \frac{1}{pq}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a^{kl}_{mn}\Big(\sum_{j=n}^{n+q-1}\sum_{i=m}^{m+p-1} x_{ij} - q\sum_{i=m}^{m+p-1} x_{in}\Big). \tag{3.29}$$

The $\sum^{kl}_{RC}$ in (3.29) we transform as follows:

$$\begin{aligned}
\Sigma^{kl}_{RC} &= \frac{1}{pq}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a^{kl}_{mn}\Big(\sum_{i=m}^{m+p-1} x_{in} + \cdots + \sum_{i=m}^{m+p-1} x_{i,n+q-1}\Big)\\
&= \frac{1}{pq}\sum_{m=1}^{\infty} a^{kl}_{m1}\Big(\sum_{i=m}^{m+p-1} x_{i1} + \cdots + \sum_{i=m}^{m+p-1} x_{i,q-1} + \sum_{i=m}^{m+p-1} x_{iq}\Big)\\
&\quad + \cdots\\
&\quad + \frac{1}{pq}\sum_{m=1}^{\infty} a^{kl}_{mq}\Big(\sum_{i=m}^{m+p-1} x_{iq} + \cdots + \sum_{i=m}^{m+p-1} x_{i,2q-1}\Big)\\
&\quad + \cdots\\
&= \frac{1}{q}\sum_{m=1}^{\infty} a^{kl}_{m1}\frac{1}{p}\sum_{i=m}^{m+p-1} x_{i1} + \frac{1}{q}\sum_{m=1}^{\infty}\big(a^{kl}_{m1} + a^{kl}_{m2}\big)\frac{1}{p}\sum_{i=m}^{m+p-1} x_{i2} + \cdots\\
&\quad + \frac{1}{q}\sum_{m=1}^{\infty}\big(a^{kl}_{m1} + \cdots + a^{kl}_{m,q-1}\big)\frac{1}{p}\sum_{i=m}^{m+p-1} x_{i,q-1}\\
&\quad + \frac{1}{q}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\big(a^{kl}_{mn} + \cdots + a^{kl}_{m,n+q-1}\big)\frac{1}{p}\sum_{i=m}^{m+p-1} x_{i,n+q-1}.
\end{aligned} \tag{3.30}$$

Denoting the sum of the first $q-1$ lines above by $\Sigma^{kl}_{RC}(I)$, and the last line by

$$\Sigma^{kl}_{RC}(II) = \frac{1}{pq}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\big(a^{kl}_{mn} + \cdots + a^{kl}_{m,n+q-1}\big)\sum_{i=m}^{m+p-1} x_{i,n+q-1} \tag{3.31}$$

we obtain

$$\Sigma^{kl}_{RC} = \Sigma^{kl}_{RC}(I) + \Sigma^{kl}_{RC}(II). \tag{3.32}$$

Splitting the sum with respect to j of Σ^{kl}_{R} in (3.29) for the part with $j \le q-1$ and the part with $j \ge q$ it follows that

$$\begin{aligned}
\Sigma^{kl}_{R}(I) &= \frac{1}{pq}\sum_{m=1}^{\infty}\sum_{j=1}^{q-1} a^{kl}_{mj} q \sum_{i=m}^{m+p-1} x_{ij},\\
\Sigma^{kl}_{R}(II) &= \frac{1}{pq}\sum_{m=1}^{\infty}\sum_{j=q}^{\infty} a^{kl}_{mj} q \sum_{i=m}^{m+p-1} x_{ij}.
\end{aligned} \tag{3.33}$$

Therefore

$$\begin{aligned}\Sigma^{kl}_{R} &= \Sigma^{kl}_{R}(I) + \Sigma^{kl}_{R}(II) \\ &= \sum_{m=1}^{\infty} a^{kl}_{m1} \frac{1}{p} \sum_{i=m}^{m+p-1} x_{ij} + \cdots \\ &\quad + \sum_{m=1}^{\infty} a^{kl}_{m,q-1} \frac{1}{p} \sum_{i=m}^{m+p-1} x_{i,q-1} + \frac{1}{pq} \sum_{m=1}^{\infty} q \sum_{j=q}^{\infty} a^{kl}_{mj} \sum_{i=m}^{m+p-1} x_{ij}\end{aligned} \tag{3.34}$$

with

$$\begin{aligned}\Sigma^{kl}_{R}(II) &= \frac{1}{pq} \sum_{m=1}^{\infty} \sum_{j=q}^{\infty} a^{kl}_{mj} q \sum_{i=m}^{m+p-1} x_{ij} \\ &= \frac{1}{pq} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q a^{kl}_{m,n+q-1} \sum_{i=m}^{m+p-1} x_{i,n+q-1}.\end{aligned} \tag{3.35}$$

We estimate (3.28) by

(3.36)

$$\begin{aligned}|\Sigma^{kl}_{R}| &\leq |\Sigma^{kl}_{RC}| + |\Sigma^{kl}_{RC} - \Sigma^{kl}_{R}| \\ &\leq |\Sigma^{kl}_{RC}| + |\Sigma^{kl}_{RC}(I)| + |\Sigma^{kl}_{R}(I)| + |\Sigma^{kl}_{RC}(II) - \Sigma^{kl}_{R}(II)|.\end{aligned}$$

Now we look at members of (3.36). First, we take N_ε defined by (1.6) and C defined by (2.2). For $p, q \geq N_\varepsilon$ we have

$$|\Sigma^{kl}_{RC}| = \Big| \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a^{kl}_{mn} \frac{1}{pq} \sum_{j=n}^{n+q-1} \sum_{i=m}^{m+p-1} x_{ij} \Big| \leq C\varepsilon. \tag{3.37}$$

For a particular n and $\varepsilon > 0$, by (2.20) and a sufficiently large k, l,

$$\sum_{m=1}^{\infty} |a^{kl}_{mn}| \leq \frac{\varepsilon}{q-1} \tag{3.38}$$

for every n (Corollary 2.5 and Remark 2.8).

By (3.30), $|\Sigma^{kl}_{RC}(I)|$ is estimated from above by

$$\begin{aligned}&(q-1) \sum_{m=1}^{\infty} |a^{kl}_{m1}| \frac{1}{q} \Big| \frac{1}{p} \sum_{i=m}^{m+p-1} x_{i1} \Big| + \ldots \\ &\cdots + \sum_{m=1}^{\infty} |a^{kl}_{m,q-1}| \frac{1}{q} \Big| \frac{1}{p} \sum_{i=m}^{m+p-1} x_{i,q-1} \Big|.\end{aligned} \tag{3.39}$$

By (3.38) and choosing k,l large enough we obtain

$$\sum_{i=1}^{\infty}|a_{ij}^{kl}|<\frac{\varepsilon}{q-1},\quad j=1,\dots,q-1. \tag{3.40}$$

With

$$\sup_{ij}|x_{ij}|=B<+\infty, \tag{3.41}$$

we have

$$\frac{1}{q}\sum_{m=1}^{\infty}|a_{mj}^{kl}|\Big|\frac{1}{p}\sum_{i=m}^{m+p-1}x_{ij}\Big|\le\frac{1}{q}B\frac{\varepsilon}{q-1},\quad j=1,\dots,q-1 \tag{3.42}$$

for every $p\in\mathbb{N}$. Estimating (3.39) we obtain

$$\big|\textstyle\sum_{RC}^{kl}(I)\big|\le\frac{(q-1)q}{2}\cdot\frac{1}{q}B\frac{\varepsilon}{q-1}=\frac{B\varepsilon}{2}. \tag{3.43}$$

The estimate for $\sum_{R}^{kl}(I)$ in (3.33) differs from (3.39) having no coefficients $\frac{q-1}{q},\dots,\frac{1}{q}$. Therefore

$$\big|\textstyle\sum_{R}^{kl}(I)\big|\le B\varepsilon. \tag{3.44}$$

To complete the estimate of (3.36) we estimate the difference of (3.31) and (3.35):

$$\begin{aligned}&\textstyle\sum_{RC}^{kl}(II)-\sum_{R}^{kl}(II)\\&=\frac{1}{q}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\big((a_{mn}^{kl}-a_{m,n+q-1}^{kl})\\&\quad+\dots+(a_{m,n+q-1}^{kl}-a_{m,n+q-1}^{kl})\big)\frac{1}{p}\sum_{i=m}^{m+p-1}x_{i,n+q-1}.\end{aligned} \tag{3.45}$$

For $q\ge 2$ the above differences are sums

$$\begin{aligned}&(a_{mn}^{kl}-a_{m,n+1}^{kl})+\dots+(a_{m,n+q-2}^{kl}-a_{m,n+q-1}^{kl})\\&(a_{m,n+1}^{kl}-a_{m,n+2}^{kl})+\dots+(a_{m,n+q-2}^{kl}-a_{m,n+q-1}^{kl})\\&\qquad\vdots\end{aligned} \tag{3.46}$$

with $q-1,q-2,\dots,1$ terms. For every m in (3.46) there are $\frac{(q-1)q}{2}$ terms with $\Delta_{01}a_{mj}^{kl}$ (counting separately members that in the sum of lines (3.46) result to be equal).

The $(v_\triangle)_1\in\mathbf{bcn}$ in (3.26) means that

$$\lim_{k,l}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}|(a_{mn}^{kl}-a_{m,n+1}^{kl})|=0. \tag{3.47}$$

The same equality is valid if we have $\triangle_{01}a^{kl}_{m,n+\varrho}$ $(\varrho = 0, 1, \dots, q-2)$ instead of $\triangle_{01}a^{kl}_{mn}$. The (3.46) is reduced to the sum of

$$\frac{1}{q}\sum_{n=1}^{\infty}\Big(\sum_{m=1}^{\infty}\triangle_{01}a^{kl}_{m,n+\varrho}\frac{1}{p}\sum_{i=m}^{m+p-1}x_{i,n+q-1}\Big) \tag{3.48}$$

with $\varrho = 0, \dots, q-2$. The absolute value of (3.48) is less or equal to

$$\frac{1}{q}\sum_{n=1}^{\infty}\Big|\sum_{m=1}^{\infty}\triangle_{01}a^{kl}_{m,n+\varrho}\Big|B, \tag{3.49}$$

where by $B = \sup_{ij}|x_{ij}|$. By the $\frac{\varepsilon}{q-1}$-estimate for (3.47) the (3.49) is $\le \frac{B\varepsilon}{q(q-1)}$. Therefore (3.45) is estimated by

$$\big|\textstyle\sum^{kl}_{RC}(II) - \sum^{kl}_{R}(II)\big| \le \dfrac{B\varepsilon}{2}. \tag{3.50}$$

By (3.50), (3.43) and (3.44)

$$|\textstyle\sum^{kl}_{RC}(II) - \sum^{kl}_{R}(II)| + |\sum^{kl}_{RC}(I)| + |\sum^{kl}_{R}(I)| \le 2B\varepsilon. \tag{3.51}$$

Therefore, with (3.37) we have the estimate for (3.36):

$$\big|\textstyle\sum^{kl}_{R}\big| \le (C+2B)\varepsilon. \tag{3.52}$$

This means that Lemma 3.4 holds for $\sum^{kl}_{R}$. Similarly for $\sum^{kl}_{C}$. ◻

Theorem 3.5. *The matrix* $A = [a^{kl}_{ij}]$ *transforms the space* **racrn** *into the space* **bcn** *if and only if*

$$(v) \in \mathbf{b}; \qquad (v_\triangle) \in \mathbf{bcn}. \tag{3.53}$$

Proof. *Sufficiency.* With (3.10) and (3.11), y^{kl} is estimated by estimates for Y^{kl}, $\sum^{kl}_{RC}$, $\sum^{kl}_{R}$ and $\sum^{kl}_{C}$. Let p, q be such that $|\sum^{kl}_{RC}| < \varepsilon C$, which is possible by the existence of the P a-limit 0 of σ^{mn}_{pq} and C from (2.2). Lemma 3.2 gives Y^{kl} as small as we please with conditions $(v) \in \mathbf{b}$ and $(v_\square) \in \mathbf{bcn}$. For $\sum^{kl}_{R}$ and $\sum^{kl}_{C}$, conditions on $(v_\triangle)_1$, resp. $(v_\triangle)_2$, are given by Lemma 3.4. Conditions (3.26) and (3.27) assure that, by $k, l \to \infty$ and every $x \in$ **racrn**, both $\sum^{kl}_{R}$ and $\sum^{kl}_{C}$ converge to 0. The same conditions imply $(v_\square) \in \mathbf{bcn}$ (Lemma 2.7). Therefore y^{kl} from (3.10) also P converges to 0.

Necessity. We assume $(v) \in \mathbf{b}$ and $(i) \in \mathbf{bcn}$ which are *N.S.* for the transformation of **rcrn** to **bcn**. The necessity of $(v_\triangle)_1 \in \mathbf{bcn}$ for the space of all double sequences with columns uniformly a-convergent to 0 and rows non-uniformly a-convergent to 0 is proved in [12, pp. 287–288]. The proof there is given for uniformly a-convergent rows. Moreover, the bounded regularity of

$[a^{kl}_{mn}]$ is assumed. It includes $(iv) \in \mathbf{bcn}$ which is not needed for the necessity proof but follows from $(v_\triangle) \in \mathbf{bcn}$ by Corollary 2.5. ◻

REMARK 3.6. The sufficiency proof in [12] uses $(v_\triangle)_1$ and $(v_\triangle)_2$ together. In our setting they are related to $\sum^{kl}_R$, resp. to $\sum^{kl}_C$. In the case of the uniformly a-convergent columns $\sum^{kl}_C$ is small for large q and Y^{kl} is estimated via Lemma 2.7. Notice that the p and q are determined by (3.37) and hereafter remain fixed.

COROLLARY 3.7. *Let subspaces of* **racrn** *be sets of all* (x_{ij}) *such that their a-convergence to 0 is*

$$\begin{aligned} &i.\ \textit{uniform by rows and columns}\\ &ii.\ \textit{uniform by columns}\\ &iii.\ \textit{uniform by rows}\\ &iv.\ \textit{without uniformity restrictions.}\end{aligned} \tag{3.54}$$

Conditions on $[a^{kl}_{ij}]$, *N.S. for the transformation of the classes above into* **bcn**, *corresponding to the respective cases are*

$$\begin{aligned} i'.\ &(v) \in \mathbf{b}, \quad (v_\square) \in \mathbf{bcn}\\ ii'.\ &(v) \in \mathbf{b}, \quad (v_\triangle)_1 \in \mathbf{bcn}\\ iii'.\ &(v) \in \mathbf{b}, \quad (v_\triangle)_2 \in \mathbf{bcn}\\ iv'.\ &(v) \in \mathbf{b}, \quad (v_\triangle) \in \mathbf{bcn}.\end{aligned} \tag{3.55}$$

THEOREM 3.8. *The matrix* $A = [a^{kl}_{ij}]$ *transforms the space* **rac** *into the space* **bc** *such that* $\lim_{kl} y^{kl}$ *of* $y = Ax$ *is equal to* $\mathrm{Lim}_{ij} x^{ij}$ *if and only if*

$$(iii) \in \mathbf{bc} \quad \lim_{kl} = 1; \quad (v) \in \mathbf{b}; \quad (v_\triangle) \in \mathbf{bcn}. \tag{3.56}$$

PROOF. *Sufficiency.* We estimate

$$y^{kl} - L = Y^{kl} - \Big(\sum{}^{kl}_{RC} - L\Big) + \Big(\sum{}^{kl}_{R} - L\Big) + \Big(\sum{}^{kl}_{C} - L\Big). \tag{3.57}$$

By the Lemma 3.2, $(v) \in \mathbf{b}$ and $(v_\square) \in \mathbf{bcn}$ are sufficient for $|Y^{kl}| \leq \varepsilon$ if k, l are sufficiently large.

The last three terms of (3.57) are

$$
\begin{aligned}
{\textstyle\sum}^{kl}_{RC} - L &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}^{kl}\Big[\frac{1}{pq}\sum_{i=m}^{m+p-1}\sum_{j=n}^{n+q-1}(x_{ij}-L)\Big] \\
&\quad + \Big(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}^{kl} - 1\Big)L, \\
{\textstyle\sum}^{kl}_{R} - L &= \sum_{m=1}^{\infty}\sum_{j=1}^{\infty} a_{mj}^{kl}\Big[\frac{1}{p}\sum_{i=m}^{m+p-1}(x_{ij}-L_i'') + \Big(\frac{1}{p}\sum_{i=m}^{m+p-1}L_i'' - L\Big)\Big] \\
&\quad + \Big(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}^{kl} - 1\Big)L, \\
{\textstyle\sum}^{kl}_{C} - L &= \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} a_{in}^{kl}\Big[\frac{1}{q}\sum_{j=n}^{n+q-1}(x_{ij}-L_j') + \Big(\frac{1}{q}\sum_{j=n}^{n+q-1}L_j' - L\Big)\Big] \\
&\quad + \Big(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}^{kl} - 1\Big)L.
\end{aligned} \tag{3.58}
$$

By $(iii) \in$ **bc**, $\lim_{kl} = 1$, increasing k, l we have $|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}^{kl} - 1||L| \le \frac{\varepsilon}{8}$. With C from $(v) \in$ **b** and the P a-convergence of x, the choice of p, q gives $|\sigma_{pq}^{mn} - L| \le \frac{\varepsilon}{8C}$ and $|{\textstyle\sum}^{kl}_{RC} - L| \le \frac{\varepsilon}{4}$. By Lemma 3.1, $\Big|\frac{1}{p}\sum_{i=m}^{m+p-1} L_i'' - L\Big|$ and $\Big|\frac{1}{q}\sum_{j=n}^{n+q-1} L_j' - L\Big|$ are also $\le \frac{\varepsilon}{8C}$. The terms $(x_{ij} - L_i'')$ resp. $(x_{ij} - L_j')$ in ${\textstyle\sum}^{kl}_{R} - L$ and ${\textstyle\sum}^{kl}_{C} - L$ a-converge to 0. Lemma 3.4 applies on $x_{ij} - L_i''$ because (x_{ij}) and (L_i'') as double sequences a-converge to L: $x \in$ **rac** and $\frac{1}{p}\sum_{i=m}^{m+p-1}\frac{1}{q}\sum_{j=n}^{n+q-1} L_i'' = \frac{1}{p}\sum_{i=m}^{m+p-1} L_i''$ converges to L if $p, q \to \infty$. By (3.26) and (3.27) i.e. by $(v) \in$ **b** and $(v_\triangle)_1 \in$ **bcn**, resp. $(v_\triangle)_2 \in$ **bcn**, if $k, l \to \infty$, the absolute value of these members becomes $\le \frac{\varepsilon}{8}$. Therefore, the linear combination of (3.58) is by absolute value $\le \varepsilon$ and $|y^{kl} - L| \le 2\varepsilon$.

Necessity. The *N.S.* conditions for the a-convergence of sequences which have uniformly a-convergent rows and columns (the class **rac un** denoted in [3] by **arc**) are necessary conditions for transformations of sequences from **rac**. This means that $(v) \in$ **b**, $(iii) \in$ **bc** with $\lim_{kl} = 1$ and $(v_\square) \in$ **bcn** are necessary. Other conditions, given in [3, Theorem 2, p. 361] follow by implications (2.23). Also, necessary are $(v_\triangle) \in$ **bcn** because of **racrn** $\subset$ **rac** and the Theorem 3.5 above. □

Similarly to Corollary 3.7, subclasses of **rac** are characterized by uniformities of row, resp. column, a-convergence.

Corollary 3.9. *Let subspaces of* **rac** *be sets of all* (x_{ij}) *such that their a-convergence is*

$$\begin{aligned} &i.\text{uniform by rows and columns}\\ &ii.\text{uniform by columns}\\ &iii.\text{uniform by rows}\\ &iv.\text{without uniformity restrictions.}\end{aligned} \tag{3.59}$$

Conditions on A, N.S. that $y = Ax$ *belongs to* **bc** *with* $\lim_{kl} y^{kl} = \operatorname{Lim}_{ij} x_{ij}$ *are*

$$(iii) \in \mathbf{bc}, \qquad \lim_{kl} = 1, \qquad (v) \in \mathbf{b} \tag{3.60}$$

and, moreover, for cases i–iv,

$$\begin{aligned} &i'.(v_\square), \quad (vi_\triangle) \in \mathbf{bcn}\\ &ii'.(v_\triangle)_1, \quad (vi_\triangle)_2 \in \mathbf{bcn}\\ &iii'.(v_\triangle)_2, \quad (vi_\triangle)_1 \in \mathbf{bcn}\\ &iv'.(v_\triangle) \in \mathbf{bcn}.\end{aligned} \tag{3.61}$$

Proof. The proof for i is given in [3]. We look at the proof for ii. *Sufficiency.* For $\Sigma^{kl}_R - L$ in (3.58) we apply (3.60) and $(v_\triangle)_1 \in \mathbf{bcn}$ as in the proof of Theorem 3.8. For $\Sigma^{kl}_C - L$ we have

$$\begin{aligned} \Sigma^{kl}_C - L =& \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} a^{kl}_{in}\Big[\Big(\frac{1}{q}\sum_{j=n}^{n+q-1} x_{ij} - L''_i\Big) + (L''_i - L)\Big]\\ &+ \Big(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a^{kl}_{mn} - 1\Big)L.\end{aligned} \tag{3.62}$$

Because of the uniformity for $\varepsilon > 0$ by increasing q we obtain $\Big|\frac{1}{q}\sum_{j=n}^{n+q-1} x_{ij} - L''_i\Big| \le \varepsilon$ for every $n, i = 1, 2, \ldots$. For the transformation of $(L''_i - L)$ we apply $(vi_\triangle)_2 \in \mathbf{bcn}$.

Necessity. For Σ^{kl}_C and Y^{kl} transformation conditions follow from [3, Theorem 2, p. 261] where *N.S.* conditions are given for the case i. Conditions (i), $(ii) \in \mathbf{bcn}$ in [3] are superfluous by Lemmas 2.2 and 2.3. For Σ^{kl}_R, as for Theorems above, necessity follows from the proof for y^{kl} in [12]. Móricz-Rhoades conditions for transformations of the class **ac** in [12] are conditions of our Theorem 3.8. For the transformations of **acn** the $(iii) \in \mathbf{b}$, $\lim_{kl} = 1$ in [12] is superfluous (cf. also [4, 35. $RCN \to BCN$, p. 49]). Conditions that follow from [12] for transformations of **acn** are the same as our conditions for transformations of **racrn** and therefore also for our **racn** (Theorem 3.5). □

Proofs of the cases (ii) and (iii) in Corollaries 3.7 and 3.9 remain valid without the a-convergence of rows, resp. columns (see Remark 3.6). We denote subspaces of **acn** and of **ac** with H a-convergence by columns, resp. by rows, by $\mathbf{r_1}$**acn** and $\mathbf{r_1}$**ac**, resp. by $\mathbf{r_2}$**acn** and $\mathbf{r_2}$**ac**. Therefore, $\mathbf{r_1}$**ac** is the class of sequences with the P a-limit L and with columns that a-converge to L_i''. By Lemma 1.1 the L_i'' a-converges to L but row a-limits don't necessarily exist. The class $\mathbf{r_1}$**acr** is a subclass of $\mathbf{r_1}$**ac** with $L_i'' = L$ for every i and $\mathbf{r_1}$**acn** is a subclass of $\mathbf{r_1}$**ac** with $L = 0$. For $x \in \mathbf{r_1}$**acn** and $i \in \mathbb{N}$ the L_i'' are not necessarily 0. The class $\mathbf{r_1}$**acrn** is a subclass of $\mathbf{r_1}$**ac** with $L_i'' = L = 0$ for every i. If the a-convergence of columns to L_i'' is uniform, to the designation of the class we add **un**. If instead of columns we have the a-convergence by rows, instead of $\mathbf{r_1}$ we have $\mathbf{r_2}$.

Because the classes $\mathbf{r_1}$**acrn** and $\mathbf{r_1}$**ac** as well as $\mathbf{r_2}$**acrn** and $\mathbf{r_2}$**ac** are in the between of **racrn** and **ac**, *N.S.* conditions for transformations of these classes to **bcn** and **bc** are conditions of Theorems 3.5 and 3.8. In case that the H a-convergence by columns or by rows is uniform, the corresponding conditions differ from conditions [12].

THEOREM 3.10. *The matrix* $[a_{ij}^{kl}]$ *transforms the space* $\mathbf{r_1}$**acrn un** *into the space* **bcn** *if and only if*

$$(3.63)\qquad (v) \in \mathbf{b}, \quad (v_\triangle)_1 \in \mathbf{bcn}.$$

Conditions N.S. for the transformation of $\mathbf{r_2}$**acrn un** *into* **bcn** *are*

$$(3.64)\qquad (v) \in \mathbf{b}, \quad (v_\triangle)_2 \in \mathbf{bcn}.$$

The matrix $A = [a_{ij}^{kl}]$ *transforms* $\mathbf{r_1}$**ac un** *into* **bc** *such that* $\lim_{kl} y^{kl}$ *of* $y = Ax$ *is equal to* $\mathrm{Lim}_{ij} x_{ij}$ *if and only if*

$$(3.65)\qquad (iii) \in \mathbf{bc},\ \lim_{kl} = 1, \quad (v) \in \mathbf{b}, \quad (v_\triangle)_1 \in \mathbf{bcn}, \quad (vi_\triangle)_2 \in \mathbf{bcn}.$$

The N.S. conditions for the analogous transformation of $\mathbf{r_2}$**ac un** *are*

$$(3.66)\qquad (iii) \in \mathbf{bc},\ \lim_{kl} = 1, \quad (v) \in \mathbf{b}, \quad (v_\triangle)_2 \in \mathbf{bcn}, \quad (vi_\triangle)_1 \in \mathbf{bcn}.$$

The N.S. conditions on A for the transformation of $\mathbf{r_1}$**acr un**, *resp.* $\mathbf{r_2}$**acr un**, *into* **bc** *with* $\lim_{kl} y_{kl} = \mathrm{Lim}_{ij} x_{ij}$ *are (3.63) resp. (3.64), with added* $(iii) \in \mathbf{bc}$, $\lim_{kl} = 1$.

The N.S. conditions on A for the transformation of $\mathbf{r_1}$**acn un**, *resp.* $\mathbf{r_2}$**acn un**, *into* **bcn** *are (3.65), resp. (3.66) without* $(iii) \in \mathbf{bc}$, $\lim_{kl} = 1$.

PROOF. With inclusions commented in the introduction of classes with $\mathbf{r_1}$ and $\mathbf{r_2}$ the various cases above are deduced by the corresponding cases in Corollaries 3.7 and 3.9. See also Corollary in [3, p. 261]. □

REMARK 3.11. In accordance with the notation for classes in Theorem 3.10 we can denote classes in Corollaries 3.7 (ii), (iii) and 3.9 (ii), (iii) by **racrn** $\mathbf{un_1}$, **racrn** $\mathbf{un_2}$ and **rac** $\mathbf{un_1}$, **rac** $\mathbf{un_2}$. The necessity counterexample in [12] is from **racrn** $\mathbf{un_2}$.

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